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Research Article

Parameter Estimation for Stochastic Differential Equations Driven by Mixed Fractional Brownian Motion

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We study the asymptotic properties of minimum distance estimator of drift parameter for a class of nonlinear scalar stochastic differential equations driven by mixed fractional Brownian motion. The consistency and limit distribution of this estimator are established as the diffusion coefficient tends to zero under some regularity conditions.

1. Introduction

Stochastic differential equations (SDEs) are a natural choice to model the time evolution of dynamic systems which are subject to random influences. For existence and uniqueness of solutions of finite dimensional stochastic differential equations and properties of stochastic integrals, we refer to [1–3] and the references therein.

It is natural that a model contains unknown parameters. The parametric estimation problems for diffusion processes satisfying SDEs driven by Brownian motion (hereafter Bm) have been studied earlier. For a more recent comprehensive discussion, we refer to [4, 5] and the references therein. In case of statistical inference for diffusion processes satisfying SDEs driven by a fractional Brownian motion (hereafter fBm), substantial progress has been made in this direction; we refer to [6–9] for more details.

The mixed fractional Brownian motion (hereafter mfBm) was introduced by Patrick [10] to present a stochastic model of the discounted stock price in some arbitrage-free and complete financial markets. As a result, in order to take into account long memory and exclude arbitrage, it is natural to use mfBm to replace the standard Brownian motion. Consequently, there has been a growing interest in parameter estimation for stochastic processes driven by mfBm.

There are several heuristic methods available for use in case of SDEs driven by mfBm, such as MLE, LSE, and sequential estimation. In the continuous case, since the MLE has desirable asymptotic properties of consistency, normality, and efficiency under broad conditions, perhaps the most direct method is the MLE. However, MLE has some shortcomings; MLE's calculation is often cumbersome as the expressions for MLE involve stochastic integrals which need good approximations for computational purposes. Moreover, the generally good asymptotic properties are not always satisfied in the discrete case. Paper [11] showed that the approach of estimating parameters of an Itô process by applying MLE to a discretization of the SDE does not yield consistent estimators. LSE is asymptotically equivalent to the MLE. It is well known that the sequential estimation methods might lead to equally efficient estimators from the process observed possibly over a shorter expected period of observation time. Although there exists a wide range of estimation techniques developed for the problem of parameters estimation for SDEs driven by mfBm, we should choose a suitable estimation method.

Though mfBm has stationary self-similar increments, it does not have stationary increments and is not a Markov process. So, state-space models and Kalman filter estimators cannot be applied to the parameters of this process. Under the circumstances, in order to overcome those difficulties, the minimum distance approach is proposed.

The interest for this method of parametric estimation is that the minimum distance estimation (hereafter MDE) method has several features. It makes MDE an attractive method. From one part it is sometimes easy to calculate.

On the other side this estimator is known to be consistent (see [12]) under some general conditions. Millar [13] studied a general framework for (MDE) of Hilbertian type and showed that the MDE is efficient and asymptotically normal in some situations. Furthermore, the MDE is a class of estimators that is automatically robust in the same sense (for more details see [14]), which is generically optimal according to some quantitative measure of robustness.

For the SDEs driven by Brownian motions, Kutoyants [15] and Kutoyants and Pilibossian [16] proved that $\varepsilon^{-1}(\theta_\varepsilon^* - \theta_0)$ converge in probability to the random variable ζ_T with L_1, L_2 or supremum norm and he also proved that ζ_T is asymptotically normal when $T \to +\infty$ and $\theta_0 > 0$. Hénaff [17] established the same results in the general case of a norm in some Banach space of functions on $[0,T]$. For the SDEs driven by fBm, Prakasa Rao [18] studied the minimum L_1-norm estimator θ_ε^* of the drift parameter of a fractional Ornstein-Uhlenbeck type process and proved that $\varepsilon^{-1}(\theta_\varepsilon^* - \theta)$ converges in probability under $\mathbb{P}_{\theta_0}$ to a random variable ζ. Kouame et al. [19] studied asymptotic properties of minimum distance estimator of the parameter of stochastic process driven by a fBm as the diffusion coefficient tends to zero.

However, it appears that there are few works studying the estimators of mfBm. Zili [20] obtained some general stochastic properties of the mfBm and treated the Hölder continuity of the sample paths and α-differentiability of the trajectories of mfBm. Miao [21] obtained the asymptotic properties of the minimum L_1-norm estimator of the drift parameter for a linear SDE driven by an mfBm. Xiao et al. [22] studied the problem of estimating the parameters for the mfBm from discrete observations based on the MLE.

In present paper, our aim is to obtain the MDE of the drift parameter for a class of nonlinear scalar SDEs driven by mfBm and study the asymptotic properties of this estimator.

The remainder of this paper proceeds as follows. Section 2 starts with a short description of definition of mfBm and then provides some basic lemmas that will be used in the forthcoming sections. And we obtain the MDE of the drift parameter for a class of nonlinear scalar SDEs driven by mfBm. In Section 3, we study the consistency of the above estimator. In Section 4, the limit distribution of this estimator is established as the diffusion coefficient tends to zero under some regularity conditions.

2. Notation and Preliminaries

Let $B_t^H := \{B^H(t), 0 \le t \le T\}$ be a fractional Brownian motion defined on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, \mathbb{P})$ and $\{\mathcal{F}_t\}_{t\ge 0}$ is a filtration of σ-algebra of $\mathcal{F}$, where the usual conditions are satisfied; that is, $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space, $\mathcal{F}_0$ contains all P-null sets of $\mathcal{F}$, and, for each $t \ge 0$, $\mathcal{F}_+ := \bigcap_{s>t} \mathcal{F}_s = \mathcal{F}_t$.

A fractional Brownian motion B_t^H of Hurst parameter $H \in (0,1)$ is a continuous and centered Gaussian process; that is, $E(B_t^H) = 0$ for all $t \ge 0$, with covariance function

$$E\left(B_s^H B_t^H\right) = \frac{1}{2}\left[s^{2H} + t^{2H} - |s-t|^{2H}\right], \quad t \ge 0,\ s \ge 0. \tag{1}$$

From (1), it is easy to obtain that $E[B_t^H]^2 = t^{2H}$, $t \ge 0$, for all $H \in (0,1)$. Moreover, the fBm B_t^H reduces to a standard Brownian motion denoted by $B_t := \{B(t), 0 \le t \le T\}$ for $H = 1/2$.

The notation $\{X_t\} \overset{d}{=} \{Y_t\}$ means that the $\{X_t\}_{t\in\mathbb{R}_+}$ and $\{Y_t\}_{t\in\mathbb{R}_+}$ have the same law. Denote by X_t^* the supremum process

$$X_t^* = \sup_{s\le t} |X_s|. \tag{2}$$

A standard fBm B_t^H has the following properties (for more details see [23], Page 5, Definition 1.1.1).

(1) B_t^H has homogeneous increments; that is, $B_{s+t}^H - B_t^H \overset{d}{=} B_t^H$ for $s, t \ge 0$.

(2) B_t^H has continuous trajectories.

Let us take a and b which are two real constants such that $(a,b) \ne (0,0)$. By Patrick [10], we introduce the following.

Definition 1. A mixed fractional Brownian motion of parameters a, b, and H is a process $M^H := \{M_t^H(a,b); t \ge 0\} = \{M_t^H; t \ge 0\}$, defined on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, \mathbb{P})$ by

$$M_t^H = M_t^H(a,b) := aB_t + bB_t^H, \quad \forall t \in \mathbb{R}_+. \tag{3}$$

Remark 2. From Zili [20], we know that mfBm is a mixed-self-similar process:

$$M_{ht}^H(a,b) \overset{d}{=} \left\{M_t^H\left(ah^{1/2}, bh^H\right)\right\}, \tag{4}$$

where $h > 0$ is a constant. Furthermore it follows for the supremum process M^* that

$$M_{ht}^{H*}(a,b) \overset{d}{=} \left\{M_t^{H*}\left(ah^{1/2}, bh^H\right)\right\}. \tag{5}$$

By using the self-similarity of mfBm, we obtain the following lemma.

Lemma 3. *Let $T > 0$ be a constant and M_t^H an mfBm with parameter a, b, H; then for every $p > 0$,*

$$\begin{aligned}\mathbf{E}\left(M_T^{H*}(a,b)\right)^p &= \mathbf{E}\left(M_1^{H*}\left(aT^{1/2}, bT^H\right)\right)^p \\ &= \mathbf{E}\left(\sup_{t\le 1}\left|aT^{1/2}B_t + bT^H B_t^H\right|\right)^p.\end{aligned} \tag{6}$$

The value of (6) is not known. However it is fortunate that we have the following two lemmas which give the bounds for the standard Bm and fBm, respectively.

Lemma 4 (Burkholder-Davis-Gundy inequalities). *For any stopping time τ with respect to the filtration generated by the Bm B_t, one has*

$$c(p)\,\mathbf{E}\left(\tau^{p/2}\right) \le \mathbf{E}\left(\left(B_\tau^*\right)^p\right) \le C(p)\,\mathbf{E}\left(\tau^{p/2}\right), \quad p > 0, \tag{7}$$

where the constants $c(p)$ and $C(p) > 0$ depend only upon the parameter p.

B-D-G have a long history and we cite only some works in this area. Maybe the first general results were due to Novikov ($p > 1/2$) and Burkholder (see [24, 25]).

Lemma 5 (see [26]). *Let τ be a stopping time with respect to the filtration generated by the fBm B_t^H. Then, for $H \in (1/2, 1)$, one has*

$$c(p,H)\,\mathbf{E}\left(\tau^{pH}\right) \le \mathbf{E}\left(\left(B_\tau^{H*}\right)^p\right) \le C(p,H)\,\mathbf{E}\left(\tau^{pH}\right), \quad \forall p \ge 0, \tag{8}$$

and, for $H \in (0, 1/2)$, one has

$$c(p,H)\,\mathbf{E}\left(\tau^{pH}\right) \le \mathbf{E}\left(\left(B_\tau^{H*}\right)^p\right), \quad \forall p > 0, \tag{9}$$

where the constants $c(p,H)$ and $C(p,H) > 0$ depend only upon the parameters p, H.

Lemma 6 (see [16, page 120]). *Let $Z_\varepsilon(u)$, $\varepsilon > 0$, $u \in \mathbb{R}$, be a sequence of continuous functions and $Z_0(u)$ a convex function which admits a unique minimum $\xi \in \mathbb{R}$. Let L_ε, $\varepsilon > 0$, be a sequence of positive numbers such that $L_\varepsilon \to +\infty$ as $\varepsilon \to 0$. We suppose that*

$$\lim_{\varepsilon \to 0} \sup_{|u|<L_\varepsilon} \left|Z_\varepsilon(u) - Z_0(u)\right| = 0. \tag{10}$$

Then

$$\lim_{\varepsilon \to 0} \arg\min_{|u|<L_\varepsilon} Z_\varepsilon(u) = \xi, \tag{11}$$

where if there are several minima of Z_ε, we choose the arbitrary one.

Now we consider the parameter estimation problem for a class of nonlinear scalar mixed SDE in the following framework:

$$dX_t = S_t(\theta, X)\,dt + \varepsilon dM_t^H, \quad X_0 = x_0, \ 0 \le t \le T, \tag{12}$$

where $S_t(\cdot, X)$ is a known measurable functional, the unknown drift parameter $\theta \in \Theta \subset \mathbb{R}$, and $\varepsilon > 0$.

Denote θ_0 by the true parameter of θ. Let $\mathbb{P}_{\theta_0}^{(\varepsilon)}$ be the probability measure induced by the process $\{X_t\}$ and $x_t(\theta)$ the solution of the differential equation (12) with $\varepsilon = 0$.

Assume that the trend functional of the above equation has the following form:

$$S_t(\theta, X) = V(\theta, t, X_t) + \int_0^t K(\theta, t, s, X_s)\,ds, \tag{13}$$

where $V(\theta, t, X_t)$ and $K(\theta, t, s, X_s)$ are two measurable functions.

The function $S_t(\theta, x)$ is measurable with respect to (t, θ) and, for any $\delta > 0$, denote

$$g(\delta) := \inf_{|\theta-\theta_0|>\delta} \left\|x_t(\theta) - x_t(\theta_0)\right\|_T. \tag{14}$$

Note that $g(\delta) > 0$ for any $\delta > 0$.

The $L_{p\ge1}$ or supremum norm can be denoted by $\|\ \|_T$ and the MDE θ_ε^* (see [15]) is defined by

$$\theta_\varepsilon^* := \arg\min_{\theta\in\Theta} \|X - x(\theta)\|_T. \tag{15}$$

We also need the following additional conditions.

($\mathscr{A}_1$) The functional $S_t(\theta, \cdot)$ is measurable and nonanticipative and satisfies the following inequalities: for all $t \in [0, T]$ and $X, Y \in \mathscr{C}[0, T]$,

$$\begin{gathered}\left|S_t(\theta, X) - S_t(\theta, Y)\right| \le L_1 \int_0^t \left|X_s - Y_s\right| ds + L_2 \left|X_t - Y_t\right|, \\ \left|S_t(\theta, X)\right| \le L_1 \int_0^t \left(1 + \left|X_s\right|\right) ds + L_2 \left(1 + \left|X_t\right|\right),\end{gathered} \tag{16}$$

where L_1, L_2 are positive constants.

($\mathscr{A}_2$) The measurable functions $V(\theta, t, X_t)$ and $K(\theta, t, s, X_s)$ are continuously twice differentiable in θ and x.

($\mathscr{A}_3$) Suppose that

$$\inf_{\theta\in\Theta} \|\dot{x}(\theta)\|_T > 0 \tag{17}$$

and define by the random variable $\xi_T = \xi_T(\theta_0)$,

$$\xi_T := \arg\min_{u\in\mathbb{R}} \left\|x^{(1)} - u\dot{x}(\theta_0)\right\|_T. \tag{18}$$

Denote $\dot{x}_t(\theta)$ by the derivatives of $x_t(\theta)$ with respect to θ and introduce a Gaussian process $x_t^{(1)} = x_t^{(1)}(\theta)$ which satisfies the equation

$$\begin{aligned} dx_t^{(1)} &= \Big[V_x'(\theta, t, x_t)\, x_t^{(1)} \\ &\quad + \int_0^t K_x'(\theta, t, s, x_s)\, x_s^{(1)} ds\Big]\, dt + dM_t^H, \\ & x_0^{(1)} = 0, \quad 0 \le t \le T, \end{aligned} \tag{19}$$

where V_x' and K_x' are the derivatives of $V(\vartheta, t, x)$ and $K(\vartheta, t, s, x)$ with respect to x and $x_t^{(1)}$ is a derivative with probability one of X_t with respect to ε as $\varepsilon = 0$.

In the paper, we will use C to denote a generic constant which may vary from place to place.

3. Consistency

Theorem 7. *If the above condition ($\mathscr{A}_1$) is satisfied, then, for any $p > 0$, there exist constants $C_1(p,H)$, $C_2(p) > 0$ such that, for every $\delta > 0$,*

$$\begin{aligned} &\mathbf{P}_{\theta_0}^{(\varepsilon)} \left\{\left|\theta_\varepsilon^* - \theta_0\right| > \delta\right\} \\ &\quad \le 2^p C_T^p C^p \varepsilon^p (g(\delta))^{-p} \left(C_1(p,H)\, T^{Hp} + C_2(p)\, T^{p/2}\right) \\ &\quad = O\left((g(\delta))^{-p} \varepsilon^p\right). \end{aligned} \tag{20}$$

Proof. Condition $(\mathscr{A}_1)$ ensures the existence and uniqueness of a strong solution of (12). It is obvious that, with $\mathbb{P}_{\theta_0}^{(\varepsilon)}$ probability one,

$$\left\|X_t - x_t(\theta_0)\right\|_\infty \le C\varepsilon \sup_{0\le t\le T}\left|M_t^H\right|. \tag{21}$$

For any $\delta > 0$,

$$\begin{aligned}
&\mathbf{P}_{\theta_0}^{(\varepsilon)}\left\{\left|\theta_\varepsilon^* - \theta_0\right| > \delta\right\} \\
&\quad\le \mathbf{P}_{\theta_0}^{(\varepsilon)}\left\{\inf_{|\theta-\theta_0|\le\delta}\left(\|X - x(\theta)\|_T\right) \le \inf_{|\theta-\theta_0|>\delta}\left(\|X - x(\theta)\|_T\right)\right\} \\
&\quad\le \mathbf{P}_{\theta_0}^{(\varepsilon)}\left\{\left\|X - x(\theta_0)\right\|_T + \inf_{|\theta-\theta_0|\le\delta}\left(\left\|x(\theta) - x(\theta_0)\right\|_T\right)\right. \\
&\qquad\qquad \left. > \inf_{|\theta-\theta_0|>\delta}\left(\left\|x(\theta) - x(\theta_0)\right\|_T\right) - \left\|X - x(\theta_0)\right\|_T\right\} \\
&\quad\le \mathbf{P}_{\theta_0}^{(\varepsilon)}\left\{2\left\|X - x(\theta_0)\right\|_T > \inf_{|\theta-\theta_0|>\delta}\left(\left\|x(\theta) - x(\theta_0)\right\|_T\right)\right\}.
\end{aligned} \tag{22}$$

By (21), we have

$$\begin{aligned}
\mathbf{P}_{\theta_0}^{(\varepsilon)}\left\{\left|\theta_\varepsilon^* - \theta_0\right| > \delta\right\} &\le \mathbf{P}_{\theta_0}^{(\varepsilon)}\left\{\|X - x(\theta)\|_\infty > \frac{g(\delta)}{2C_T}\right\} \\
&\le \mathbf{P}_{\theta_0}^{(\varepsilon)}\left\{\sup_{0\le t\le T}\left|M_t^H\right| > \frac{g(\delta)}{2\varepsilon C C_T}\right\} \\
&= \mathbf{P}_{\theta_0}^{(\varepsilon)}\left\{M_T^{H*} > \frac{g(\delta)}{2\varepsilon C C_T}\right\} \\
&\le \frac{\mathbb{E}\left(M_T^{H*}\right)^p}{\left(g(\delta)/2\varepsilon C C_T\right)^p}.
\end{aligned} \tag{23}$$

Then using Lemmas 3, 4, and 5, we obtain

$$\begin{aligned}
&\mathbf{P}_{\theta_0}^{(\varepsilon)}\left\{\left|\theta_\varepsilon^* - \theta_0\right| > \delta\right\} \\
&\quad\le 2^p C_T^p C^p \varepsilon^p \left(g(\delta)\right)^{-p}\left(C_1(p,H)T^{Hp} + C_2(p)T^{p/2}\right) \\
&\quad= O\left(\left(g(\delta)\right)^{-p}\varepsilon^p\right).
\end{aligned} \tag{24}$$

□

Remark 8. As a consequence of the above theorem, we obtain that θ_ε^* converges in probability to θ_0 under $P_{\theta_0}^{(\varepsilon)}$-measure as $\varepsilon \to 0$. Furthermore, the rate of convergence is of order $O(\varepsilon^p)$ for every $p > 0$.

4. Asymptotic Distribution

Theorem 9. *Under conditions $(\mathscr{A}_2)$ and $(\mathscr{A}_3)$, we have that the random variable $\varepsilon^{-1}(\theta_\varepsilon^* - \theta)$ converges in probability to a random variable whose probability distribution is the same as that of ξ_T under $\mathbb{P}_{\theta_0}$.*

Proof. Let $\nu = \nu_\varepsilon = \varepsilon\lambda_\varepsilon \to 0$ and $\lambda_\varepsilon \to +\infty$ as $\varepsilon \to 0$.

Introduce the set

$$\begin{aligned}
H_0 = H_0(\nu) = \Big\{\omega : &\inf_{|\theta-\theta_0|<\nu}\|X - x(\theta)\|_T \\
&< \inf_{|\theta-\theta_0|\ge\nu}\|X - x(\theta)\|_T\Big\}.
\end{aligned} \tag{25}$$

For any $\omega \in H_0$, we have $|\theta_\varepsilon^* - \theta_0| < \nu_\varepsilon$.

In fact, according to (17) of condition $(\mathscr{A}_3)$ and Taylor formula, we obtain

$$\begin{aligned}
g(\nu_\varepsilon) &= \inf_{|u|>\nu_\varepsilon}\left\|x(\theta_0 + u) - x(\theta_0)\right\|_T \\
&= \inf_{|u|>\nu_\varepsilon}\left\|u\dot{x}(\widetilde{\theta}_0)\right\|_T \\
&= \left(\inf_{|u|>\nu_\varepsilon}|u|\right)\left\|\dot{x}(\widetilde{\theta}_0)\right\|_T \ge \kappa_0\nu_\varepsilon,
\end{aligned} \tag{26}$$

where $u \in \mathbb{R}$, $\kappa_0 > 0$, $\widetilde{\theta}_0 = \theta_0 + \alpha u$, and $0 < \alpha < 1$.

Then, by Theorem 7, we have

$$\mathbf{P}_{\theta_0}^{(\varepsilon)}\left\{\left|\theta_\varepsilon^* - \theta_0\right| > \nu_\varepsilon\right\} \longrightarrow 0, \quad \varepsilon \longrightarrow 0. \tag{27}$$

Therefore, we just need to consider the behavior of the norm $\|X - x(\theta)\|_T$, for $\theta \in \{\theta : |\theta_\varepsilon^* - \theta_0| < \nu_\varepsilon\}$.

We have

$$\begin{aligned}
\varepsilon^{-1}\|X - x(\theta)\|_T &= \left\|\frac{X - x(\theta_0)}{\varepsilon} - \frac{x(\theta) - x(\theta_0)}{\varepsilon}\right\|_T \\
&= \left\|x^{(1)} - u\dot{x}(\theta_0) + r + q\right\|_T,
\end{aligned} \tag{28}$$

where

$$\begin{aligned}
\theta &= \theta_0 + \varepsilon u, \qquad |u| < \lambda_\varepsilon, \\
q_t &= \frac{X_t - x_t(\theta_0)}{\varepsilon} - x_t^{(1)}, \\
r_t &= \frac{X_t(\theta_0 + \varepsilon u) - x_t(\theta_0)}{\varepsilon} - u\dot{x}(\theta_0).
\end{aligned} \tag{29}$$

Then using (12), (13), (19), and (21),

$$\begin{aligned}
&\left|q_t\right| \\
&\quad= \left|\int_0^t\left[\frac{S_v(\theta_0, X) - S_v(\theta_0, x)}{\varepsilon} - V_x'(\theta_0, v, x_v)x_v^{(1)}\right.\right. \\
&\qquad\qquad \left.\left. - \int_0^v K_x'(\theta_0, v, h, x_h)x_h^{(1)}dh\right]dv\right| \\
&\quad\le \int_0^t\left|\frac{V(\theta_0, v, X_v) - V(\theta_0, v, x_v)}{\varepsilon}\right. \\
&\qquad\qquad \left. - V_x'(\theta_0, v, x_v)x_v^{(1)}\right|dv
\end{aligned}$$

$$
\begin{aligned}
&+ \int_0^t \int_0^v \left| \frac{K(\theta_0, v, h, X_h) - K(\theta_0, v, h, x_h)}{\varepsilon} \right. \\
&\qquad \left. - K_x'(\theta_0, v, h, x_h)\, x_h^{(1)} \right| dh\, dv \\
&\leq \int_0^t \left| V_x'\left(\theta_0, v, \widetilde{X}_v\right) \frac{X_v - x_v}{\varepsilon} - V_x'(\theta_0, v, x_v)\, x_v^{(1)} \right| dv \\
&+ \int_0^t \int_0^v \left| K_x'\left(\theta_0, v, h, \widetilde{X}_h\right) \frac{X_h - x_h}{\varepsilon} \right. \\
&\qquad \left. - K_x'(\theta_0, v, h, x_h)\, x_h^{(1)} \right| dh\, dv \\
&\leq \int_0^t \left| V_{x'}\left(\theta_0, v, \widetilde{X}_v\right) \right| \left| \frac{X_v - x_v}{\varepsilon} - x_v^{(1)} \right| dv \\
&+ \int_0^t \left| V_x'\left(\theta_0, v, \widetilde{X}_v\right) - V_x'(\theta_0, v, x_v) \right| \left| x_v^{(1)} \right| dv \\
&+ \int_0^t \int_0^v \left| K_x'\left(\theta_0, v, h, \widetilde{X}_h\right) \right| \left| \frac{X_h - x_h}{\varepsilon} - x_h^{(1)} \right| dh\, dv \\
&+ \int_0^t \int_0^v \left| K_x'\left(\theta_0, s, v, \widetilde{X}_v\right) - K_x'(\theta_0, s, v, x_v) \right| \left| x_v^{(1)} \right| dh\, dv \\
&\leq C_1 \int_0^t |q_v|\, dv + C_2 \int_0^t \int_0^v |q_h|\, dh\, dv \\
&+ \varepsilon C_3 \sup_{0 \leq t \leq T} \left| M_T^H \right| \sup_{0 \leq t \leq T} \left| x_t^{(1)} \right|
\end{aligned}
\tag{30}
$$

with some constants $C_i > 0, i = 1, 2, 3$. From (19), condition $(\mathscr{A}_2)$, and Lemma 4.13 (see [27]), we obtain

$$
\sup_{0 \leq t \leq T} \left| x_t^{(1)} \right| \leq C \sup_{0 \leq t \leq T} \left| M_T^H \right|; \tag{31}
$$

then,

$$
|q_t| \leq C_1 \int_0^t |q_s|\, ds + C_2 \int_0^t \int_0^s |q_v|\, dv\, ds + \varepsilon C_4 \sup_{0 \leq t \leq T} \left| M_T^H \right|^2. \tag{32}
$$

By using Lemma 4.13 once more, we get

$$
\sup_{0 \leq t \leq T} |q_t| \leq C\varepsilon \sup_{0 \leq t \leq T} \left| M_T^H \right|^2. \tag{33}
$$

Now consider r_t; using the Taylor formula, we obtain

$$
\begin{aligned}
\sup_{|u| < \lambda_\varepsilon} \sup_{0 \leq t \leq T} |r_t| &= \sup_{|u| < \lambda_\varepsilon} \sup_{0 \leq t \leq T} \left| u\left(\dot{x}_t\left(\widehat{\theta}_0\right) - \dot{x}_t(\theta)\right) \right| \\
&\leq \sup_{|u| < \lambda_\varepsilon} |u| \sup_{0 \leq t \leq T} \left| \dot{x}_t\left(\widehat{\theta}_0\right) - \dot{x}_t(\theta) \right| \leq C\varepsilon \lambda_\varepsilon^2,
\end{aligned}
\tag{34}
$$

where $\widehat{\theta}_0 = \theta_0 + \beta\varepsilon u$, $\beta \in (0, 1)$.

Introduce the functions

$$
Z_\varepsilon(u) = \left\| X - x(\theta_0 + \varepsilon u) \right\|_T, \quad Z_0(u) = \left\| x^{(1)} - u\dot{x}(\theta_0) \right\|_T. \tag{35}
$$

By (33) and (34),

$$
\begin{aligned}
&\sup_{|u| < \lambda_\varepsilon} \left| Z_\varepsilon(u) - Z_0(u) \right| \\
&\quad = \sup_{|u| < \lambda_\varepsilon} \left| \left\| \frac{1}{\varepsilon} (X - x(\theta_0 + \varepsilon u)) \right\|_T - \left\| x^{(1)} - u\dot{x}(\theta_0) \right\|_T \right| \\
&\quad \leq \sup_{|u| < \lambda_\varepsilon} \left\| \frac{1}{\varepsilon} (X - x(\theta_0 + \varepsilon u)) - \left(x^{(1)} - u\dot{x}(\theta_0) \right) \right\|_T \\
&\quad \leq C_T \sup_{|u| < \lambda_\varepsilon} \left\| \frac{1}{\varepsilon} (X - x(\theta_0 + \varepsilon u)) - \left(x^{(1)} - u\dot{x}(\theta_0) \right) \right\|_\infty \\
&\quad \leq C_T \left(\sup_{0 \leq t \leq T} |q_t| + \sup_{|u| < \lambda_\varepsilon} \sup_{0 \leq t \leq T} |r_t| \right) \\
&\quad \leq C\varepsilon \sup_{0 \leq t \leq T} \left| M_T^H \right|^2 + C\varepsilon \lambda_\varepsilon^2.
\end{aligned}
\tag{36}
$$

Therefore, if we choose λ_ε such that $\varepsilon\lambda_\varepsilon^2 \to 0$ when $\varepsilon \to 0$, then, with probability one, we have

$$
\sup_{|u| < \lambda_\varepsilon} \left| Z_\varepsilon(u) - Z_0(u) \right| \longrightarrow 0. \tag{37}
$$

Using Lemma 6, we obtain the result. The proof completes. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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