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Research Article

Polar Functions for Anisotropic Gaussian Random Fields

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Let X be an (N, d)-anisotropic Gaussian random field. Under some general conditions on X, we establish a relationship between a class of continuous functions satisfying the Lipschitz condition and a class of polar functions of X. We prove upper and lower bounds for the intersection probability for a nonpolar function and X in terms of Hausdorff measure and capacity, respectively. We also determine the Hausdorff and packing dimensions of the times set for a nonpolar function intersecting X. The class of Gaussian random fields that satisfy our conditions includes not only fractional Brownian motion and the Brownian sheet, but also such anisotropic fields as fractional Brownian sheets, solutions to stochastic heat equation driven by space-time white noise, and the operator-scaling Gaussian random field with stationary increments.

1. Introduction

Gaussian random fields have been extensively studied in probability theory and applied in a wide range of scientific areas including physics, engineering, hydrology, biology, economics, and finance. Two of the most important Gaussian random fields are, respectively, the Brownian sheet and fractional Brownian motion.

On the other hand, many data sets from various areas such as image processing, hydrology, geostatistics, and spatial statistics have anisotropic nature in the sense that they have different geometric and probabilistic characteristics along different directions. Hence fractional Brownian motion, which is isotropic in the sense that the distribution of its increments depends only on the Euclidean distance of the time interval, is not adequate for modelling such phenomena. Many people have proposed to apply anisotropic Gaussian random fields as more realistic models; see [1, 2] and the references therein for more information.

Typical examples of anisotropic Gaussian random fields are fractional Brownian sheets and the solution to the stochastic heat equation. It has been known that the sample path properties such as fractal dimensions of these anisotropic Gaussian random fields can be very different from those of isotropic ones such as Levy's fractional Brownian motion; see, for example, [3–7]. Recently, Xiao [2] systematically studied the analytic and geometric properties of anisotropic Gaussian random fields under certain general conditions. Biermé et al. [1] studied the hitting probabilities and the Hausdorff dimension of the inverse of anisotropic Gaussian random fields under some conditions. Their main goal is to characterize the anisotropic nature of the Gaussian random fields by a multiparameter index $H = (H_1, \dots, H_N) \in (0,1)^N$. This index is often related to the operator-self-similarity or multi-self-similarity of the Gaussian random field under study. In this paper, we further discuss the polar functions of anisotropic Gaussian random fields.

We will continue to use the same setting as in Biermé et al. [1]. Let $H = (H_1, \dots, H_N) \in (0,1)^N$ be a fixed vector and, for $a, b \in \mathbb{R}^N$ with $a_j < b_j$ $(j = 1, \dots, N)$, let $I = [a, b] := \prod_{j=1}^{N} [a_j, b_j] \subseteq \mathbb{R}^N$ denote a compact interval (or a rectangle). For example, we may take $I = [\epsilon_0, 1]^N$, where $\epsilon_0 \in (0,1)$ is a fixed constant.

Let $X(t) = (X_1(t), \dots, X_d(t))$, $t \in \mathbb{R}^N$, be a Gaussian random field on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with mean zero and whose components X_k, $k = 1, \dots, d$, are independent.

Suppose that for each $k = 1, \dots, d$, X_k satisfies the following general conditions.

(C1) There exist positive and finite constants $c_{1.1}$, $c_{1.2}$, and $c_{1.3}$ such that $\mathbb{E}[X_k(t)]^2 \geq c_{1.1}$ for all $t \in I$ and

$$c_{1.2} \sum_{\ell=1}^{N} \left| s_\ell - t_\ell \right|^{2H_\ell} \leq \mathbb{E}\left[X_k(s) - X_k(t) \right]^2 \leq c_{1.3} \sum_{\ell=1}^{N} \left| s_\ell - t_\ell \right|^{2H_\ell}, \quad \forall s, t \in I. \tag{1}$$

(C2) There exists a positive and finite constant $c_{1.4}$ such that, for all $s, t \in I$,

$$\operatorname{Var}\left(X_k(t) \mid X_k(s) \right) \geq c_{1.4} \sum_{\ell=1}^{N} \left| s_\ell - t_\ell \right|^{2H_\ell}. \tag{2}$$

Here $\operatorname{Var}(X_k(t)|X_k(s))$ denotes the conditional variance of $X_k(t)$ given $X_k(s)$. We will call X an (N, d)-Gaussian random field. Xiao [2] and Biermé et al. [1] gave some remarks on the above conditions. We point out that the class of Gaussian random fields that satisfy conditions (C1) and (C2) is large. It includes not only the well-known fractional Brownian motion and the Brownian sheet, but also such anisotropic random fields as fractional Brownian sheets (cf. [3, 4, 7]), solutions to stochastic heat equation driven by space-time white noise (cf. [5, 6, 8–10]), and many more.

In the following, we present some notations about several classes of functions satisfying certain conditions. The relationship between them will be studied in Section 3.

Let $\mathscr{C} = \{f : f$ is a continuous function on $\mathbb{R}^N$ with values in $\mathbb{R}^d\}$. As usual, a function $f \in \mathscr{C}$ is said to be a polar function for the random field $X(t)$ if

$$\mathbb{P}\left\{ \exists\, t \in \mathbb{R}^N \text{ such that } X(t) = f(t) \right\} = 0. \tag{3}$$

Let $\mathscr{P}$ denote the collection of the continuous functions satisfying (3).

Let $K = (K_1, \dots, K_N) \in (0,1)^N$ be a fixed vector, and let $\mathscr{L}(K)$ denote the collection of all Hölder continuous functions of any order less than K_ℓ along the ℓth direction in time; that is, there exists a finite and positive constant $c_{1.5}$, depending only on I and K_ℓ $(0 \leq \ell \leq N)$, such that for all $0 < \delta_\ell < K_\ell$ $(0 \leq \ell \leq N)$, $s, t \in I$, and $f \in \mathscr{C}$,

$$\left| f(s) - f(t) \right| \leq c_{1.5} \sum_{\ell=1}^{N} \left| s_\ell - t_\ell \right|^{K_\ell - \delta_\ell}. \tag{4}$$

Moreover, let $\mathscr{Q}(K)$ denote the collection of all functions satisfying the following condition: there exist finite and positive constants $c_{1.6}$ and $c_{1.7}$, depending only on I and K_ℓ $(0 \leq \ell \leq N)$, such that for all $s, t \in I$ and $f \in \mathscr{C}$,

$$c_{1.6} \sum_{\ell=1}^{N} \left| s_\ell - t_\ell \right|^{K_\ell} \leq \left| f(s) - f(t) \right| \leq c_{1.7} \sum_{\ell=1}^{N} \left| s_\ell - t_\ell \right|^{K_\ell}. \tag{5}$$

Note that if $K = (\alpha, \dots, \alpha)$, then the functions in $\mathscr{L}(K)$ are called Hölder continuous of any order less than α, and the functions in $\mathscr{Q}(K)$ are called quasi-spiral with order α; see Kahane [11]. Hence $\mathscr{L}(K)$ and $\mathscr{Q}(K)$ can be regarded as a nature generalization of Hölder continuous function and quasi-spiral, respectively.

In the studies of random fields, it is interesting to consider the following questions.

(i) Given a nonrandom continuous function $f \in \mathscr{C}$, when is it nonpolar for X in the sense that $\mathbb{P}\{\exists t \in \mathbb{R}^N$ such that $X(t) = f(t)\} > 0$? When is it polar for X in the sense that $\mathbb{P}\{\exists t \in \mathbb{R}^N$ such that $X(t) = f(t)\} = 0$?

(ii) Given a nonrandom Borel set $E \in \mathbb{R}^N$, what is the probability for the random set $\{\exists t \in I$ such that $X(t) = f(t)\}$? What is the Hausdorff and packing dimensions of the set $\{t \in \mathbb{R}^N : X(t) = f(t)\}$ if f is nonpolar f or X?

The above questions are some important questions in fractal theory of random fields and the related results have only been known for a few types of random fields. For example, Graversen [12] studied the characteristics of the polar functions for the two dimensional Brownian motions. Le Gall [13] made a further discussion for the d-dimensional Brownian motion and proposed an open problem about the existence of its no-polar continuous function satisfying the Hölder condition. Some of these results have been extended partially to fractional Brownian motion with stationary increments by Xiao [14], to the Brownian sheet with independent increments by Chen [15], and recently to the fractional Brownian sheets with anisotropy by Chen [4].

In all these papers, the isotropic properties of the Brownian sheet and fractional Brownian motion have played crucial roles. Since, in general, the anisotropic random fields have neither the isotropic properties nor the properties of independent increment and stationary increments due to their general dependence structure, it is more difficult to investigate fine properties of their sample paths. The main objective of this paper is to further investigate the characteristics of the polar functions and the intersection probabilities for X satisfying conditions (C1) and (C2) by using the approach of Biermé et al. in [1] and Xiao in [2]. Our main results, in some cases, strengthen the results in the aforementioned works, and their proofs are different from the proofs for the Brownian sheet and the fractional Brownian motion. Of particular significance, we determine the exact Hausdorff and packing dimensions of the times set for a nonpolar function intersecting X. However, for the intersection probability, we can only establish an inequality in terms of Hausdorff measure and capacity, respectively; see Theorem 16. It is still an open problem to prove the best upper bound in terms of capacity. We should also point out that, compared with the isotropic case, the anisotropic nature of X induces far richer fractal structure into the properties of the nonpolar functions for X.

The rest of the paper is organized as follows. In Section 2, we derive a few preliminary estimates and lemmas for X that will be useful to our arguments. In Section 3, we obtain the relationship between the class of continuous functions satisfying Lipschitz condition and the class of polar functions of X. We also give upper and lower bounds for the probabilities for a nonpolar function intersecting X and determine the Hausdorff and packing dimensions of the times points for a nonpolar function intersecting X. A question proposed by Le Gall [13] about the existence of no-polar, continuous Hölder functions for the Brownian motion is also solved. Finally in Section 4, we show that our main results in Section 3 can be applied to solutions to stochastic partial differential equations.

Throughout this paper we will use c to denote unspecified positive and finite constant whose precise values are not important and may be different in each appearance. More specific constants in Section i are numbered as $c_{i.1}, c_{i.2}, \ldots$.

2. Some Preliminary Estimates

Because of the complex dependence structure for the anisotropic Gaussian random fields, the proofs of the main results in Sections 3 and 4 are quite involved. Therefore, we split the proofs into several lemmas to be used in Sections 3 and 4.

Let E be a compact set in $\mathbb{R}^N$. $\operatorname{Cov}(X_k(s), X_k(t))$ denotes the covariance matrix of the random vector $(X_k(s), X_k(t))$. Then, for all $s, t \in E$,

$$\det\operatorname{Cov}\left(X_k(s), X_k(t)\right) = \mathbb{E}\left[X_k^2(s)\right]\mathbb{E}\left[X_k^2(t)\right] - \left[\mathbb{E}\left(X_k(s)X_k(t)\right)\right]^2, \tag{6}$$

where $k = 1, \ldots, d$.

We need to estimate upper and lower bounds of the covariance determinant in (6). For the sake of completeness, we provide a simple proof by using the expression for the characteristic functions and the density functions of Gaussian random fields.

Lemma 1. *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be an (N, d)-Gaussian random field satisfying conditions (C1) and (C2) and let E be a compact set on $\mathbb{R}^N$. Then there exist positive constants $c_{2.1}$ and $c_{2.2}$, such that for all $s = (s_1, \ldots, s_N)$, $t = (t_1, \ldots, t_N) \in E$,*

$$c_{2.1}\sum_{\ell=1}^{N}\left|s_\ell - t_\ell\right|^{2H_\ell} \le \det\operatorname{Cov}\left(X_k(s), X_k(t)\right) \le c_{2.2}\sum_{\ell=1}^{N}\left|s_\ell - t_\ell\right|^{2H_\ell}, \quad 1 \le k \le d. \tag{7}$$

Proof. Since E is a compact set in $\mathbb{R}^N$, then there exists a positive constant c, such that $E \subset [-c, c]^N$. In order to prove (7), it suffices to show that (7) holds for all $s, t \in E$ with $s \ne t$. We claim that for all, $s, t \in E$ with $s \ne t$,

$$\det\operatorname{Cov}\left(X_k(s), X_k(t)\right) = \operatorname{Var}\left(X_k(t)\right)\operatorname{Var}\left(X_k(s) \mid X_k(t)\right). \tag{8}$$

If $\det\operatorname{Cov}(X_k(s), X_k(t)) > 0$, then by using the expression for the characteristic functions and the density functions of Gaussian random fields, it turns out that

$$\frac{2\pi}{\det\operatorname{Cov}\left(X_k(s), X_k(t)\right)} = \int_{\mathbb{R}^2}\mathbb{E}\exp\left(-i\left(uX_k(s) + vX_k(t)\right)\right)du\,dv. \tag{9}$$

By applying the fact that the conditional distribution of $X_k(s)$ given $X_k(t)$ is still Gaussian with mean $\mathbb{E}(X_k(s) \mid X_k(t))$ and variance $\operatorname{Var}(X_k(s) \mid X_k(t))$, one can evaluate the integral in the right-hand side of (9) and thus deduce that (8) holds.

If $\det\operatorname{Cov}(X_k(s), X_k(t)) = 0$, then we can deduce that the related coefficient of $X(s)$ and $X(t)$ is equal to 0, so there exists $\lambda \in \mathbb{R}$ such that $X(s) = \lambda X(t)$ a.s., and, in particular, a simply estimation implies that (8) still holds in this case.

We now prove the upper bound in (7). Note that $(X_k(s), X_k(t))$ is a mean zero Gaussian vector. Since $t \mapsto \operatorname{Var}(X_k(t))$ is a positive continuous function on E, then there exists a positive constant $c_{2.3}$ such that, for all $t \in E$,

$$c_{1.1} \le \operatorname{Var}\left(X_k(t)\right) \le c_{2.3}. \tag{10}$$

This, together with (1), (2), and (8), implies that the upper bound in (7) holds. The lower bound in (7) follows from (2), (8), and (10). This completes the proof of Lemma 1. □

Similar to the argument of Testard in [16], we will provide a proof of the following lemma.

Lemma 2. *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be an (N, d)-Gaussian random field satisfying conditions (C1) and (C2) and let E be a compact set on $\mathbb{R}^N$. Then there exist positive constants $c_{2.4}$ and $c_{2.5}$, such that, for all $s, t \in E$, we have*

$$c_{2.4}\left(f_k(s) - f_k(t)\right)^2 \le \mathbb{E}\left[f_k(t)X_k(s) - f_k(s)X_k(t)\right]^2 \le c_{2.5}\left(f_k(s) - f_k(t)\right)^2, \tag{11}$$

where $f \in \mathscr{C}$, $k = 1, \ldots, d$.

Proof. Since E is a compact set on $\mathbb{R}^N$, then there exists a positive constant c, such that $E \subset [-c, c]^N$. As usual, the proof is divided into proving the lower and upper bounds separately. We first prove the lower bound in (11). By (1) and (7), we have

$$\begin{aligned}\det\operatorname{Cov}\left(X_k(s), X_k(t)\right) &= \mathbb{E}\left[X_k^2(s)\right]\mathbb{E}\left[X_k^2(t)\right] - \left[\mathbb{E}\left(X_k(s)X_k(t)\right)\right]^2\\ &\ge c_{2.1}\sum_{\ell=1}^{N}\left|s_\ell - t_\ell\right|^{2H_\ell}\\ &\ge \frac{c_{2.1}}{c_{1.3}}\mathbb{E}\left[X_k(s) - X_k(t)\right]^2.\end{aligned} \tag{12}$$

By taking $c_{2.4} = \min\{c_{1.1}, c_{2.1}/c_{1.3}\}$, then for all $s, t \in E$,

$$\mathbb{E}\left[X_k^2(s)\right] - c_{2.4} \ge 0, \qquad \mathbb{E}\left[X_k^2(t)\right] - c_{2.4} \ge 0. \tag{13}$$

It follows from (12) and (13) that

$$\begin{aligned}&\left|\mathbb{E}\left[X_k(s)X_k(t)\right]-c_{2.4}\right|\\&\quad\le\left(\mathbb{E}\left[X_k^2(s)\right]-c_{2.4}\right)^{1/2}\left(\mathbb{E}\left[X_k^2(t)\right]-c_{2.4}\right)^{1/2}.\end{aligned}\tag{14}$$

Note that

$$\begin{aligned}&\mathbb{E}\left[f_k(t)X_k(s)-f_k(s)X_k(t)\right]^2-c_{2.4}\left(f_k(s)-f_k(t)\right)^2\\&\quad=f_k^2(t)\left(\mathbb{E}\left[X_k^2(s)\right]-c_{2.4}\right)+f_k^2(s)\left(\mathbb{E}\left[X_k^2(t)\right]-c_{2.4}\right)\\&\qquad-2f_k(s)f_k(t)\left(\mathbb{E}\left[X_k(s)X_k(t)\right]-c_{2.4}\right).\end{aligned}\tag{15}$$

Then inequalities (14) and (15) imply

$$\begin{aligned}&\mathbb{E}\left[f_k(t)X_k(s)-f_k(s)X_k(t)\right]^2-c_{2.4}\left(f_k(s)-f_k(t)\right)^2\\&\quad\ge f_k^2(t)\left(\mathbb{E}\left[X_k^2(s)\right]-c_{2.4}\right)+f_k^2(s)\left(\mathbb{E}\left[X_k^2(t)\right]-c_{2.4}\right)\\&\qquad-2\left|f_k(s)\right|\left|f_k(t)\right|\left(\mathbb{E}\left[X_k^2(s)\right]-c_{2.4}\right)^{1/2}\\&\qquad\times\left(\mathbb{E}\left[X_k^2(t)\right]-c_{2.4}\right)^{1/2}\\&\quad=\Big[\left|f_k(t)\right|\left(\mathbb{E}\left[X_k^2(s)\right]-c_{2.4}\right)^{1/2}\\&\qquad-\left|f_k(s)\right|\left(\mathbb{E}\left[X_k^2(t)\right]-c_{2.4}\right)^{1/2}\Big]^2\\&\quad\ge 0.\end{aligned}\tag{16}$$

Now we prove the upper bound in (11). By using (1) and (7) and repeating the procedure in (12), we can derive

$$\begin{aligned}&\mathbb{E}\left[X_k^2(s)\right]\mathbb{E}\left[X_k^2(t)\right]-\left[\mathbb{E}\left(X_k(s)X_k(t)\right)\right]^2\\&\quad\le\frac{c_{2.2}}{c_{1.2}}\mathbb{E}\left[X_k(s)-X_k(t)\right]^2.\end{aligned}\tag{17}$$

By taking $c_{2.5}=\max\{c_{2.3},c_{2.2}/c_{1.2}\}$, then for all $s,t\in E$,

$$c_{2.5}-\mathbb{E}\left[X_k^2(s)\right]\ge 0,\qquad c_{2.5}-\mathbb{E}\left[X_k^2(t)\right]\ge 0.\tag{18}$$

It follows from (17) and (18) that

$$\begin{aligned}&\left|c_{2.5}-\mathbb{E}\left[X_k(s)X_k(t)\right]\right|\\&\quad\ge\left(c_{2.5}-\mathbb{E}\left[X_k^2(s)\right]\right)^{1/2}\left(c_{2.5}-\mathbb{E}\left[X_k^2(t)\right]\right)^{1/2}.\end{aligned}\tag{19}$$

Combining (15), (18), and (19), we obtain

$$\begin{aligned}&\mathbb{E}\left[f_k(t)X_k(s)-f_k(s)X_k(t)\right]^2-c_{2.5}\left(f_k(s)-f_k(t)\right)^2\\&\quad\le-f_k^2(t)\left(c_{2.5}-\mathbb{E}\left[X_k^2(s)\right]\right)-f_k^2(s)\left(c_{2.5}-\mathbb{E}\left[X_k^2(t)\right]\right)\\&\qquad+2\left|f_k(s)\right|\left|f_k(t)\right|\left(c_{2.5}-\mathbb{E}\left[X_k^2(s)\right]\right)^{1/2}\\&\qquad\times\left(c_{2.5}-\mathbb{E}\left[X_k^2(t)\right]\right)^{1/2}\\&\quad=-\Big[\left|f_k(t)\right|\left(\mathbb{E}\left[X_k^2(s)\right]-c_{2.4}\right)^{1/2}\\&\qquad-\left|f_k(s)\right|\left(\mathbb{E}\left[X_k^2(t)\right]-c_{2.4}\right)^{1/2}\Big]^2\\&\quad\le 0.\end{aligned}\tag{20}$$

By inequalities (16) and (20), we finish the proof of Lemma 2. □

Let ρ be a metric on $\mathbb{R}^N$ defined by

$$\rho(s,t)=\sum_{\ell=1}^{N}\left|s_\ell-t_\ell\right|^{H_\ell\wedge K_\ell},\quad\forall s,t\in\mathbb{R}^N.\tag{21}$$

In the following, we will provide a slightly more general result in the proof of Proposition 4.4 by modifying the argument [8].

Lemma 3. *Let $X=\{X(t),t\in\mathbb{R}^N\}$ be an (N,d)-Gaussian random field satisfying conditions (C1) and (C2). Then there exist positive constants $c_{2.6}$ and δ, such that, for all $r\in(0,\delta)$, $s\in E$, and all $f\in\mathscr{L}(K)$,*

$$\mathbb{P}\left\{\inf_{t\in B_\rho(s,r)}\left|X(t)-f(t)\right|<r\right\}\le c_{2.6}r^d,\tag{22}$$

where $B_\rho(s,r)$ denotes the ball of radius r centered at s in the metric ρ defined by (21).

Proof. Using the Gaussian regressions, we have

$$\begin{aligned}\mathbb{E}\left(X_k(t)\mid X_k(s)\right)&=\frac{\mathbb{E}\left[X_k(s)X_k(t)\right]}{\mathbb{E}\left[X_k(s)\right]^2}X_k(s)\\&\widehat{=}\,k(s,t)X_k(s).\end{aligned}\tag{23}$$

Note that, for all $t\in E$, the Gaussian random variables $X_k(t)-k(s,t)X_k(s)$ $(s\in E)$ and $X_k(s)$ are independent. By using the triangle inequality, we can deduce that, for all $1\le k\le d$,

$$\begin{aligned}&\left\{\inf_{t\in B_\rho(s,r)}\left|X_k(t)-f_k(t)\right|\ge r\right\}\\&\quad\supseteq\left\{\inf_{t\in B_\rho(s,r)}\left|k(s,t)\left(X_k(s)-f_k(s)\right)\right|\ge 2r\right\}\end{aligned}$$

$$\cap \left\{ 2Z_k(s,x,r) \le \inf_{t\in B_\rho(s,r)} \left| k(s,t)\left(X_k(s)-f_k(s)\right) \right| \right\}, \tag{24}$$

where $Z_k(s,r) = \sup_{t\in B_\rho(s,r)} |(X_k(t)-f_k(t)) - k(s,t)(X_k(s)-f_k(s))|$. Then

$$\begin{aligned} &\mathbb{P}\left\{ \inf_{y\in B_\rho(s,r)} \left|X_k(t)-f_k(t)\right| < r \right\} \\ &\le \mathbb{P}\left\{ \inf_{t\in B_\rho(s,r)} \left| k(s,t)\left(X_k(s)-f_k(s)\right)\right| < 2r \right\} \\ &\quad + \mathbb{P}\left\{ 2Z_k(s,r) > \inf_{t\in B_\rho(s,r)} \left| k(s,t)\left(X_k(s)-f_k(s)\right)\right| \right\}. \end{aligned} \tag{25}$$

By the Cauchy-Schwarz inequality, (1), and (23), we have

$$\begin{aligned} |1-k(s,t)| &= \frac{\left|\mathbb{E}\left[X_k(s)\left(X_k(s)-X_k(t)\right)\right]\right|}{\mathbb{E}\left[X_k(s)\right]^2} \\ &\le \left(\frac{c_{1.3}}{c_{1.1}} \sum_{\ell=1}^{N} \left|s_\ell - t_\ell\right|^{2H_\ell} \right)^{1/2}. \end{aligned} \tag{26}$$

Therefore, there exists a positive constant δ such that, for all $r \in (0,\delta)$ and $t \in B_\rho(s,r)$, we can deduce that $1/2 \le k(s,t) \le 3/2$. Recall that, for the unimodality of the centered Gaussian process $k(s,t)X_k(s)$, we have

$$\begin{aligned} &\mathbb{P}\left\{ \inf_{t\in B_\rho(s,r)} \left| k(s,t)\left(X_k(s)-f_k(s)\right)\right| < 2r \right\} \\ &\le \mathbb{P}\left\{ \left|X_k(s)-f_k(s)\right| < 4r \right\} \\ &\le \mathbb{P}\left\{ \left|X_k(s)\right| < 4r \right\} \le c_{2.7} r. \end{aligned} \tag{27}$$

Note that $Z_k(s,r)$ and $k(s,t)X_k(s)$ are independent. It follows from (27) that

$$\begin{aligned} &\mathbb{P}\left\{ 2Z_k(s,r) > \inf_{t\in B_\rho(s,r)} \left| k(s,t)\left(X_k(s)-f_k(s)\right)\right| \right\} \\ &= \int_0^\infty \mathbb{P}\left\{ \left| k(s,t)\left(X_k(s)-f_k(s)\right)\right| < 4y \mid Z_k(s,r) = y \right\} \\ &\quad \times \mathbb{P}\left\{ Z_k(s,r) \in dy \right\} \\ &= c_{2.7} \int_0^\infty y \mathbb{P}\left\{ Z_k(s,r) \in dy \right\} \\ &\le c_{2.7} \mathbb{E}\left[Z_k(s,r) \right]. \end{aligned} \tag{28}$$

In order to estimate $\mathbb{E}[Z_k(s,r)]$, we denote that the Gaussian process $Y_k(t) = X_k(t) - f_k(t) - k(s,t)(X_k(s) - f_k(s))$ ($t \in B_\rho(s,r)$) and note that $Y_k(s) = 0$ and the canonical metric

$$d\left(t,t'\right) \mathrel{\hat{=}} \left[\mathbb{E}\left(Y_k(t) - Y_k\left(t'\right)\right)^2 \right]^{1/2}, \quad \forall t,t' \in B_\rho(s,r). \tag{29}$$

Therefore, by the Hölder inequality and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} d^2\left(t,t'\right) &\le c_{2.8} \Big(\mathbb{E}\left(X_k(t) - X_k\left(t'\right)\right)^2 \\ &\quad + \left(f_k(t) - f_k\left(t'\right)\right)^2 + \left(k(s,t) - k\left(s,t'\right)\right)^2 \\ &\quad \times \left(\mathbb{E}\left[X_k^2(s)\right] + f_k^2(s) \right) \Big), \\ \left(k(s,t) - k\left(s,t'\right)\right)^2 &= \frac{\left[\mathbb{E}\left[X_k(s)\left(X_k(t) - X_k\left(t'\right)\right)\right]\right]^2}{\left[\mathbb{E}\left[X_k(s)\right]^2 \right]^2} \\ &\le \frac{c_{1.3}}{c_{1.1}} \sum_{\ell=1}^{N} \left| t_\ell - t'_\ell \right|^{2H_\ell}. \end{aligned} \tag{30}$$

By using (1), (10), (30), and the fact that $f \in \mathcal{Q}(K)$, we have

$$\begin{aligned} d\left(t,t'\right) &\le c_{2.9} \sum_{\ell=1}^{N} \left| t_\ell - t'_\ell \right|^{H_\ell \wedge K_\ell} \\ &= c_{2.9} \rho\left(t,t'\right) \le c_{2.9} r, \\ &\quad \forall t,t' \in B_\rho(s,r). \end{aligned} \tag{31}$$

Then

$$\begin{aligned} D &\mathrel{\hat{=}} \sup_{t,t'\in B_\rho(s,r)} d\left(t,t'\right) \le c_{2.9} r, \\ N_d\left(B_\rho(s,r),\varepsilon\right) &\le c_{2.10} \left(\frac{r}{\varepsilon}\right)^Q, \end{aligned} \tag{32}$$

where $N_d(B_\rho(s,r),\varepsilon)$ is the metric entropy number of $B_\rho(s,r)$ and $Q = \sum_{\ell=1}^{N}(1/(H_\ell \wedge K_\ell))$. It follows from Dudley's theorem of Kahane [11] that

$$\mathbb{E}\left[Z_k(s,r)\right] \le c_{2.11} \int_0^D \sqrt{\log N_d\left(B_\rho(s,r),\varepsilon\right)} d\varepsilon \le c_{2.12} r. \tag{33}$$

Combining (25), (27), (28), and (33) and using the coordinate processes independence of X, we have

$$\begin{aligned} &\mathbb{P}\left\{ \inf_{t\in B_\rho(s,r)} \left|X(t)-f(t)\right| < r \right\} \\ &\le \prod_{k=1}^{d} \mathbb{P}\left\{ \inf_{t\in B_\rho(s,r)} \left|X_k(t)-f_k(t)\right| \le r \right\} \\ &\le c_{2.6} r^d. \end{aligned} \tag{34}$$

This finishes the proof of Lemma 3. □

Lemma 4. *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be an (N,d)-Gaussian random field satisfying conditions (C1) and (C2) and let E be a compact set on $\mathbb{R}^N$. Then there exists a positive constant $c_{2.13}$ such that, for all $f \in \mathcal{Q}(K)$, $\varepsilon > 0$ and $s, t \in E$,*

$$\mathbb{E}\left(\frac{2\pi}{\varepsilon}\right)^d \exp\left(-\frac{|X(s)-f(s)|^2+|X(t)-f(t)|^2}{2\varepsilon}\right) \leq c_{2.13}\left(\sum_{\ell=1}^N |s_\ell - t_\ell|^{H_\ell \wedge K_\ell}\right)^{-d}. \tag{35}$$

Proof. Note that

$$\begin{aligned}\left(\frac{2\pi}{\varepsilon}\right)^{d/2} \exp\left(-\frac{|X(t)-f(t)|^2}{2\varepsilon}\right) &= \int_{\mathbb{R}^d} \exp\left(-\frac{\varepsilon}{2}|u|^2 + i\langle u, X(t)-f(t)\rangle\right) du \\ &= \prod_{k=1}^d \int_{\mathbb{R}} \exp\left(-\frac{\varepsilon}{2}|u_k|^2 + i\langle u_k, X_k(t)-f_k(t)\rangle\right) du_k.\end{aligned} \tag{36}$$

Denote by I_2 the identity matrix of order 2 and let $\Gamma_k(s,t) = \varepsilon I_2 + \mathrm{Cov}(X_k(s), X_k(t))$. Then the inverse of $\Gamma_k(s,t)$ is given by

$$\Gamma_k^{-1}(s,t) = \frac{1}{\det \Gamma_k(s,t)} \times \begin{pmatrix} \varepsilon + \mathbb{E}\left[X_k^2(t)\right] & -\mathbb{E}\left(X_k(s)X_k(t)\right) \\ -\mathbb{E}\left(X_k(s)X_k(t)\right) & \varepsilon + \mathbb{E}\left[X_k^2(s)\right] \end{pmatrix}, \tag{37}$$

where $\det \Gamma_k(s,t)$ denotes the determinant of $\Gamma_k(s,t)$.

By (36), Lemma 2, Fubini's theorem, and some elementary calculations, we derive

$$\begin{aligned}&\mathbb{E}\left(\frac{2\pi}{\varepsilon}\right)^d \exp\left(-\frac{|X(s)-f(s)|^2+|X(t)-f(t)|^2}{2\varepsilon}\right) \\ &= \prod_{k=1}^d \int_{\mathbb{R}}\int_{\mathbb{R}} \exp\left(-i\left(\langle u_k, f_k(s)\rangle + \langle v_k, f_k(t)\rangle\right)\right) \\ &\quad \cdot \exp\left(-\frac{\varepsilon}{2}\left(|u_k|^2+|v_k|^2\right)\right) \\ &\quad \times \mathbb{E}\exp\left(i\langle u_k, X_k(s)\rangle + i\langle v_k, X_k(t)\rangle\right) du_k dv_k \\ &= \prod_{k=1}^d \int_{\mathbb{R}}\int_{\mathbb{R}} \exp\left(-i\left(\langle u_k, f_k(s)\rangle + \langle v_k, f_k(t)\rangle\right)\right) \\ &\quad \times \exp\left(-\frac{1}{2}(u_k, v_k)\Gamma_k(s,t)(u_k,v_k)'\right) du_k dv_k \\ &= \prod_{k=1}^d \frac{2\pi}{\sqrt{\det\Gamma_k(s,t)}} \\ &\quad \times \exp\left(-\frac{1}{2}(f_k(s), f_k(t))\Gamma_k^{-1}(s,t)(f_k(s), f_k(t))'\right) \\ &= \prod_{k=1}^d \frac{2\pi}{\sqrt{\det\Gamma_k(s,t)}} \exp\left(-\frac{f_k^2(s)+f_k^2(t)}{2\varepsilon\det\Gamma_k(s,t)}\right) \\ &\quad \cdot \exp\left(-\frac{\mathbb{E}\left(f_k(t)X_k(s)-f_k(s)X_k(t)\right)^2}{2\det\Gamma_k(s,t)}\right) \\ &\leq \prod_{k=1}^d \frac{2\pi}{\sqrt{\det\Gamma_k(s,t)}} \\ &\quad \times \exp\left(-\frac{c_{2.4}|f_k(s)-f_k(t)|^2}{2\det\Gamma_k(s,t)}\right).\end{aligned} \tag{38}$$

If $\det\Gamma_k(s,t) \geq |f_k(s)-f_k(t)|^2$, then

$$\frac{1}{\sqrt{\det\Gamma_k(s,t)}} \exp\left(-\frac{c_{2.4}|f_k(s)-f_k(t)|^2}{2\det\Gamma_k(s,t)}\right) \leq \frac{1}{\max\left\{\sqrt{\det\Gamma_k(s,t)}, |f_k(s)-f_k(t)|\right\}}. \tag{39}$$

For all $x \geq 0$, we can deduce

$$\exp\left(-\frac{c_{2.4}}{2}x\right) \leq (ec_{2.4})^{-1/2}x^{-1/2}. \tag{40}$$

If $\det\Gamma_k(s,t) < |f_k(s)-f_k(t)|^2$, by the inequality above and taking $c_{2.14} = (ec_{2.4})^{-1/2}$ and $x = |f_k(s)-f_k(t)|^2/\det\Gamma_k(s,t)$, we have

$$\frac{1}{\sqrt{\det\Gamma_k(s,t)}} \exp\left(-\frac{c_{2.4}|f_k(s)-f_k(t)|^2}{2\det\Gamma_k(s,t)}\right) \leq \frac{c_{2.14}}{\max\left\{\sqrt{\det\Gamma_k(s,t)}, |f_k(s)-f_k(t)|\right\}}. \tag{41}$$

It follows from Lemma 1 that, for all $s, t \in E$,

$$\det\Gamma_k(s,t) \geq \det\mathrm{Cov}\left(X_k(s), X_k(t)\right) \geq c_{2.15}\left(\sum_{\ell=1}^N |s_\ell - t_\ell|^{H_\ell}\right)^2. \tag{42}$$

By using (42) and the fact that $f \in \mathcal{Q}(K)$, we have

$$\max\left\{\sqrt{\det\Gamma_k(s,t)}, |f_k(s)-f_k(t)|\right\} \geq c_{2.16}\sum_{\ell=1}^N |s_\ell - t_\ell|^{H_\ell\wedge K_\ell}. \tag{43}$$

Combining (36) through (43), we prove that Lemma 4 holds. □

For proving the lower bound in Theorem 11, we will use two lemmas below, which are slightly more general results, by modifying the argument in [3, 17].

Lemma 5. *Let $0 < \alpha < 1$, $b_1 > a_1$ and $b_2 > a_2$ be given constants. Then for all constants p, $M > 0$ and $\delta > 2\alpha$, there exists a positive and finite constant $c_{2.17}$, depending on a_1, b_1, a_2, b_2, δ, p, α, and M only, such that, for all $0 < A \leq M$,*

$$\begin{aligned} \mathcal{I}(A) &\mathrel{\widehat=} \int_{a_1}^{b_1} ds \int_{a_2}^{b_2} \left(A + |s-t|^{2\alpha}\right)^{-p} dt \\ &\leq c_{2.17} \left(A^{-(p-1/\delta)} + 1\right). \end{aligned} \tag{44}$$

Proof. Let $a = \min\{a_1, a_2\}$ and $b = \max\{b_1, b_2\}$. By the symmetry of the integrand, we get

$$\mathcal{I}(A) \leq 2 \int_a^b ds \int_s^b \left(A + (t-s)^{2\alpha}\right)^{-p} dt. \tag{45}$$

Putting $u = t - s$ and using the fact that $a \leq s \leq t \leq b$, we see that the above integral is bound by

$$\begin{aligned} &2 \int_a^b ds \int_0^{b-s} \left(A + u^{2\alpha}\right)^{-p} du \\ &\leq 2(b-a) \int_0^{b-a} \left(A + u^{2\alpha}\right)^{-p} du \\ &= \frac{b-a}{\alpha} \int_A^{A+(b-a)^{2\alpha}} v^{-p} (v-A)^{(1-2\alpha)/2\alpha} dv, \end{aligned} \tag{46}$$

where we have used the substitution $v = A + u^{2\alpha}$.

Let $B \mathrel{\widehat=} A + (b-a)^{2\alpha}$. If $\alpha > 1/2$, then for $\delta > 2\alpha$, it follows from (46) and Hölder's inequality that there exists a positive and finite constant $c_{2.18}$, which depends on a_1, b_1, a_2, b_2, δ, p, α, and M only, such that

$$\begin{aligned} \mathcal{I}(A) &\leq \frac{b-a}{\alpha} \left(\int_A^B v^{-p\delta} dv \right)^{1/\delta} \\ &\quad \times \left(\int_A^B (v-A)^{-((2\alpha-1)/2\alpha)(\delta/(\delta-1))} dv \right)^{(\delta-1)/\delta} \\ &\leq c_{2.18} \left(A^{-(p-1/\delta)} + 1\right), \end{aligned} \tag{47}$$

where we have used the fact that $((2\alpha-1)/2\alpha)(\delta/(\delta-1)) < 1$.

If $0 < \alpha \leq 1/2$, then some elementary calculations imply that, for all $0 < A \leq M$,

$$\begin{aligned} \mathcal{I}(A) &\leq \frac{b-a}{\alpha} \int_A^B v^{-p} (v-A)^{(1-2\alpha)/2\alpha} dv \\ &\leq \frac{b-a}{\alpha} \int_A^B v^{-p-1+1/2\alpha} dv \\ &\leq c_{2.19} \left(A^{-(p-1/\delta)} + 1\right), \end{aligned} \tag{48}$$

where $c_{2.19}$ depends on a_1, b_1, a_2, b_2, δ, p, α, and M only. By (47) and (48), the proof of Lemma 5 is finished. □

Lemma 6. *Let α, β, η, and b be positive constants. For $A > 0$ and $B > 0$, let*

$$\mathcal{J}(A, B) \mathrel{\widehat=} \int_0^b \frac{dt}{(A + t^\alpha)^\beta (B + t)^\eta}. \tag{49}$$

Then there exist positive and finite constants $c_{2.20}$ and $c_{2.21}$, depending on α, β, η, and b only, such that the following hold for all reals $A, B > 0$ satisfying $A^{1/\alpha} \leq c_{2.20} B$:

(i) *if $\alpha\beta > 1$, then*

$$\mathcal{J}(A, B) \leq c_{2.21} \frac{1}{A^{\beta-\alpha^{-1}} B^\eta}; \tag{50}$$

(ii) *if $\alpha\beta = 1$, then*

$$\mathcal{J}(A, B) \leq c_{2.21} \frac{1}{B^\eta} \log\left(1 + BA^{-1/\alpha}\right); \tag{51}$$

(iii) *if $0 < \alpha\beta < 1$ and $\alpha\beta + \eta \neq 1$, then*

$$\mathcal{J}(A, B) \leq c_{2.21} \left(\frac{1}{B^{\alpha\beta+\eta-1}} + 1 \right). \tag{52}$$

Proof. If $b \leq 1$, by using Lemma 10 in [3], we can prove that inequalities (50), (51), and (52) hold. If $b > 1$, then we can split the integral in (49) so that

$$\begin{aligned} \mathcal{J}(A, B) &= \int_0^1 \frac{dt}{(A + t^\alpha)^\beta (B + t)^\eta} \\ &\quad + \int_1^b \frac{dt}{(A + t^\alpha)^\beta (B + t)^\eta} \\ &\mathrel{\widehat=} \mathcal{I}_1 + \mathcal{I}_2. \end{aligned} \tag{53}$$

Let $t = bs$. Since $b > 1$ and α, β, and η are positive constants, we get

$$\begin{aligned} \mathcal{I}_2 &= b \int_{1/b}^1 \frac{ds}{(A + (bs)^\alpha)^\beta (B + bs)^\eta} \\ &\leq b \int_0^1 \frac{ds}{(A + s^\alpha)^\beta (B + s)^\eta}. \end{aligned} \tag{54}$$

By using (53), (54), and Lemma 10 in [3] again, we can also prove (50), (51), and (52); in this case $b > 1$. Thus, the proof of Lemma 6 is finished. □

Let $H = (H_1, \ldots, H_N) \in (0,1)^N$ and $K = (K_1, \ldots, K_N) \in (0,1)^N$ be given vectors. For convenience, we may further assume

$$0 < H_1 K_1 \leq H_2 K_2 \leq \cdots \leq H_N K_N < 1. \tag{55}$$

Lemma 7. *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be an (N, d)-Gaussian random field satisfying conditions (C1) and (C2). If $\sum_{\ell=1}^N (1/(H_\ell \wedge K_\ell)) > d$, then there exists a positive and finite* *constant $c_{2.22}$, depending on a, b, H, K, N, and d only, such that, for all $f \in \mathcal{Q}(K)$,*

$$\int_I \int_I \prod_{k=1}^{d} \left[\max\left\{ \det \operatorname{Cov}\left(X_k(s), X_k(t)\right), \left|f_k(s) - f_k(t)\right|^2 \right\}\right]^{-1/2} dt\, ds \le c_{2.22}. \tag{56}$$

Proof. Note that $f \in \mathcal{Q}(K)$. Then, by using Lemma 1, we have

$$\begin{aligned} \mathcal{J} &\mathrel{\hat=} \int_I \int_I \prod_{k=1}^{d} \left[\max\left\{ \det \operatorname{Cov}\left(X_k(s), X_k(t)\right), \left|f_k(s) - f_k(t)\right|^2 \right\}\right]^{-1/2} dt\, ds \\ &\le c_{2.23} \int_I \int_I \left[\sum_{\ell=1}^{N} \left|s_\ell - t_\ell\right|^{H_\ell \wedge K_\ell}\right]^{-d} dt\, ds. \end{aligned} \tag{57}$$

Let $k \in \{1, \ldots, N\}$ be the unique positive integer such that

$$\sum_{\ell=1}^{k-1} \frac{1}{H_\ell \wedge K_\ell} \le d < \sum_{\ell=1}^{k} \frac{1}{H_\ell \wedge K_\ell}, \qquad \sum_{\ell=1}^{0} \frac{1}{H_\ell \wedge K_\ell} \mathrel{\hat=} 0. \tag{58}$$

Then, we choose positive constants $\delta_1, \ldots, \delta_{N-1}$ such that $\delta_\ell > H_\ell \wedge K_\ell$ for each $1 \le \ell \le N-1$ and

$$\frac{1}{\delta_1} + \frac{1}{\delta_2} + \cdots + \frac{1}{\delta_{N-1}} < d < \frac{1}{\delta_1} + \frac{1}{\delta_2} + \cdots + \frac{1}{\delta_{N-1}} + \frac{1}{H_N \wedge K_N}. \tag{59}$$

Applying Lemma 5 to (57) with

$$A = \sum_{\ell=2}^{N} \left|s_\ell - t_\ell\right|^{H_\ell \wedge K_\ell}, \qquad p = d, \tag{60}$$

we obtain that

$$\mathcal{J} \le c_{2.24} + c_{2.24} \int_{a_1}^{b_1} ds_2 \int_{a_2}^{b_2} dt_2 \cdots \int_{a_N}^{b_N} ds_N \times \int_{a_N}^{b_N} \left[\sum_{\ell=2}^{N} \left|s_\ell - t_\ell\right|^{H_\ell \wedge K_\ell}\right]^{-(d - 1/\delta_1)} dt_N. \tag{61}$$

By repeatedly using Lemma 5 to the integral in above inequality for $N-2$ steps, we have

$$\mathcal{J} \le c_{2.25} + c_{2.25} \int_{a_N}^{b_N} ds_N \times \int_{a_N}^{b_N} \left[\left|s_N - t_N\right|^{(H_N \wedge K_N)}\right]^{-(d - (1/\delta_1 + \cdots + 1/\delta_{N-1}))} dt_N. \tag{62}$$

Since the $\delta' s$ satisfy (59), we have

$$\left(\left(H_N \wedge K_N\right)\right)\left(d - \left(\frac{1}{\delta_1} + \cdots + \frac{1}{\delta_{N-1}}\right)\right) < 1. \tag{63}$$

Thus, the integral in the right-hand side of (62) is finite. This completes the proof of Lemma 7. □

Lemma 8. *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be an (N, d)-Gaussian random field satisfying conditions (C1) and (C2). If $\sum_{\ell=1}^{N}(1/(H_\ell \wedge K_\ell)) > d$, then there exist positive and finite constants δ_0 and $c_{2.26}$, such that for all $f \in \mathcal{Q}(K)$, $\delta \in (0, \delta_0)$,*

$$\int_I \int_I |t - s|^{-\gamma} \prod_{k=1}^{d} \left[\max\left\{ \det \operatorname{Cov}\left(X_k(s), X_k(t)\right), \left|f_k(s) - f_k(t)\right|^2 \right\}\right]^{-1/2} dt\, ds \le c_{2.26}, \tag{64}$$

where $\gamma = \min_{1 \le k \le N}\{\sum_{\ell=1}^{k}((H_k \wedge K_k)/(H_\ell \wedge K_\ell)) + N - k - (H_k \wedge K_k)(1 + \delta)d\}$ and $c_{2.26}$ depends on a, b, H, K, δ_0, N, and d only.

Proof. For our purpose, let us note that (55) implies

$$\begin{aligned} &\sum_{\ell=1}^{k} \frac{H_k \wedge K_k}{H_\ell \wedge K_\ell} + N - k - \left(H_k \wedge K_k\right) d \\ &\quad = \left(H_k \wedge K_k\right)\left(\left(\sum_{\ell=1}^{N} \frac{1}{H_\ell \wedge K_\ell} - d\right) + \sum_{\ell=k+1}^{N} \left(\frac{1}{H_k \wedge K_k} - \frac{1}{H_\ell \wedge K_\ell}\right)\right) > 0. \end{aligned} \tag{65}$$

Then, there exist $\delta_0 > 0$ such that for all $k \in \{1, \ldots, N\}$ and $\delta \in (0, \delta_0)$, we have $\gamma > 0$. By using Lemma 1 and $f \in \mathcal{Q}(K)$, we have

$$\begin{aligned} \mathcal{J} &\mathrel{\hat=} \int_I \int_I |t - s|^{-\gamma} \times \prod_{k=1}^{d} \left[\max\left\{ \det\operatorname{Cov}\left(X_k(s), X_k(t)\right), \left|f_k(s) - f_k(t)\right|^2 \right\}\right]^{-1/2} dt\, ds \\ &\le c_{2.27} \int_I \int_I \frac{1}{\left(\sum_{\ell=1}^{N} \left|s_\ell - t_\ell\right|\right)^{\gamma} \left(\sum_{\ell=1}^{N} \left|s_\ell - t_\ell\right|^{H_\ell \wedge K_\ell}\right)^{d}} ds\, dt. \end{aligned} \tag{66}$$

By a change of variable, we have

$$\mathcal{J} \le c_{2.28} \int_0^{b_N - a_N} dt_N \cdots \int_0^{b_1 - a_1} \frac{1}{\left(\sum_{\ell=1}^{N} t_\ell\right)^{\gamma} \left(\sum_{\ell=1}^{N} t_\ell^{H_\ell \wedge K_\ell}\right)^{d}} dt_1. \tag{67}$$

In order to show the integral in (67) is finite, we will integrate $[dt_1], [dt_2], \ldots, [dt_k]$ iteratively. We only need to consider the case when

$$\sum_{\ell=1}^{k-1} \frac{1}{H_\ell \wedge K_\ell} \le d < \sum_{\ell=1}^{k} \frac{1}{H_\ell \wedge K_\ell}, \quad \text{for some } 1 \le k \le N. \tag{68}$$

Here and in the sequel $\sum_{\ell=1}^{0}(1/(H_\ell \wedge \beta)) \mathrel{\hat{=}} 0$. Then, by using (55), we can deduce that

$$\gamma = \sum_{\ell=1}^{k} \frac{H_k \wedge K_k}{H_\ell \wedge K_\ell} + N - k - (H_k \wedge K_k)(1+\delta) d, \tag{69}$$

where k is the unique integer satisfying (68).

If $k = 1$ in (68), we integrate $[dt_1]$. Note that $0 < (H_1 \wedge K_1)d < 1$ and $(H_1 \wedge K_1)d + \gamma = N - (H_1 \wedge K_1)\delta d \neq 1$. Then we can use (52) of Lemma 6 with $A = \sum_{\ell=2}^{N} t_\ell^{H_\ell \wedge K_\ell}$ and $B = \sum_{\ell=2}^{N} t_\ell$ to get

$$\mathcal{J} \le c_{2.29} \left[\int_0^{b_N - a_N} dt_N \cdots \int_0^{b_2 - a_2} \frac{1}{\left(\sum_{\ell=2}^{N} t_\ell\right)^{(H_1 \wedge K_1)d + \gamma - 1}} dt_2 + 1 \right] < \infty, \tag{70}$$

since $(H_1 \wedge K_1)d + \gamma - 1 < N - 1$.

If $k > 1$ in (68), we integrate $[dt_1]$ first. Since $(H_1 \wedge K_1)d > 1$, we can use (50) of Lemma 6 with $A = \sum_{\ell=2}^{N} t_\ell^{H_\ell \wedge K_\ell}$ and $B = \sum_{\ell=2}^{N} t_\ell$ to get

$$\mathcal{J} \le c_{2.30} \int_0^{b_N - a_N} dt_N \cdots \int_0^{b_2 - a_2} \frac{1}{\left(\sum_{\ell=2}^{N} t_\ell^{H_\ell \wedge K_\ell}\right)^{d - 1/(H_1 \wedge K_1)} \left(\sum_{\ell=2}^{N} t_\ell\right)^{\gamma}} dt_2. \tag{71}$$

We can repeat this procedure for integrating $dt_2, \ldots, dt_{k-1}$.

Note that if $d = \sum_{\ell=1}^{k-1}(1/(H_\ell \wedge K_\ell))$, then $(H_{k-1} \wedge K_{k-1})(d - \sum_{\ell=1}^{k-2}(1/(H_\ell \wedge K_\ell))) = 1$. We need to use (51) of Lemma 6 with $A = \sum_{\ell=k}^{N} t_\ell^{H_\ell \wedge K_\ell}$ and $B = \sum_{\ell=k}^{N} t_\ell$ to integrate dt_{k-1} and obtain

$$\mathcal{J} \le c_{2.31} \int_0^{b_N - a_N} dt_N \cdots \int_0^{b_k - a_k} \frac{1}{\left(\sum_{\ell=k}^{N} t_\ell\right)^{\gamma}} \times \log\left(1 + \frac{1}{\sum_{\ell=k}^{N} t_\ell}\right) dt_k < \infty, \tag{72}$$

since $\gamma < N - k + 1$.

On the other hand, if $d > \sum_{\ell=1}^{k-1}(1/(H_\ell \wedge K_\ell))$, then $(H_{k-1} \wedge K_{k-1})(d - \sum_{\ell=1}^{k-2}(1/(H_\ell \wedge K_\ell))) > 1$. By using (50) of Lemma 6 with $A = \sum_{\ell=k}^{N} t_\ell^{H_\ell \wedge K_\ell}$ and $B = \sum_{\ell=k}^{N} t_\ell$ to integrate dt_{k-1}, we can deduce

$$\mathcal{J} \le c_{2.32} \int_0^{b_N - a_N} dt_N \cdots \int_0^{b_k - a_k} \frac{1}{\left(\sum_{\ell=k}^{N} t_\ell^{H_\ell \wedge K_\ell}\right)^{d - \sum_{\ell=1}^{k-1}(1/(H_\ell \wedge K_\ell))} \left(\sum_{\ell=k}^{N} t_\ell\right)^{\gamma}} dt_k. \tag{73}$$

Note that $0 < (H_k \wedge K_k)(d - \sum_{\ell=1}^{k-1}(1/(H_\ell \wedge K_\ell))) < 1$ and $(H_k \wedge K_k)(d - \sum_{\ell=1}^{k-1}(1/(H_\ell \wedge K_\ell))) + \gamma = N - k + 1 - (H_k \wedge K_k)\delta d \neq 1$ for a small enough δ_0. Applying (52) to integrate dt_k in (73), we see that

$$\mathcal{J} \le c_{2.33} \times \left[\int_0^{b_N - a_N} dt_N \cdots \int_0^{b_{k+1} - a_{k+1}} \frac{1}{\left(\sum_{\ell=k+1}^{N} t_\ell\right)^{\gamma + (H_k \wedge K_k)(d - \sum_{\ell=1}^{k-1}(1/(H_\ell \wedge K_\ell))) - 1}} dt_{k+1} + 1 \right] < \infty, \tag{74}$$

since $\gamma + (H_k \wedge K_k)(d - \sum_{\ell=1}^{k-1}(1/(H_\ell \wedge K_\ell))) - 1 < N - k$. Combining (70) through (74) yields (64). This completes the proof of Lemma 8. □

3. Characteristics of Polar Functions

In this section, we provide some necessary conditions and sufficient conditions for a function $f \in \mathscr{C}$ to be polar for X. We also give the intersection probabilities for a nonpolar function and X and determine the Hausdorff and packing dimensions of the set $\{t \in I, X(t) = f(t)\}$.

Let us note that

$$\mathbb{P}\left\{\exists t \in \mathbb{R}^N, X(t) = f(t)\right\} > 0. \tag{75}$$

If and only if there exists a rectangle $I \subset \mathbb{R}^N$, such that

$$\mathbb{P}\{\exists t \in I, X(t) = f(t)\} > 0. \tag{76}$$

For our purpose, it suffices to consider the polar functions of X in a rectangle $I = \Pi_{\ell=1}^{N}[a_\ell, b_\ell] \subseteq \mathbb{R}^N$ with $a_\ell < b_\ell$ ($\ell = 1, \ldots, N$).

Theorem 9. *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be an (N, d)-Gaussian random field satisfying conditions (C1) and (C2) on I. If $\sum_{\ell=1}^{N}(1/(H_\ell \wedge K_\ell)) < d$, then $\mathscr{L}(K) \subset \mathscr{P}$.*

Proof. For any constants $0 < \delta < H_\ell$ $(1 \le \ell \le N)$ and any rectangle $I = \Pi_{\ell=1}^{N}[a_\ell, b_\ell] \subseteq \mathbb{R}^N$ with $a_\ell < b_\ell$ ($\ell = 1, \ldots, N$), it follows from a similar argument as in the proof of Theorem 4.2 in [2] that there is a random variable A_1 of finite moments of all orders and an event Ω_1^* of probability 1 such that, for all $\omega \in \Omega_1^*$,

$$\sup_{s,t \in I} \frac{|X(s,\omega) - X(t,\omega)|}{\sum_{\ell=1}^{N} |s - t|^{H_\ell - \delta}} \le A_1(\omega). \tag{77}$$

For any $f \in \mathscr{L}(K)$, in order to prove $f \in \mathscr{P}$, it suffices to prove that, for any $\varepsilon > 0$ and any rectangle $I \subset \mathbb{R}^N$,

$$\mathbb{P}\{\omega : \exists t \in I, X(t,\omega) = f(t)\} \le \varepsilon. \tag{78}$$

Fix ε, δ and choose η such that $\sum_{\ell=1}^{N}(1/(H_\ell \wedge K_\ell)) < (1-\eta)d$. By $A_1 < \infty$, a.s., then there exist $\Omega_\varepsilon \subset \Omega_1^*$ and $c_{3.1} > 0$ such that $\mathbb{P}(\Omega_\varepsilon) > 1 - \varepsilon/2$ and for any $\omega \in \Omega_\varepsilon$, $A_1(\omega) \le c_{3.1}$. For any integer $n \ge 1$, divide the rectangle I into $m_n = (\prod_{\ell=1}^{N}(b_\ell - a_\ell))n^{\sum_{\ell=1}^{N}(1/(H_\ell \wedge K_\ell))}$ subrectangles $I_{n,\flat}$ with sides parallel to the axes and side lengths $n^{-1/(H_\ell \wedge K_\ell)}$ ($\ell = 1, \dots, N$). Let $\tau_{n,\flat}$ be the lower-left vertex of $I_{n,\flat}$.

Let $\omega \in \Omega_\varepsilon$, $\flat$ be fixed. If there exists $t \in I_{n,\flat}$ such that $X(t,\omega) = f(t)$, then by (77) and $f \in \mathscr{L}(K)$,

$$\begin{aligned} \left|X(\tau_{n,\flat},\omega) - f(\tau_{n,\flat})\right| &\le \left|X(\tau_{n,\flat},\omega) - X(t,\omega)\right| \\ &\quad + \left|f(\tau_{n,\flat},\omega) - f(t)\right| \\ &\le c_{3.1}\sum_{\ell=1}^{N}\left|\tau_{n,\flat,\ell} - t_\ell\right|^{H_\ell-\delta} \\ &\quad + c_{1.5}\sum_{\ell=1}^{N}\left|\tau_{n,\flat,\ell} - t_\ell\right|^{K_\ell-\delta} \\ &\le c_{3.2}\sum_{\ell=1}^{N}\left|\tau_{n,\flat,\ell} - t_\ell\right|^{(H_\ell\wedge K_\ell)-\delta} \\ &\le c_{3.3}n^{-1+\gamma}, \end{aligned} \tag{79}$$

where $\gamma = \max\{\delta/(H_\ell \wedge K_\ell), \ell = 1, \dots, N\}$.

We can choose a positive δ such that $0 < \gamma < \eta$. It follows from (79) that

$$\begin{aligned} &\mathbb{P}\{\exists t \in I \text{ such that } X(t) = f(t)\} \\ &\quad < \frac{\varepsilon}{2} + \mathbb{P}\{\omega \in \Omega_\varepsilon : \exists t \in I \text{ such that } X(t,\omega) = f(t)\} \\ &\quad \le \frac{\varepsilon}{2} + \mathbb{P}\left\{\bigcup_{i=1}^{m_n}\{\omega \in \Omega_\varepsilon : \exists t \in I_{n,\flat} \right. \\ &\qquad\qquad \left. \text{such that } X(t,\omega) = f(t)\}\right\} \\ &\quad \le \frac{\varepsilon}{2} + \sum_{i=1}^{m_n}\mathbb{P}\left\{\omega \in \Omega_\varepsilon : \left|X(\tau_{n,\flat},\omega) - f(\tau_{n,\flat})\right| \right. \\ &\qquad\qquad \left. \le c_{3.3}n^{-1+\gamma}\right\} \\ &\quad \le \frac{\varepsilon}{2} + c_{3.4}n^{\sum_{\ell=1}^{N}(1/(H_\ell\wedge K_\ell))-(1-\gamma)d} \le \varepsilon. \end{aligned} \tag{80}$$

In the above, we can get the last inequality as n is big enough. This proves Theorem 9. □

Theorem 10. *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be an (N,d)-Gaussian random field satisfying conditions (C1) and (C2) and $f \in \mathscr{Q}(K)$. If $\sum_{\ell=1}^{N}(1/(H_\ell \wedge K_\ell)) > d$, then with probability 1,*

$$\begin{aligned} &\dim\{t \in I : X(t) = f(t)\} \\ &\quad \le \min_{1\le k\le N}\left\{\sum_{\ell=1}^{k}\frac{H_k\wedge K_k}{H_\ell\wedge K_\ell} + N - k - (H_k\wedge K_k)d\right\} \\ &\quad = \sum_{\ell=1}^{k}\frac{H_k\wedge K_k}{H_\ell\wedge K_\ell} + N - k - (H_k\wedge K_k)d, \\ &\qquad \text{if } \sum_{\ell=1}^{k-1}\frac{1}{H_\ell\wedge K_\ell} \le d < \sum_{\ell=1}^{k}\frac{1}{H_\ell\wedge K_\ell}. \end{aligned} \tag{81}$$

Proof. Let $F(\omega) = \{t \in I : X(t,\omega) = f(t)\}$, $\omega \in \Omega$. In order to prove inequality (81), it suffices to prove that, for any $\varepsilon > 0$ and any $\zeta > \sum_{\ell=1}^{k}((H_k\wedge K_k)/(H_\ell\wedge K_\ell)) + N - k - (H_k\wedge K_k)d$,

$$\mathbb{P}\{\omega : \dim(F(\omega)) \le \zeta\} \ge 1 - \varepsilon. \tag{82}$$

We choose $0 < \eta < 1$ such that $\zeta > \sum_{\ell=1}^{k}((H_k\wedge K_k)/(H_\ell\wedge K_\ell)) + N - k - (H_k\wedge K_k)(1-\eta)d$. Then for all $0 < \lambda < \eta$, we have

$$\begin{aligned} &\sum_{\ell=1}^{N}\frac{1}{H_\ell\wedge K_\ell} + \sum_{\ell=k+1}^{N}\left(\frac{1}{H_k\wedge K_k} - \frac{1}{H_\ell\wedge K_\ell}\right) \\ &\qquad - \frac{\zeta}{H_k\wedge K_k} - (1-\lambda)d < 0. \end{aligned} \tag{83}$$

For any integer $n \ge 1$, divide the interval I into $m_n = [\prod_{\ell=1}^{N}(b_\ell - a_\ell)] \cdot n^{\sum_{\ell=1}^{N}(1/(H_\ell\wedge K_\ell))}$ subrectangles $I_{n,\flat}$ of side lengths $n^{-1/(H_\ell\wedge K_\ell)}$ ($\ell = 1, \dots, N$). Let $\tau_{n,\flat}$ be the lower-left vertex of $I_{n,\flat}$. It follows from (83) that

$$\begin{aligned} &\lim_{n\to\infty} n^{\sum_{\ell=1}^{N}(1/(H_\ell\wedge K_\ell))+\sum_{\ell=k+1}^{N}(1/(H_k\wedge K_k)-1/(H_\ell\wedge K_\ell))-\zeta/(H_k\wedge K_k)-(1-\eta)d} \\ &\quad = 0. \end{aligned} \tag{84}$$

Let

$$I'_{n,\flat}(\omega) = \begin{cases} I_{n,\flat}, & \text{if } \exists t \in I_{n,\flat},\ X(t,\omega) = f(t), \\ \emptyset & \text{otherwise}. \end{cases} \tag{85}$$

Then $F(\omega)$ can be covered by $\{I'_{n,\flat}(\omega)\}$. For every $1 \le k \le N$, $I_{n,\flat}(\omega)$ can be covered by

$$\begin{aligned} \prod_{\ell=1}^{N} n^{(1/(H_k\wedge K_k)-1/(H_\ell\wedge K_\ell))} &= n^{\sum_{\ell=1}^{N}(1/(H_k\wedge K_k)-1/(H_\ell\wedge K_\ell))} \\ &\le n^{\sum_{\ell=k+1}^{N}(1/(H_k\wedge K_k)-1/(H_\ell\wedge K_\ell))} \end{aligned} \tag{86}$$

cubes of side length $n^{-1/(H_k\wedge K_k)}$. Then, we can cover the $F(\omega)$ by a sequence of cubes of side length $n^{-1/(H_k\wedge K_k)}$.

Repeating this procedure in (77) and (79) in Theorem 9, we can deduce that, for all $\omega \in \Omega_\varepsilon$, if there exists $t \in I_{n,\flat}$ such that $X(t,\omega) = f(t)$, then

$$\left|X(\tau_{n,\flat},\omega) - f(\tau_{n,\flat})\right| \le c_{3.3} n^{-1+\gamma}, \tag{87}$$

where $\gamma = \max\{\delta/(H_\ell \wedge K_\ell), \ell = 1,\dots,N\}$. Hence,

$$\begin{aligned} &\varliminf_{n\to\infty} \sum_{\flat(\omega)} \left[\operatorname{diam} I'_{n,\flat}(\omega)\right]^\zeta \\ &\quad \le \varliminf_{n\to\infty} \sum_{\flat\in T_n(\omega)} \left[\operatorname{diam} I_{n,\flat}\right]^\zeta, \quad \omega \in \Omega_\varepsilon, \end{aligned} \tag{88}$$

where $T_n(\omega) = \{\flat : |X(\tau_{n,\flat},\omega) - f(\tau_{n,\flat})| \le c_{3.3} n^{-1+\gamma}\}$. We can choose a positive δ such that $0 < \gamma < \eta$. It follows from (84)~(88) and lemma of Fatou that

$$\begin{aligned} &\mathbb{E}\left[\varliminf_{n\to\infty}\sum_{\flat(\omega)}\left[\operatorname{diam} I'_{n,\flat}(\omega)\right]^\zeta I_{\Omega_\varepsilon}\right] \\ &\le \varliminf_{n\to\infty}\mathbb{E}\left[\sum_{\flat\in T_n(\omega)}\left[\operatorname{diam} I'_{n,\flat}\right]^\zeta I_{\Omega_\varepsilon}\right] \\ &\le c_{3.5}\varliminf_{n\to\infty}\sum_{\flat=1}^{m_n}\left[\operatorname{diam} I_{n,\flat}\right]^\zeta \\ &\quad \times \mathbb{P}\left\{\omega\in\Omega_\varepsilon : \left|X(\tau_{n,\flat},\omega) - f(\tau_{n,\flat})\right| \le c_{3.3}n^{-1+\gamma}\right\} \\ &\le c_{3.6} \\ &\quad \times \varliminf_{n\to\infty} n^{\sum_{\ell=1}^{N}(1/(H_\ell\wedge K_\ell)) + \sum_{\ell=k+1}^{N}(1/(H_k\wedge K_k) - 1/(H_\ell\wedge K_\ell)) - \zeta/(H_k\wedge K_k) - (1-\gamma)d} \\ &= 0. \end{aligned} \tag{89}$$

Therefore, there exists $\Omega_0 \subset \Omega$ such that $\mathbb{P}(\Omega_0) = 1$, and for all $\omega \in \Omega_0 \cap \Omega_\varepsilon$ we have $\varliminf_{n\to\infty}\sum_{\flat(\omega)}[\operatorname{diam} I'_{n,\flat}(\omega)]^\zeta = 0$. Then, $\dim(F(\omega)) \le \zeta$. Since $\mathbb{P}(\Omega_0 \cap \Omega_\varepsilon) > 1 - \varepsilon/2$, we obtain (82). □

Theorem 11. *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be an (N,d)-Gaussian random field satisfying conditions (C1) and (C2) and $f \in \mathcal{Q}(K)$. If $\sum_{\ell=1}^{N}(1/(H_\ell \wedge K_\ell)) > d$, then with positive probability,*

$$\begin{aligned} &\dim\{t \in I : X(t) = f(t)\} \\ &\quad \ge \min_{1\le k\le N}\left\{\sum_{\ell=1}^{k}\frac{H_k\wedge K_k}{H_\ell\wedge K_\ell} + N - k - (H_k\wedge K_k)\,d\right\} \\ &\quad = \sum_{\ell=1}^{k}\frac{H_k\wedge K_k}{H_\ell\wedge K_\ell} + N - k - (H_k\wedge K_k)\,d, \\ &\qquad \textit{if } \sum_{\ell=1}^{k-1}\frac{1}{H_\ell\wedge K_\ell} \le d < \sum_{\ell=1}^{k}\frac{1}{H_\ell\wedge K_\ell}. \end{aligned} \tag{90}$$

Proof. Let us assume that $\sum_{\ell=1}^{k-1}(1/(H_\ell \wedge K_\ell)) \le d < \sum_{\ell=1}^{k}(1/(H_\ell \wedge K_\ell))$ for some $1 \le k \le N$, and let δ be a positive constant such that

$$\sum_{\ell=1}^{k-1}\frac{1}{H_\ell\wedge K_\ell} < (1+\delta)\,d < \sum_{\ell=1}^{k}\frac{1}{H_\ell\wedge K_\ell} \tag{91}$$

and hence

$$\begin{aligned} \beta &\mathrel{\hat=} \min_{1\le k\le N}\left\{\sum_{\ell=1}^{k}\frac{H_k\wedge K_k}{H_\ell\wedge K_\ell} + N - k - (H_k\wedge K_k)(1+\delta)\,d\right\} \\ &= \sum_{\ell=1}^{k}\frac{H_k\wedge K_k}{H_\ell\wedge K_\ell} + N - k - (H_k\wedge K_k)(1+\delta)\,d. \end{aligned} \tag{92}$$

Note that, if we can prove that there is a constant $c_{3.7} > 0$, independent of δ and β, such that

$$\mathbb{P}\left\{\omega : \dim\{t \in I : X(t,\omega) = f(t)\} \ge \beta\right\} \ge c_{3.7}, \tag{93}$$

then the lower bound in (90) will follow by letting $\delta \downarrow 0$. The proof of (93) is based on the capacity argument due to Kahane [11].

Let $\mathcal{M}_\beta^+$ be the space of all nonnegative measures on $\mathbb{R}^N$ with finite β-energy. It is known that $\mathcal{M}_\beta^+$ is a complete metric space under the metric

$$\|\mu\|_\beta = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\mu(dt)\,\mu(ds)}{|t-s|^\beta}. \tag{94}$$

We define a sequence of random positive measures μ_ε on the Borel sets C of I by

$$\mu_\varepsilon(C) = \int_C \left(\frac{2\pi}{\varepsilon}\right)^{d/2} \exp\left(-\frac{|X(t) - f(t)|^2}{2\varepsilon}\right) dt. \tag{95}$$

It follows from Kahane [11] or Testard [16] that if there are positive constants $c_{3.8}$ and $c_{3.9}$ such that

$$\mathbb{E}\left(\|\mu_\varepsilon\|\right) \ge c_{3.8}, \qquad \mathbb{E}\left(\|\mu_\varepsilon\|^2\right) \le c_{3.9}, \qquad \mathbb{E}\left(\|\mu_\varepsilon\|_\beta\right) < \infty, \tag{96}$$

where $\|\mu_\varepsilon\| = \mu_\varepsilon(I)$, then there is a subsequence of $\{\mu_\varepsilon\}$, say $\{\mu_{\varepsilon_k}\}$, such that $\mu_{\varepsilon_k} \to \mu$ in $\mathcal{M}_\beta^+$ and μ is strictly positive with probability $\ge c_{3.8}^2/(2c_{3.9})$. In this case, it follows from (95) that the measure μ has its support in F a.s. Hence, Frostman's theorem yields (93) with $c_{3.7} = c_{3.8}^2/(2c_{3.9})$. It remains to verify (96).

By using Fubini's theorem and (10), for all $\varepsilon \in (0,1)$, we have

$$\begin{aligned}\mathbb{E}(\|\mu_\varepsilon\|) &= \int_I \int_{\mathbb{R}^d} \exp\left(-\frac{|u|^2}{2\varepsilon}\right) \times \mathbb{E}(\exp(i\langle u, X(t)-f(t)\rangle))\,du\,dt \\ &= \int_I \left(\prod_{k=1}^d \int_{\mathbb{R}} \exp(-i\langle u_k, f_k(t)\rangle) \times \exp\left(-\frac{1}{2}(\varepsilon^{-1}+\operatorname{Var}(X_k(t))) \times |u_k|^2\right) du_k\right) dt \\ &= \int_I \prod_{k=1}^d \left(\frac{2\pi}{\varepsilon^{-1}+\operatorname{Var}(X_k(t))}\right)^{1/2} \times \exp\left(-\frac{|f_k(t)|^2}{2(\varepsilon^{-1}+\operatorname{Var}(X_k(t)))}\right) dt \\ &\geq \int_I \prod_{k=1}^d \left(\frac{2\pi}{1+\operatorname{Var}(X_k(t))}\right)^{1/2} \times \exp\left(-\frac{|f_k(t)|^2}{2\operatorname{Var}(X_k(t))}\right) dt \,\widehat{=}\, c_{3.8}.\end{aligned} \tag{97}$$

Using Lemmas 4 and 7 and Fubini's theorem, we can deduce that

$$\begin{aligned}\mathbb{E}(\|\mu_\varepsilon\|^2) &= \int_I \int_I \mathbb{E}\left(\frac{2\pi}{\varepsilon}\right)^d \times \exp\left(-\left(|X(s)-f(s)|^2 + |X(t)-f(t)|^2\right)(2\varepsilon)^{-1}\right) dt\,ds \\ &\leq c_{3.10}\int_I\int_I \prod_{k=1}^d \left[\max\left\{\det\operatorname{Cov}(X_k(s),X_k(t)), |f_k(s)-f_k(t)|^2\right\}\right]^{-(1/2)} dt\,ds \\ &\leq c_{3.9}.\end{aligned} \tag{98}$$

Similar to (98), we have

$$\begin{aligned}\mathbb{E}(\|\mu_\varepsilon\|_\beta) &= \int_I\int_I \frac{1}{|s-t|^\beta}\mathbb{E}\left(\frac{2\pi}{\varepsilon}\right)^d \times \exp\left(-\frac{|X(s)-f(s)|^2+|X(t)-f(t)|^2}{2\varepsilon}\right) ds\,dt \\ &\leq c_{3.11}\int_I\int_I |s-t|^{-\beta} \times \prod_{k=1}^d \left[\max\left\{\det\operatorname{Cov}(X_k(s),X_k(t)), |f_k(s)-f_k(t)|^2\right\}\right]^{-(1/2)} dt\,ds \\ &\leq c_{3.12},\end{aligned} \tag{99}$$

where the last inequality follows from Lemma 8. This completes the proof of Theorem 11. □

By using Theorems 10 and 11, we can derive the following corollaries.

Corollary 12. *If $\sum_{\ell=1}^N (1/(H_\ell \wedge K_\ell)) > d$ and $f \in \mathcal{Q}(K)$, then with positive probability,*

$$\begin{aligned}\dim\{t \in I : X(t) = f(t)\} &= \min_{1\leq k\leq N}\left\{\sum_{\ell=1}^k \frac{H_k\wedge K_k}{H_\ell \wedge K_\ell} + N - k - (H_k\wedge K_k)d\right\} \\ &= \sum_{\ell=1}^k \frac{H_k\wedge K_k}{H_\ell\wedge K_\ell} + N - k - (H_k\wedge K_k)d, \\ &\quad \text{if } \sum_{\ell=1}^{k-1}\frac{1}{H_\ell\wedge K_\ell} \leq d < \sum_{\ell=1}^k \frac{1}{H_\ell\wedge K_\ell}.\end{aligned} \tag{100}$$

In particular, we have the following.

Corollary 13. *If $\sum_{\ell=1}^N (1/(H_\ell \wedge K_\ell)) > d$, then $\mathcal{Q}(K) \subset \mathcal{C} \setminus \mathcal{P}$.*

The following corollary presents the Hausdorff dimension about the fixed points of $X(t)$, $t \in I$.

Corollary 14. *Let $F \,\widehat{=}\, \{t \in I, X(t) = t\}$ and $N = d$. Then, with positive probability,*

$$\begin{aligned}\dim F &= \min_{1\leq k\leq N}\left\{\sum_{\ell=1}^k \left(\frac{H_k}{H_\ell} - 1\right) + (1-H_k)d\right\} \\ &= \sum_{\ell=1}^k \left(\frac{H_k}{H_\ell}-1\right) + (1-H_k)d, \\ &\quad \text{if } \sum_{\ell=1}^{k-1}\frac{1}{H_\ell} \leq d < \sum_{\ell=1}^k \frac{1}{H_\ell}.\end{aligned} \tag{101}$$

The following corollary also solves the question proposed by Le Gall [13] about the existence of nonpolar, continuous functions satisfying the Hölder condition for the Brownian motion.

Corollary 15. *Let $B = \{B(t), t \in \mathbb{R}_+\}$ be $(1,d)$ Brownian motion. Then for any $0 < \beta < 1/d$, there exists a function f satisfying the Hölder condition with index β such that $f \in \mathcal{C} \setminus \mathcal{P}$.*

The following theorem provides the intersection probability for a nonpolar function and X in terms of Hausdorff measure and capacity, respectively.

Theorem 16. *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be an (N, d)-Gaussian random field satisfying conditions (C1) and (C2). If $E \subseteq I$ is a compact set on $\mathbb{R}^N$ and $f \in \mathcal{Q}(K)$, then there exist positive constants $c_{3.13}$ and $c_{3.14}$ depending on E, K, and H only, such that*

$$c_{3.13}\mathcal{C}_d^{\rho}(E) \le \mathbb{P}\{\exists t \in E \text{ such that } X(t) = f(t)\} \le c_{3.14}\mathcal{H}_d^{\rho}(E), \tag{102}$$

where the metric ρ is defined in (21), $\mathcal{C}_d^{\rho}(E)$ is the capacity of E on $\mathbb{R}^N$ generated by the kernel function $\rho^{-d}(s,t)$, and $\mathcal{H}_d^{\rho}(E)$ is defined as the d-dimensional Hausdorff measure of E in the metric space $(\mathbb{R}^N, \rho)$.

Proof. We first prove the lower bound in (102). When $\mathcal{C}_d^{\rho}(E) = 0$, the lower bound in (102) holds automatically. On the other hand, when $\mathcal{C}_d^{\rho}(E) > 0$, by the definition of $\mathcal{C}_d^{\rho}(E)$, then there exists a finite positive measure σ supported on E, such that

$$\mathcal{I}_d^{\rho}(\sigma) \mathrel{\hat=} \int_E\int_E \frac{\sigma(dt)\,\sigma(ds)}{[\rho(s,t)]^d} < \frac{1}{\mathcal{C}_d^{\rho}(E)}. \tag{103}$$

For all $\varepsilon > 0$, we define a family of random measures $\mu_\varepsilon \mathrel{\hat=} \mu_\varepsilon(x,\cdot)$ on the Borel sets C of E by

$$\begin{aligned}\mu_\varepsilon(C) &= \int_C \left(\frac{2\pi}{\varepsilon}\right)^{d/2} \exp\left(-\frac{|X(s) - f(s) - x|^2}{2\varepsilon}\right)\sigma(ds)\\ &= \int_C\int_{\mathbb{R}^d} \exp\left(-\frac{\varepsilon|\xi|^2}{2} + i\langle \xi, X(s) - f(s) - x\rangle\right) d\xi\,\sigma(ds).\end{aligned} \tag{104}$$

We claim that there are positive constants $c_{3.15}$ and $c_{3.16}$, such that

$$\mathbb{E}(\|\mu_\varepsilon\|) \ge c_{3.15}, \qquad \mathbb{E}(\|\mu_\varepsilon\|^2) \le c_{3.16}\mathcal{I}_d^{\rho}(\sigma), \tag{105}$$

where $\|\mu_\varepsilon\| = \mu_\varepsilon(E)$.

For any $\varepsilon \in (0,1)$, by Fubini's theorem and (10), we have

$$\begin{aligned}\mathbb{E}(\|\mu_\varepsilon\|) &= \int_E\int_{\mathbb{R}^d} \exp\left(-\frac{\varepsilon}{2}|\xi|^2\right) \times \mathbb{E}(\exp(i\langle \xi, X(s) - f(s) - x\rangle))\, d\xi\,\sigma(ds)\\ &= \int_E \prod_{k=1}^d \left(\frac{2\pi}{\varepsilon + \operatorname{Var}(X_k(s) - f_k(s))}\right)^{1/2} \times \exp\left(-\frac{x_k^2}{2(\varepsilon + \operatorname{Var}(X_k(s) - f_k(s)))}\right)\sigma(ds)\\ &\ge \int_E \prod_{k=1}^d \left(\frac{2\pi}{1 + \operatorname{Var}(X_k(s))}\right)^{1/2} \times \exp\left(-\frac{x_k^2}{2\operatorname{Var}(X_k(s))}\right)\sigma(ds)\\ &\mathrel{\hat=} c_{3.15}.\end{aligned} \tag{106}$$

Let $M_k^{\varepsilon}(s,t) = \varepsilon I_2 + \operatorname{Cov}(X_k(s) - f_k(s), X_k(t) - f_k(t))$. It follows from Lemmas 1 and 2 that

$$\begin{aligned}\det M_k^{\varepsilon}(s,t) &\ge \operatorname{Cov}(X_k(s) - f_k(s), X_k(t) - f_k(t))\\ &= \det\operatorname{Cov}(X_k(s), X_k(t)) + \mathbb{E}[f_k(t)X_k(s) - f_k(s)X_k(t)]^2\\ &\ge c_{3.17}\sum_{\ell=1}^N |s_\ell - t_\ell|^{2(H_\ell \wedge K_\ell)}\\ &\mathrel{\hat=} c_{3.18}\rho^2(s,t).\end{aligned} \tag{107}$$

By Fubini's theorem, (103), and (107), we have

$$\begin{aligned}\mathbb{E}(\|\mu_\varepsilon\|^2) &= \int_E\int_E \mathbb{E}\left(\frac{2\pi}{\varepsilon}\right)^d \times \exp\left(-\left(|X(s) - f(s) - x|^2 + |X(t) - f(t) - x|^2\right)(2\varepsilon)^{-1}\right)\\ &\quad \times \sigma(ds)\,\sigma(dt)\end{aligned}$$

$$= \int_E \int_E \prod_{k=1}^{d} \frac{2\pi}{\left(\det M_k^{\varepsilon}(s,t)\right)^{1/2}} \times \exp\left(-\frac{1}{2}(x_k,x_k)\left(M_k^{\varepsilon}(s,t)\right)^{-1}(x_k,x_k)'\right) \times \sigma(ds)\,\sigma(dt)$$
$$\leq \int_E \int_E \frac{c_{3.16}}{\rho^d(s,t)}\sigma(ds)\,\sigma(dt)$$
$$< c_{3.16}\mathcal{I}_d^{\rho}(\sigma) < \frac{c_{3.16}}{\mathcal{C}_d^{\rho}(E)}. \tag{108}$$

By modifying an argument from Kahane [11] or Testard [16], we can verify that there is a subsequence of $\{\mu_\varepsilon\}$, say $\{\mu_{\varepsilon_n}\}$, such that $\mu_{\varepsilon_n} \to \mu$ and μ is strictly positive with probability that at least $c_{3.15}^2/(c_{3.16}\mathcal{I}_d^{\rho}(\sigma))$. It follows from (104) and the continuity of X that μ has its support on $\{t \in E : X(t)-f(t) = x\}$ a.s. Then, we apply the Paley-Zygmund inequality, (103), and (105), to deduce that

$$\mathbb{P}\{t \in E : X(t) = f(t)\} \geq \frac{[\mathbb{E}(\|\mu_\varepsilon\|)]^2}{\mathbb{E}\left(\|\mu_\varepsilon\|^2\right)} \geq \frac{c_{3.15}^2}{c_{3.16} I_d^{\rho}(\sigma)} \geq \frac{c_{3.15}^2}{c_{3.16}}\mathcal{C}_d^{\rho}(E). \tag{109}$$

Next we prove that the upper of (102) holds. When $\mathcal{H}_d^{\rho}(E) = \infty$, the result follows immediately; when $0 < \mathcal{H}_d^{\rho}(E) < \infty$, we can choose and fix an arbitrary constant $\gamma > \mathcal{H}_d^{\rho}(E)$. By using the definition of $\mathcal{H}_d^{\rho}(E)$ and modifying an argument from Theorem 32 in Rogers [18], there is a sequence of balls $B_\rho(s_n, r_n)$ $(0 < r_n < \delta, n = 1, 2, \ldots)$ in the metric space $(\mathbb{R}^N, \rho)$ such that

$$E \subseteq \bigcup_{n=1}^{\infty} B_\rho(s_n, r_n), \qquad \sum_{n=1}^{\infty} r_n^d < \gamma. \tag{110}$$

By (110) and Lemma 3, we have

$$\mathbb{P}\{t \in E : X(t) = f(t)\} \leq \sum_{n=1}^{\infty}\mathbb{P}\left\{t \in B_\rho(s_n, r_n) : X(t) = f(t)\right\} \leq c_{3.19}\sum_{n=1}^{\infty} r_n^d < c_{3.19}\gamma. \tag{111}$$

This implies that the upper of (102) holds in this case.

When $\mathcal{H}_d^{\rho}(E) = 0$, by using Theorem 32 in Rogers [18] again, we can deduce that there exist sequences of open balls $B_\rho(s_n, r_n)$ $(0 < r_n < \delta, n = 1, 2, \ldots)$ in the metric space $(\mathbb{R}^N, \rho)$ such that

$$E \subseteq \limsup_{n\to\infty} B_\rho(s_n, r_n), \qquad \sum_{n=1}^{\infty} r_n^d < \infty. \tag{112}$$

Let

$$D_n = \left\{t \in B_\rho(s_n, r_n) : X(t) = f(t)\right\}, \tag{113}$$

and then by Lemma 3 and (112) we have

$$\sum_{n=1}^{\infty}\mathbb{P}(D_n) \leq c_{3.20}\sum_{n=1}^{\infty} r_n^d < \infty. \tag{114}$$

Therefore the Borel-Cantelli Lemma implies

$$\mathbb{P}\left\{\limsup_{n\to\infty} D_n\right\} = 0. \tag{115}$$

On the other hand, by (112) we have

$$\{t \in E : X(t) = f(t)\} \subseteq \limsup_{n\to\infty} D_n. \tag{116}$$

Then (115) and (116) imply the upper bound of (102) when $\mathcal{H}_d^{\rho}(E) = 0$. Thus, the proof of Theorem 16 is finished. □

Finally, we discuss the packing dimension for the (N, d)-anisotropic Gaussian random fields.

For any $\varepsilon > 0$ and any bounded set $E \subset \mathbb{R}^N$, we use $M(\varepsilon, E)$ to denote the smallest number of cubes of side lengths ε that are needed to cover E. Then the upper box-counting dimension of E is defined as

$$\Delta(E) = \limsup_{\varepsilon\to 0}\frac{\log M(\varepsilon, E)}{-\log\varepsilon}. \tag{117}$$

The packing dimension of E is defined as

$$\operatorname{Dim}(E) = \inf\left\{\sup\Delta(E_n),\ E \subset \bigcup_{n=1}^{\infty} E_n\right\}. \tag{118}$$

It is proved in Tricot Jr. [19] that, for any bounded set $E \subset \mathbb{R}^N$,

$$0 \leq \dim(E) \leq \operatorname{Dim}(E) \leq \Delta(E) \leq N. \tag{119}$$

Theorem 17. *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be an (N, d)-Gaussian random field satisfying conditions (C1) and (C2). If $\sum_{\ell=1}^{N}(1/(H_\ell \wedge K_\ell)) > d$, then for any $f \in \mathcal{Q}(K)$, with positive probability*

$$\operatorname{Dim}\{t \in I, X(t) = f(t)\} = \min_{1\leq k\leq N}\left\{\sum_{\ell=1}^{k}\frac{H_k \wedge K_k}{H_\ell \wedge K_\ell} + N - k - (H_k \wedge K_k)\,d\right\} = \sum_{\ell=1}^{k}\frac{H_k \wedge K_k}{H_\ell \wedge K_\ell} + N - k - (H_k \wedge K_k)\,d,$$
$$\text{if } \sum_{\ell=1}^{k-1}\frac{1}{H_\ell \wedge K_\ell} \leq d < \sum_{\ell=1}^{k}\frac{1}{H_\ell \wedge K_\ell}. \tag{120}$$

Proof. The lower bound of (120) follows from (90) and (119). In order to prove the upper bound in (120), let us assume that

$\sum_{\ell=1}^{k-1}(1/(H_\ell \wedge K_\ell)) \le d < \sum_{\ell=1}^{k}(1/(H_\ell \wedge K_\ell))$ for some $1 \le k \le N$. Then, by (119) we only need to prove that

$$\Delta(F) \le \sum_{\ell=1}^{k} \frac{H_k \wedge K_k}{H_\ell \wedge K_\ell} + N - k - (H_k \wedge K_k) d \quad \text{a.s.}, \tag{121}$$

where $F = \{t \in I : X(t) = f(t)\}$.

For any integer $n \ge 1$, divide the I into $m_n = [\prod_{\ell=1}^{N}(b_\ell - a_\ell)] \cdot n^{\sum_{\ell=1}^{N}(1/(H_\ell \wedge K_\ell))}$ subrectangles $I_{n,\flat}$ with sides parallel to the axes and side lengths $n^{-(1/(H_\ell \wedge K_\ell))}$ $(1 \le \ell \le N)$. Then I can be covered by $\{I_{n,\flat}\}$ and each $I_{n,\flat}$ is equivalent to a ball of radius n^{-1} under the metric ρ. It follows from Lemma 3 that

$$\begin{aligned} &\mathbb{P}\{\exists t \in I_{n,\flat} : X(t) = f(t)\} \\ &\quad \le \mathbb{P}\left\{\inf_{t \in I_{n,\flat}} |X(t) - f(t)| < \frac{1}{n}\right\} \\ &\quad \le c_{3.21} n^{-d}. \end{aligned} \tag{122}$$

For every $\flat$, let

$$I'_{n,\flat} = \begin{cases} I_{n,\flat}, & \text{if } \exists t \in I_{n,\flat},\ X(t) = f(t) \\ \emptyset, & \text{otherwise.} \end{cases} \tag{123}$$

Then F can be covered by $\{I'_{n,\flat}\}$. For every $1 \le k \le N$, $I'_{n,\flat}$ can be covered by

$$\prod_{\ell=1}^{N} n^{(1/(H_k \wedge K_k) - 1/(H_\ell \wedge K_\ell))} \le n^{\sum_{\ell=k+1}^{N}(1/(H_k \wedge K_k) - 1/(H_\ell \wedge K_\ell))} \tag{124}$$

cubes of side length $n^{-1/(H_k \wedge K_k)}$. Thus, we can cover the F by a sequence of cubes of side length $n^{-1/(H_k \wedge K_k)}$. Denote the number of such cubes by $M_{n,k}$. Using (122) and (124), we have

$$\begin{aligned} &\mathbb{E}\left[M\left(n^{-(1/(H_k \wedge K_k))}, F\right)\right] \\ &\quad \le \mathbb{E}[M_{n,k}] \\ &\quad \le c_{3.22} n^{\sum_{\ell=1}^{N}(1/(H_\ell \wedge K_\ell)) + \sum_{\ell=k+1}^{N}(1/(H_k \wedge K_k) - 1/(H_\ell \wedge K_\ell))} \cdot n^{-d} \\ &\quad = c_{3.22} n^{\sum_{\ell=1}^{k}(1/(H_\ell \wedge K_\ell) + (N-K)/(H_k \wedge K_k) - d)}. \end{aligned} \tag{125}$$

Now let $0 < \delta < 1$ be fixed and let η be the constant defined by

$$\eta = \sum_{\ell=1}^{k} \frac{1}{H_\ell \wedge K_\ell} + \frac{N-k}{H_k \wedge K_k} - (1-\delta) d. \tag{126}$$

We consider the sequence of integers $n_i = 2^i$ $(i \ge 1)$. By using (125) and Markov inequality, we have

$$\begin{aligned} &\sum_{i=1}^{\infty} \mathbb{P}\left\{M\left(n_i^{-1/(H_k \wedge K_k)}, F\right) > c n_i^{\eta}\right\} \\ &\quad \le c_{3.23} \sum_{i=1}^{\infty} 2^{-\delta d i} < \infty. \end{aligned} \tag{127}$$

Then it follows from the Borel-Cantelli lemma that a.s.

$$M\left(n_i^{-1/(H_k \wedge K_k)}, F\right) \le c n_i^{\eta} \quad \text{for all } i \text{ large enough.} \tag{128}$$

For any $0 < \varepsilon < 1$, we can choose some positive integer i such that $2^{-i-1} < \varepsilon \le 2^{-i}$. Then, this, together with (128), implies that a.s.

$$\begin{aligned} \Delta(F) &= \overline{\lim_{\varepsilon \to 0}} \frac{\log\left[M\left(\varepsilon^{1/(H_k \wedge K_k)}, F\right)\right]}{-\log \varepsilon^{1/(H_k \wedge K_k)}} \\ &\le \overline{\lim_{i \to \infty}} \frac{\log\left[M\left(n_{i+1}^{-1/(H_k \wedge K_k)}, F\right)\right]}{-\log n_i^{-1/(H_k \wedge K_k)}} \\ &\le \sum_{\ell=1}^{k} \frac{H_k \wedge K_k}{H_\ell \wedge K_\ell} \\ &\quad + N - k - (1-\delta)(H_k \wedge K_k) d. \end{aligned} \tag{129}$$

Letting $\delta \searrow 0$ along rational numbers and optimizing over $k = 1, \ldots, N$, we can deduce that (121) holds. □

4. Applications to SPDEs

These results in this paper are applicable to solutions of SPDEs such as the linear string process considered by Mueller and Tribe [6], linear hyperbolic SPDEs considered by Dalang and Nualart [10], and nonlinear stochastic heat equations considered by Dalang et al. [8]. In this section, we only consider the Hausdorff and packing dimensions of the set $\{t \in I : X(t) = f(t)\}$ for nonlinear stochastic heat equations in [8].

Let $\dot{W} = (\dot{W}_1, \ldots, \dot{W}_d)$ be a space-time white noise in $\mathbb{R}^d$. That is, the components $\dot{W}_1(x,s), \ldots, \dot{W}_d(x,s)$ are independent space-time white noises, which are generalized Gaussian processes with covariance given by

$$\mathbb{E}\left[\dot{W}_i(x,s)\dot{W}_j(y,t)\right] = \delta(x-y)\delta(s-t), \quad (i = 1, \ldots, d), \tag{130}$$

where $\delta(\cdot)$ is the Dirac delta function. For all $1 \le j \le d$, let $b_j : \mathbb{R}^d \mapsto \mathbb{R}$ be globally Lipschitz and bounded functions, and let $\sigma \mathrel{\widehat{=}} (\sigma_{ij})$ be a deterministic $d \times d$ invertible matrix.

Consider the system of SPDEs

$$\frac{\partial u_i(s,x)}{\partial s} = \frac{\partial^2 u_i(s,x)}{\partial x^2} + \sum_{j=1}^{d} \sigma_{i,j} \dot{W}_j(s,x) + b_i(u(s,x)), \tag{131}$$

for $1 \le i \le d$, $s \in [0,T]$ and $x \in [0,1]$, with the initial conditions $u(0,x) = 0$ for all $x \in [0,1]$, and the Neumann boundary conditions

$$\frac{\partial u_i(s,0)}{\partial x} = \frac{\partial u_i(s,1)}{\partial x} = 0, \quad 0 \le s \le T, \tag{132}$$

where $u(s,x) = (u_1(s,x), \ldots, u_d(s,x))$. Equation (131) can be interpreted rigorously as in Dalang et al. [8].

A random field $u = \{u(s,x), s \in [0,T], x \in [0,1]\}$ is a solution of (131) if u is adapted to $(\mathscr{F}_s)$ and if for every $i \in \{1,\dots,d\}$, $s \in [0,T]$ and $x \in [0,1]$,

$$u_i(s,x) = \int_0^s \int_0^1 G_{s-r}(x,v) \sum_{j=1}^{d} \sigma_{i,j} W_j(dr,dv) + \int_0^s \int_0^1 G_{s-r}(x,v)\, b_i(u(s,x))\, dr\, dv, \tag{133}$$

where $G_s(x,y)$ is the Green kernel for the heat equation with Neumann boundary conditions (see Walsh [20]).

For the linear form of (131) (i.e., $b \equiv 0$ and $\sigma \equiv I_d$ (the $d \times d$ identity matrix)), Mueller and Tribe [6] found necessary and sufficient conditions (in terms of the dimension d) for its solution u to hit points or to have double points of various types. Wu and Xiao [21] further studied the fractal properties of the sample paths of u and obtained the Hausdorff dimensions of the level sets and the set of double times of u. Recently, Chen [5] studied the fractal properties of the algebraic sum of the image sets for u and obtained the Hausdorff and packing dimensions of the algebraic sum of the image sets of the string. More generally, Dalang et al. [8] studied hitting probabilities for the nonlinear equation (131). They also determined the Hausdorff dimensions of the range and level sets of these processes.

In the following, we show the Hausdorff dimension and the packing dimension of the intersecting sets $\{t \in I : X(t) = f(t)\}$ of the nonpolar functions for nonlinear stochastic heat equations in [8]. As shown by [8, Proposition 4.1], it is sufficient to consider these problems for the solution of (131) in the following drift-free case (i.e., $b \equiv 0$):

$$\frac{\partial u}{\partial s}(s,x) = \frac{\partial^2 u}{\partial x^2}(s,x) + \sigma \dot{W}. \tag{134}$$

The solution of (134) is the mean zero Gaussian random field $u = \{u(s,x), s \in [0,T], x \in [0,1]\}$ with values in $\mathbb{R}^d$ defined by

$$u(s,x) = \int_0^s \int_0^1 G_{s-r}(x,y)\, \sigma W(dr,dy) \qquad s \in [0,T], x \in [0,1]. \tag{135}$$

Moreover, since the matrix σ is invertible, a change of variables shows (see proof of Proposition 4.1 in [8]) that $v \,\widehat{=}\, \sigma^{-1} u$ solves the following uncoupled system of SPDE:

$$\frac{\partial v}{\partial s}(s,x) = \frac{\partial^2 v}{\partial x^2}(s,x) + \sigma \dot{W}. \tag{136}$$

Note that $f \in \mathcal{Q}(K)$ if and only if $\sigma^{-1} f \in \mathcal{Q}(K)$; that is, they belong or do not belong to the same functional class $\mathcal{Q}(K)$. Thus, both processes u and v have the same intersection probability, Hausdorff dimension and packing dimension properties. Therefore, without loss of generality, we will assume that $\sigma = I_d$ in (134).

The following is a consequence of Lemmas 4.2 and 4.3 of Dalang et al. [8] or Lemma 4.1, in Biermé et al. [1], which indicates that the Gaussian random field u satisfies conditions (C1) and (C2) with $H_1 = 1/4$ and $H_2 = 1/2$.

Lemma 18. *Let $u = \{u(s,x), s \in [0,T], x \in [0,1]\}$ be the solution of* (134). *Then for any compact set $E \subseteq (0,T] \times [0,1]$, there exist positive and finite constants $c_{4.1},\dots,c_{4.5}$ such that the following hold.*

(i) *For all $(s,x) \in E$, $c_{4.1} \le \mathbb{E}[u(s,x)]^2 \le c_{4.2}$ and for all $(s,x),(t,y) \in E$,*

$$c_{4.3}\left(|s-t|^{1/2} + |x-y|\right) \le \mathbb{E}[u(s,x) - u(t,y)]^2 \le c_{4.4}\left(|s-t|^{1/2} + |x-y|\right). \tag{137}$$

(ii) *For all $(s,x),(t,y) \in E$,*

$$\operatorname{Var}(u(s,x) \mid u(t,y)) \ge c_{4.5}\left(|s-t|^{1/2} + |x-y|\right). \tag{138}$$

Proof. Since E is a compact set on $(0,T] \times [0,1]$, then there exists a positive constant c, such that $E \subset [c,T] \times [0,1]$. By (135), we have

$$\mathbb{E}[u(s,x)]^2 = \int_0^s dr \int_0^1 (G_{s-r}(x,y))^2 dy. \tag{139}$$

Note that $\mathbb{E}[u(s,x)]^2$ is a continuous function in (s,x) and positive on E. This implies the first conclusion of the lemma. Inequality (137) follows (4.11) in Lemma 4.2 of Dalang et al. [8].

It follows from Lemma 4.3 of Dalang et al. [8] that

$$\det \operatorname{Cov}(u(s,x), u(t,y)) \ge c_{4.5}\left(|s-t|^{1/2} + |x-y|\right). \tag{140}$$

By using (8), (140), and the first inequality in Lemma 18, we can deduce that (138) holds. This finishes the proof of Lemma 18. □

Therefore, Lemma 18 shows that Theorem 16 includes the corresponding conclusion of solutions of nonlinear stochastic heat equations in [8]. The following theorems, which are two new results in [6, 8, 10], are the consequences of Theorems 10, 11, and 17 with $K = (K_1, K_2) \in (0,1)^2$. Moreover, we can obtain very different results when the parameters K take different values.

Theorem 19. *Let $u = \{u(s,x), s \in [0,T], x \in [0,1]\}$ be the solution of* (131) *and let I be a rectangle on $(0,T] \times [0,1]$. The following conclusions hold.*

(i) *If $d > \sum_{\ell=1}^{2}(4/(\ell \wedge (4K_\ell)))$, then for all $f \in \mathscr{L}(K)$, we have $f \in \mathscr{P}$ and*

$$\dim\{(s,x) \in I : u(s,x) = f(s,x)\} = 0 \quad a.s. \tag{141}$$

(ii) *If* $d < \sum_{\ell=1}^{2}(4/(\ell \wedge (4K_\ell)))$, *then for all* $f \in \mathcal{Q}(K)$, *we have* $f \in \mathscr{C} \setminus \mathscr{P}$, *and*

$$\begin{aligned}\operatorname{Dim}&\{(s,x) \in I : u(s,x) = f(s,x)\}\\ &= \dim\{(s,x) \in I : u(s,x) = f(s,x)\}\\ &= \begin{cases} 2 - \left(\frac{1}{4} \wedge K_1\right)d, & \text{if } \le d < \frac{4}{1 \wedge (4K_1)},\\ 1 + \frac{2 \wedge (4K_2)}{1 \wedge (4K_1)} \\ \quad - \left(\frac{1}{2} \wedge K_2\right)d, & \text{if } \frac{4}{1 \wedge (4K_1)} \le d < \sum_{\ell=1}^{2}\frac{4}{\ell \wedge (4K_\ell)},\end{cases}\end{aligned} \tag{142}$$

on an event of positive probability.

(iii) *If* $E \subseteq I$ *is a Borel set and* $f \in \mathcal{Q}(K)$, *then there exist positive constants* $c_{4.6}$ *and* $c_{4.7}$, *such that*

$$\begin{aligned}c_{4.6}\mathscr{C}_d^{\rho}(E) &\le \mathbb{P}\{\exists\ (s,x) \in E \text{ such that } X(s,x) = f(s,x)\}\\ &\le c_{4.7}\mathscr{H}_d^{\rho}(E),\end{aligned} \tag{143}$$

where ρ *is the metric on* $[0,T] \times [0,1]$ *defined by*

$$\rho((s,x),(t,y)) = |s-t|^{(1/4)\wedge K_1} + |x-y|^{(1/2)\wedge K_2}. \tag{144}$$

Proof. As shown by Proposition 4.1 in [8], it is sufficient to prove the results for the case of $b \equiv 0$ and $\sigma = I_d$ in (131). Note that $K = (K_1, K_2) \in (0,1)^2$ and $H = (1/4, 1/2)$. Therefore, the conclusions follow from Theorems 9, 10, 11, 16, 17 and Lemma 18. □

As we showed, in Theorem 19, we can also apply these theorems in this paper to recover the same results such as the Brownian sheet [15], fractional Brownian motion [14], fractional Brownian sheets [4], linear hyperbolic SPDEs considered by Dalang and Nualart [10], linear SPDEs considered by Mueller and Tribe [6], and operator-scaling stable Gaussian random fields with stationary increments constructed in [22].

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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