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Research Article

Some Weighted Norm Estimates for the Composition of the Homotopy and Green's Operator

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We establish the $A_r(D)$-weighted integral inequality for the composition of the Homotopy T and Green's operator G on a bounded convex domain and also motivated it to the global domain by the Whitney cover. At the same time, we also obtain some (p, q)-type norm inequalities. Finally, as applications of above results, we obtain the upper bound for the L^p norms of $T(G(u))$ or $(T(G(u)))_B$ in terms of L^q norms of u or du.

1. Introduction

Our purpose is to study the L^p theory of the composition of the Homotopy T and Green's operator G acting on differential forms on a bounded convex domain. Both operators play an important role in many fields, including harmonic analysis, potential theory, and partial equations (see [1–6]). In the present paper, we will obtain some (p, q)-type norm inequalities for the composition of the Homotopy T and Green's operator G and also prove the $A_r(D)$-weighted integral inequality on a bounded convex domain. These results will provide effective tools for studying behavior of solutions of A-harmonic equations and related differential systems on manifolds.

We start this paper by introducing some notations and definitions. Let M be a Riemannian, compact, oriented, and C^∞-smooth manifold without boundary on R^n and let Ω be an open subset of R^n. Also, we use G to denote Green's operator throughout this paper. Furthermore, we use B to denote a ball and ρB to denote the ball with the same center as B and with diameter $(\rho B) = \rho$ diameter (B). We do not distinguish balls from cubs in this paper.

We assume that $\wedge^k = \wedge^k(R^n)$ $(k = 0, 1, 2, \ldots, n)$ is the linear space of all k-forms $\omega(x) = \sum_I (x) dx_I = \sum \omega_{i_1, i_2, \ldots, i_k} dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$ with summation over all ordered k-tuples $I = (i_1, i_2, \ldots, i_k)$, $1 \le i_1 \le i_2 \le \cdots \le i_k \le n$. If the coefficient $\omega_I(x)$ of k-form $\omega(x)$ is differential on M, then we call $\omega(x)$ a differential k-form on M. A differential k-form $\omega(x)$ on M is a de Rham current (see [7]) on M with values in $\wedge^k(R^n)$. Let $\wedge^k M$ be the kth exterior power of the cotangent bundle and $C^\infty(\wedge^k M)$ be the space of smooth k-forms on M. As usual, we use $D'(M, \wedge^k)$ to denote the space of all differential k-forms and $L^p(\wedge^k M)$ to denote the k-form $\omega(x)$ with the norm

$$\|\omega(x)\|_{p,M} = \left(\int_M |\omega(x)|^p dx\right)^{1/p} = \left(\int_M \left(\sum_I |\omega_I(x)|^2\right)^{p/2} dx\right)^{1/p} \quad (1)$$

on M. Thus $L^p(\wedge^k M)$ is a Banach space. As usual, we still use $\star$ to denote the Hodge star operator. Also, we use $d : D'(M, \wedge^k) \to D'(M, \wedge^{k+1})$ to denote the differential operator and use $d^\star : D'(M, \wedge^{k+1}) \to D'(M, \wedge^k)$ to denote the Hodge codifferential operator which is defined by $d^\star = (-1)^{nk+1} \star d \star$ on $D'(M, \wedge^{k+1})$. The n-dimensional Lebesgue measure of a set $E \subseteq R^n$ is denoted by $|E|$. We call w a weight if $w \in L^1_{\text{loc}}(R^n)$ and $w > 0$, a.e. For $0 < p < 1$, we denote the weighted L^p-norm of a measurable function f over M by

$$\|f\|_{p,M,w^\alpha} = \left(\int_M |f|^p w^\alpha dx\right)^{1/p}, \quad (2)$$

where α is a real number.

Let $D \subset R^n$ be a bounded, convex domain. Iwaniec and Lutoborski in [1] first introduced a linear operator $K_y : C^\infty(D, \wedge^k) \to C^\infty(D, \wedge^{k-1})$ satisfying that

$$\left(K_y\omega\right)(x; \xi_1, \xi_2, \ldots, \xi_{k-1}) = \int_0^1 t^{k-1}\omega\left(tx + y - ty; x - y, \xi_1, \xi_2, \ldots, \xi_{k-1}\right) dt \tag{3}$$

and the decomposition $\omega = d(K_y\omega) + K_y(d\omega)$. Then by averaging K_y over all points y in D, they constructed a Homotopy operator $T : C^\infty(D, \wedge^k) \to C^\infty(D, \wedge^{k-1})$ satisfying that $T\omega = \int_D \varphi(y) K_y(\omega) dy$, where $\varphi \in C_0^\infty(D)$ is normalized by $\int_D \varphi(y) dy = 1$. The k-form $\omega_D \in D'(D, \wedge^k)$ is defined by $\omega_D = (1/|D|)\int_D \omega(y) dy$, if $k = 0$, and if $k = 1, 2, \ldots, n$, then

$$\omega_D = d(T\omega) = \omega - T(d\omega), \tag{4}$$

$$|T\omega(x)| \le C\int_D \frac{|\omega(y)|}{|y - x|^{n-1}} dy. \tag{5}$$

2. Boundedness of the Composition of the Homotopy and Green's Operator in L^p Space

In this section, we will prove the $A_r(D)$-weighted norm inequality for the composition of the Homotopy T and Green's operator G on a bounded convex domain. Then using the Whitney cover, we develop the local result to the global domain. In [8], Gol'dshtein and Troyanov proved the following lemma.

Lemma 1. *Let $D \subset R^n$ be a bounded convex domain. The operator T maps $L^p(D, \wedge^k)$ continuously to $L^q(D, \wedge^{k-1})$ in the following cases:*

$$\begin{aligned} &\text{Either } 1 \le p,\ q \le \infty, \quad \frac{1}{p} - \frac{1}{q} < \frac{1}{n}, \\ &\text{Or } 1 < p,\ q \le \infty, \quad \frac{1}{p} - \frac{1}{q} \le \frac{1}{n}. \end{aligned} \tag{6}$$

From [3], we have the following lemma about L^s-estimates for Green's operator.

Lemma 2. *Let $u \in C^\infty(\wedge^k M)$ $(k = 0, 1, 2, \ldots, n)$ and $1 < s < \infty$. Then there exists a constant C, independent of u, such that*

$$\begin{aligned} &\left\|dd^\star G(u)\right\|_{s,M} + \left\|d^\star dG(u)\right\|_{s,M} + \|dG(u)\|_{s,M} \\ &\quad + \left\|d^\star G(u)\right\|_{s,M} + \|G(u)\|_{s,M} \le C\|u\|_{s,M}. \end{aligned} \tag{7}$$

Definition 3. We say that a weight $w(x)$ satisfies the $A_r(D)$ condition for $r > 1$ and write $w(x) \in A_r(D)$, if $w > 0$ a.e. and

$$\sup_{B \subset D}\left(\frac{1}{|B|}\int_B w dx\right)\left(\frac{1}{|B|}\int_B \left(\frac{1}{w}\right)^{1/(r-1)} dx\right)^{r-1} < \infty. \tag{8}$$

For $A_r(D)$ weight, we also need the following result which appears in [9].

Lemma 4. *If $w(x) \in A_r(D)$, then there exist constants $\beta > 1$ and C, independent of w, such that*

$$\|w\|_{\beta,B} \le C|B|^{(1-\beta)/\beta}\|w\|_{1,B} \tag{9}$$

for all balls $B \subset D$.

Theorem 5. *Let $D \subset R^n$ be a bounded convex domain, $n < p < \infty$, and let $T : L^p(D, \wedge^k) \to L^p(D, \wedge^{k-1})$ be the Homotopy operator, $k = 1, 2, \ldots, n$. Then there exists a constant C, independent of u, such that*

$$\|T(G(u))\|_{p,B,w} \le C\|u\|_{p,B,w} \tag{10}$$

for any ball $B \subset D$, $w(x) \in A_r(D)$, and $1 < r < p/n$.

Proof. Since $w(x) \in A_r(D)$, by Lemma 4, there exist constants $\beta > 1$ and C_1, independent of w, such that

$$\|w\|_{\beta,B} \le C_1|B|^{(1-\beta)/\beta}\|w\|_{1,B} \tag{11}$$

for any ball $B \subset D$.

Choosing $k = \beta p/(\beta - 1)$, then by Hölder inequality with $1/k + 1/\beta p = 1/p$, we have

$$\begin{aligned} \|T(G(u))\|_{p,B,w} &= \left(\int_B |T(G(u))|^p w(x) dx\right)^{1/p} \\ &\le \left(\int_B |T(G(u))|^k dx\right)^{1/k}\left(\int_B w^\beta dx\right)^{1/\beta p} \\ &= \|T(G(u))\|_{k,B}\|w(x)\|_{\beta,B}^{1/p}. \end{aligned} \tag{12}$$

Thus, substituting (11) into (12), we obtain

$$\|T(G(u))\|_{p,B,w} \le C_1|B|^{(1-\beta)/\beta p}\|T(G(u))\|_{k,B}\|w(x)\|_{1,B}^{1/p}. \tag{13}$$

Taking $m = p/r$, it is easy to see that $m > 1$ and $(1/m) - (1/k) < (1/m) < (1/n)$. Hence communicating Lemmas 1 and 2, we have

$$\|T(G(u))\|_{k,B} \le C_2\|G(u)\|_{m,B} \le C_3\|u\|_{m,B}. \tag{14}$$

Combining (13) and (14), we have

$$\|T(G(u))\|_{p,B,w} \le C_4|B|^{(1-\beta)/\beta p}\|u\|_{m,B}\|w(x)\|_{1,B}^{1/p}. \tag{15}$$

Using Hölder inequality with $1/p + (r-1)/p = r/p$, we have

$$\begin{aligned} \|u\|_{m,B} &\le \left(\int_B \left(|u|\, w^{1/p}\right)^p dx\right)^{1/p}\left(\int_B \left(\frac{1}{w}\right)^{1/(r-1)} dx\right)^{(r-1)/p} \\ &= \|u\|_{p,B,w}\left(\int_B \left(\frac{1}{w}\right)^{1/(r-1)} dx\right)^{(r-1)/p}. \end{aligned} \tag{16}$$

Note $w(x) \in A_r(D)$; then,

$$\sup_{B \subset D}\left(\frac{1}{|B|}\int_B w dx\right)\left(\frac{1}{|B|}\int_B \left(\frac{1}{w}\right)^{1/(r-1)} dx\right)^{r-1} < C_5 < \infty. \tag{17}$$

Thus, observing (15) and (16), we immediately obtain that

$$\|T(G(u))\|_{p,B,w} \le C_6|B|^{(1-\beta)/\beta p+(r/p)}\|u\|_{p,B,w} \le C_6|D|^{(1-\beta)/\beta p+(r/p)}\|u\|_{p,B,w} \le C_7\|u\|_{p,B,w}. \tag{18}$$

Here C_7 is a constant independent of u. Thus we complete the proof of Theorem 5. □

Furthermore, if u is an A-harmonic tensor on D, $\rho > 1$ and $0 < s, t < \infty$, then there exists a constant C, independent of u, such that

$$\|u\|_{s,B} \le C|B|^{(t-s)/ts}\|u\|_{t,\rho B} \tag{19}$$

for all balls or cubs B with $\rho B \subset D$ (for more details about A-harmonic tensors, see [10]). By the property of A-harmonic tensor, using the same method developed in the proof of Theorem 5, we can easily extend into the following $A_r(D)$-weighted version.

Corollary 6. *Let $D \subset R^n$ be a bounded convex domain, $n < p < \infty$, u be an A-harmonic tensor, and $T : L^p(D, \wedge^k) \to L^p(D, \wedge^{k-1})$ be the Homotopy operator, $k = 1, 2, \ldots, n$. Then there exists a constant C, independent of u, such that*

$$\|T(G(u))\|_{p,B,w^\alpha} \le C\|u\|_{p,\rho B,w^\alpha} \tag{20}$$

for any ball $B \subset D$, $w(x) \in A_r(D)$, and $1 < r < p/n$, $0 < \alpha \le 1$, $\rho > 1$.

In order to obtain the boundedness of the composition $T \circ G$, we need the following modified Whitney cover in [10] and see [11] for more details about Whitney cover.

Lemma 7. *Each open subset $E \subset R^n$ has a modified Whitney cover of cubs $W = \{Q_i\}$ satisfying $\bigcup_i Q_i = E$ and $\sum_{Q_i\in W}\chi_{\sqrt{5/4}Q_i} \le N \cdot \chi_E(x)$, for all $x \in R^n$ and some $N > 1$, where $\chi_E(x)$ is the characteristic function for the set E.*

Theorem 8. *Let $D \subset R^n$ be a bounded convex domain, $n < p < \infty$. Then the composite operator $T \circ G : L^p(D, \wedge^k, w) \to L^p(D, \wedge^{k-1}, w)$ is bounded, $k = 1, 2, \ldots, n$. Here $w(x) \in A_r(D)$ and $1 < r < p/n$.*

Proof. From Lemma 7, we know that there exists a sequence of cubs $W = \{Q_i\}$ such that $\bigcup_i Q_i = D$ and $\sum_{Q_i\in W}\chi_{\sqrt{5/4}Q_i} \le N\cdot\chi_E(x)$ for all $x \in D$, where $N > 1$ is some constant. Hence, for $u \in L^p(D, \wedge^k, w)$, we have

$$\begin{aligned}\|T(G(u))\|^p_{p,D,w} &= \int_D |T(G(u))|^p d\mu \le \sum_{Q_i\in W}\int_{Q_i}|T(G(u))|^p d\mu \\ &\le \sum_{Q_i\in W} C_1\int_{Q_i}|u|^p d\mu \le \sum_{Q_i\in W} C_1\int_D |u|^p\chi_{Q_i}(x)\,d\mu \\ &\le C_1\int_D\sum_{Q_i\in W}|u|^p\chi_{Q_i}(x)\,d\mu \le C_1\int_D N\cdot|u|^p\chi_D(x)\,d\mu \\ &\le C_1 N\int_D |u|^p d\mu = C_2\int_D|u|^p d\mu = C_2\|u\|^p_{p,D,w},\end{aligned} \tag{21}$$

where $d\mu = w(x)dx$ and $C_2 = C_1N$ is independent of u and each Q_i. Thus, we complete the proof of Theorem 8. □

3. Norm Estimates with Power-Type Weights

Let $S \subset R^n$ be a bounded domain and D be a nonempty of $\overline{S} = S\bigcup\partial S$. If we use $\operatorname{dist}(x, D)$ to denote the distance of the point x from the set D, then $\omega(x) = (\operatorname{dist}(x, D))^\varepsilon$ for $\varepsilon \in R$ is called power-type weight. In this section, we will establish some strong (p, q)-type norm inequalities with power-type weights for the composition of the Homotopy T and Green's operator G acting on differential form. In the following proof, we will use the following Lemma which appears in [8].

Lemma 9. *The operator $T : \Omega_{p,r}(D, \wedge^k) \to \Omega_{q,p}(D, \wedge^{k-1})$ is bounded provided that*

$$\begin{aligned}&\text{Either } 1 \le p, q, r \le \infty, \quad \frac{1}{p} - \frac{1}{q} < \frac{1}{n}, \quad \frac{1}{r} - \frac{1}{p} < \frac{1}{n}, \\ &\text{Or } 1 < p, q, r \le \infty, \quad \frac{1}{p} - \frac{1}{q} \le \frac{1}{n}, \quad \frac{1}{r} - \frac{1}{p} \le \frac{1}{n}.\end{aligned} \tag{22}$$

Theorem 10. *Let $D \subset R^n$ be a bounded convex domain, $1 < p$, $q < \infty$, $0 \le 1/p - 1/q \le 1/n$, and let $T : L^p(D, \wedge^k) \to L^q(D, \wedge^{k-1})$ be the Homotopy operator, $k = 1, 2, \ldots, n$. Then there exists a constant C, independent of u, such that*

$$\left\|T(G(u)) - (T(G(u)))_D\right\|_{q,D} \le C(1 + \operatorname{diam}(D))\|u\|_{p,D} \tag{23}$$

for any $u \in \Omega_{p,p}(D, \wedge^k)$.

Proof. From (4), we have the following decomposition:

$$G(u) = T(d(G(u))) + d(T(G(u))) \tag{24}$$

for any differential form $u \in \Omega_{p,p}(D, \wedge^k)$, $k = 1, 2, \ldots, n$.

Note that u is an element of $\Omega_{p,p}(D, \wedge^k)$, $k = 1, 2, \ldots, n$. From (4) and Lemmas 1 and 9, we have

$$\begin{aligned}&\left\|T(G(u)) - (T(G(u)))_D\right\|_{q,D} \\ &\quad = \|T(d(T(G(u))))\|_{q,D} \\ &\quad \le C_1\|d(T(G(u)))\|_{p,D}.\end{aligned} \tag{25}$$

Here C_1 is a constant independent of u. Applying (24) and (5), we have

$$\begin{aligned}&\|d(T(G(u)))\|_{p,D} \\ &\quad = \|G(u) - T(d(G(u)))\|_{p,D} \\ &\quad \le \|G(u)\|_{p,D} + \|T(d(G(u)))\|_{p,D} \\ &\quad \le \|G(u)\|_{p,D} + C_2\operatorname{diam}(D)\|d(G(u))\|_{p,D}.\end{aligned} \tag{26}$$

Applying Lemma 2 into (26), we obtain

$$\|d(T(G(u)))\|_{p,D} \le (C_3 + C_4 \operatorname{diam}(D)) \|u\|_{p,D}. \tag{27}$$

Thus

$$\begin{aligned} \|T(G(u)) - (T(G(u)))_D\|_{q,D} &\le (C_5 + C_6 \operatorname{diam}(D)) \|u\|_{p,D} \\ &\le C_7 (1 + \operatorname{diam}(D)) \|u\|_{p,D}. \end{aligned} \tag{28}$$

Here $C_7 = \max\{C_5, C_6\}$ is independent of u. Thus, we complete the proof of Theorem 10. □

Next, we consider the following norm comparison equipped with power-type weights.

Theorem 11. *Let $D \subset R^n$ be a bounded convex domain, $1 < p, q < \infty$, $0 \le 1/p - 1/q \le 1/n$, let $T : L^p(D, \wedge^k) \to L^q(D, \wedge^{k-1})$ be the Homotopy operator, $k = 1, 2, \ldots, n$, and that continuous functions h and g defined in $(0, +\infty)$ satisfy (1) $\lim_{t\to 0} h(t) = 0$; (2) $\lim_{t\to 0} g(t) = \infty$. Then there exists a constant C, independent of u, such that*

$$\|T(G(u)) - (T(G(u)))_D\|_{q,D,\mu_1} \le C(1 + \operatorname{diam}(D)) \|u\|_{p,D,\mu_2} \tag{29}$$

for any $u \in \Omega_{p,p}(D, \wedge^k)$, $d\mu_1 = h(\operatorname{dist}(x, \partial D))dx$, $d\mu_2 = g(\operatorname{dist}(x, \partial D))dx$.

Proof. From Theorem 10, we know that there exists a constant C_1, independent of u, such that

$$\|T(G(u)) - (T(G(u)))_D\|_{q,D} \le C_1 (1 + \operatorname{diam}(D)) \|u\|_{p,D}. \tag{30}$$

Fixing $\varepsilon > 0$, then there exists $\delta_1(\varepsilon) > 0$ such that $h(\operatorname{dist}(x, \partial D)) < \varepsilon$ for all $x \in D$ with $\operatorname{dist}(x, \partial D) < \delta_1$. Let $D_1 = \{x \in D, \operatorname{dist}(x, \partial D) < \delta_1\}$ and $D_2 = D - D_1$. Then for all $x \in D_2$, we have

$$\delta_1 \le \operatorname{dist}(x, \partial D) < \operatorname{diam}(D). \tag{31}$$

Therefore, by the continuity of h, we know that there exists $M_1 > 0$, such that

$$h(\operatorname{dist}(x, \partial D)) < M_1 \tag{32}$$

for all $x \in D_2$. Thus we have

$$\begin{aligned} &\|T(G(u)) - (T(G(u)))_D\|_{q,D,\mu_1} \\ &\quad = \left(\int_D |T(G(u)) - (T(G(u)))_D|^q \cdot h(\operatorname{dist}(x, \partial D))\, dx\right)^{1/q} \\ &\quad \le \left(\varepsilon \int_{D_1} |T(G(u)) - (T(G(u)))_D|^q dx \right. \\ &\qquad \left. + M_1 \int_{D_2} |T(G(u)) - (T(G(u)))_D|^q dx\right)^{1/q} \\ &\quad \le C_2 \left(\int_D |T(G(u)) - (T(G(u)))_D|^q dx\right)^{1/q}. \end{aligned} \tag{33}$$

Here $C_2 = \max\{\varepsilon^{1/q}, M_1^{1/q}\}$. Communicating (30) and (33), we have

$$\begin{aligned} \|T(G(u)) - (T(G(u)))_D\|_{q,D,\mu_1} &\le C_2 \|T(G(u)) - (T(G(u)))_D\|_{q,D} \\ &\le C_3 (1 + \operatorname{diam}(D)) \|u\|_{p,D}. \end{aligned} \tag{34}$$

Note that $\lim_{t\to 0}(1/g(t)) = 0$. Then there exists $\delta_2(\varepsilon) > 0$ such that $1/g(\operatorname{dist}(x, \partial D)) < \varepsilon$ for all $x \in D$ with $\operatorname{dist}(x, \partial D) < \delta_2$. Let $D_1' = \{x \in D, \operatorname{dist}(x, \partial D) < \delta_2\}$ and $D_2' = D - D_1'$. Then for all $x \in D_2'$, we have

$$\delta_2 \le \operatorname{dist}(x, \partial D) < \operatorname{diam}(D). \tag{35}$$

Therefore, by the continuity of g, we know that there exists $M_2 > 0$, such that

$$\frac{1}{g(\operatorname{dist}(x, \partial D))} < M_2 \tag{36}$$

for all $x \in D_2'$. Therefore, we obtain

$$\begin{aligned} \|u\|_{p,D} &= \left(\int_D |u|^p \frac{1}{g(\operatorname{dist}(x, \partial D))} d\mu_2\right)^{1/p} \\ &\le \left(\varepsilon \int_{D_1'} |u|^p d\mu_2 + M_2 \int_{D_2'} |u|^p d\mu_2\right)^{1/p} \\ &\le C_4 \left(\int_D |u|^p d\mu_2\right)^{1/p} = C_4 \|u\|_{p,D,\mu_2}. \end{aligned} \tag{37}$$

Here $C_4 = \max\{\varepsilon^{1/p}, M_2^{1/p}\}$. By (34) and (37), we have

$$\begin{aligned} &\|T(G(u)) - (T(G(u)))_D\|_{q,D,\mu_1} \\ &\quad \le C_5 (1 + \operatorname{diam}(D)) \|u\|_{p,D,\mu_2}. \end{aligned} \tag{38}$$

Here C_5 is independent of u. Thus, we complete the proof of Theorem 11. □

In Theorem 11, if we choose $h(t) = t^r$ and $g(t) = t^{-s}$, $0 < r, s < \infty$, we can easily obtain the following corollary.

Corollary 12. *Let $D \subset R^n$ be a bounded convex domain, $1 < p, q < \infty$, $0 \le 1/p - 1/q \le 1/n$, and let $T : L^p(D, \wedge^k) \to L^q(D, \wedge^{k-1})$ be the Homotopy operator, $k = 1, 2, \ldots, n$. Then there exists a constant C, independent of u, such that*

$$\begin{aligned} &\int_D |T(G(u)) - (T(G(u)))_D|^q \cdot (\operatorname{dist}(x, \partial D))^r dx \\ &\quad \le C(1 + \operatorname{diam}(D)) \left(\int_D |u|^p \frac{1}{(\operatorname{dist}(x, \partial D))^s} dx\right)^{1/p}. \end{aligned} \tag{39}$$

Here $0 < r, s < \infty$.

Note that, in the proof of Theorem 11, if we let the composite operator $T \circ G$ act on the solution of nonhomogeneous A-harmonic equation, then we can drop $\lim_{t\to 0} h(t) = 0$. Next, we state the result as follows.

Corollary 13. *Let $D \subset R^n$ be a bounded convex domain, $1 < p, q < \infty$, $0 \le 1/p - 1/q \le 1/n$, let $T : L^p(D, \wedge^k) \to L^q(D, \wedge^{k-1})$ be the Homotopy operator, and $u \in \Omega_{p,p}(D, \wedge^k)$ is a solution of nonhomogeneous A-harmonic equation, $k = 1, 2, \ldots, n$. If continuous functions h and g defined in $(0, +\infty)$ satisfy that $\lim_{t\to 0} g(t) = \infty$, $d\mu_1 = h(\text{dist}(x, \partial D))dx$ and $d\mu_2 = g(\text{dist}(x, \partial D))dx$. Then there exists a constant C, independent of u, such that*

$$\|T(G(u)) - (T(G(u)))_D\|_{q,B,\mu_1} \le C(1 + \text{diam}(D))\|u\|_{p,\rho B,\mu_2} \tag{40}$$

for all balls B with $\rho B \subset D$. Here $\rho > 1$ is some constant.

It is easy to find that the above corollary does not hold for balls $B \subset D$ with $\partial B \bigcap \partial D \neq \Phi$ but holds for those balls with $\rho B \subset D$. Next, we introduce the following singular integral inequality.

Theorem 14. *Let $D \subset R^n$ be a bounded convex domain, $1 < p$, $q < \infty$, $0 \le 1/p - 1/q \le 1/n$, let $T : L^p(D, \wedge^k) \to L^q(D, \wedge^{k-1})$ be the Homotopy operator, and $u \in \Omega_{p,p}(D, \wedge^k)$ is a solution of nonhomogeneous A-harmonic equation, $k = 1, 2, \ldots, n$. If continuous functions h and g defined in $(0, +\infty)$ and $h(t)$ is an increasing function, then there exists a constant C, independent of u, such that*

$$\begin{aligned}\left(\int_B |T(G(u)) - (T(G(u)))_B|^q \frac{1}{g(\text{dist}(x, \partial D))} dx\right)^{1/q} &\le C(1 + \text{diam}(B))\left|\rho B\right|^{(p-q)/pq} \\ &\quad \times \left(\int_{\rho B} \frac{|u|^p}{(h(\text{dist}(x, \partial D)))^\lambda} dx\right)^{1/p}\end{aligned} \tag{41}$$

for all balls B with $\rho B \subset D$ and $0 < \lambda < 1$. Here $\rho > 1$ is some constant.

Proof. Let $k = q/(1-\lambda)$. From $0 < \lambda < 1$, it is easy to see that $k > q$. Using the Hölder inequality, we have

$$\begin{aligned}\left(\int_B |T(G(u)) - (T(G(u)))_B|^q \frac{1}{g(\text{dist}(x, \partial D))} dx\right)^{1/q} &\le \left(\int_B |T(G(u)) - (T(G(u)))_B|^k dx\right)^{1/k} \\ &\quad \times \left(\int_B \frac{1}{(g(\text{dist}(x, \partial D)))^{k/(k-q)}} dx\right)^{(k-q)/kq} \\ &= \|T(G(u)) - (T(G(u)))_B\|_{k,B} \\ &\quad \times \left(\int_B \frac{1}{(g(\text{dist}(x, \partial D)))^{k/(k-q)}} dx\right)^{(k-q)/kq}.\end{aligned} \tag{42}$$

Note that $\rho B \subset D$. Therefore, there exists a positive number c such that

$$c < \text{dist}(x, \partial D) \le \text{diam}(D) \tag{43}$$

for all $x \in B$. Furthermore, by the continuity of function g in $(0, +\infty)$, $g(\text{dist}(x, \partial D))$ has a positive lower bound M in B. Thus, from Theorem 10 and (42), we have

$$\begin{aligned}\left(\int_B |T(G(u)) - (T(G(u)))_B|^q \frac{1}{g(\text{dist}(x, \partial D))} dx\right)^{1/q} &\le \left(\frac{1}{M}\right)^{1/q} |B|^{(k-q)/kq} \|T(G(u)) - (T(G(u)))_B\|_{k,B} \\ &\le C_1 |B|^{(k-q)/kq} (1 + \text{diam}(B)) \|u\|_{k,B} \\ &\le C_1 |B|^{(k-q)/kq} (1 + \text{diam}(B)) \|u\|_{k,\rho_1 B},\end{aligned} \tag{44}$$

where $\rho_1 > 1$ is a constant. Let $\varepsilon \in (1/p, 1)$ and $m = \varepsilon p$. Since u is the solution of nonhomogenous A-harmonic equation. By (19), we know

$$\|u\|_{k,\rho_1 B} \le C_2 \left|\rho_1 B\right|^{(m-k)/mk} \|u\|_{m,\rho B}, \tag{45}$$

where $\rho > \rho_1 > 1$ is a constant. It is easy to find that $1 < m < p$. Using the Hölder inequality, we have

$$\begin{aligned}\|u\|_{m,\rho B} &= \left(\int_{\rho B} |u|^m \frac{1}{(h(\text{dist}(x, \partial D)))^{m\lambda/p}} \cdot (h(\text{dist}(x, \partial D)))^{m\lambda/p} dx\right)^{1/m} \\ &\le \left(\int_{\rho B} \frac{|u|^p}{(h(\text{dist}(x, \partial D)))^\lambda} dx\right)^{1/p} \\ &\quad \times \left(\int_{\rho B} \left((h(\text{dist}(x, \partial D)))^{\lambda/p}\right)^{mp/(p-m)} dx\right)^{(p-m)/mp}.\end{aligned} \tag{46}$$

The continuity and monotonicity of function h imply that

$$\begin{aligned}\left(\int_{\rho B} \left((h(\text{dist}(x, \partial D)))^{\lambda/p}\right)^{mp/(p-m)} dx\right)^{(p-m)/mp} &= \left(\int_{\rho B} (h(\text{dist}(x, \partial D)))^{\varepsilon\lambda/(1-\varepsilon)} dx\right)^{(1-\varepsilon)/\varepsilon p} \\ &\le \left|\rho B\right|^{(1-\varepsilon)/\varepsilon p} (h(\text{diam}(D)))^{\lambda/p}.\end{aligned} \tag{47}$$

Hence, combining (41)–(47), we have

$$\begin{aligned}\left(\int_B |T(G(u)) - (T(G(u)))_B|^q \frac{1}{g(\text{dist}(x, \partial D))} dx\right)^{1/q} &\le C_3 |B|^{(k-q)/kq} (1 + \text{diam}(B)) \left|\rho_1 B\right|^{(m-k)/mk} \left|\rho B\right|^{(1-\varepsilon)/\varepsilon p} \\ &\quad \times (h(\text{diam}(D)))^{\lambda/p} \left(\int_{\rho B} \frac{|u|^p}{(h(\text{dist}(x, \partial D)))^\lambda} dx\right)^{1/p} \\ &\le C_4 (1 + \text{diam}(B)) \left|\rho B\right|^{(p-q)/pq} \\ &\quad \times \left(\int_{\rho B} \frac{|u|^p}{(h(\text{dist}(x, \partial D)))^\lambda} dx\right)^{1/p}.\end{aligned} \tag{48}$$

Here C_4 is dependent of B and h but independent of u. Thus, we complete the proof of Theorem 11. □

4. Application

In this section, we will use the estimates in Section 3 to obtain the upper bound for the L^p norms of $T(G(u))$ or $(T(G(u)))_B$ in terms of L^q norms of u or du.

Example 15. For $n \geq 2$, let u be a $(n-1)$-form defined in R^n by

$$\begin{aligned} u = {} & \frac{x_1}{\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}} dx_2 \wedge dx_3 \wedge \cdots \wedge dx_n \\ & - \frac{x_2}{\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}} dx_1 \wedge dx_3 \wedge \cdots \wedge dx_n \\ & + \cdots + (-1)^{n-1} \\ & \times \frac{x_n}{\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}} dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{n-1}. \end{aligned} \tag{49}$$

It is easy to find that

$$|u| = 1, \quad du = \frac{n-1}{\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}} dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n. \tag{50}$$

If we choose the usual (p, p)-type norm inequality to estimate $T(G(u)) - (T(G(u)))_B$ and take $p = n$, where $B = B(O, r) \subset R^n$ is a ball, then by Theorem 10, we have

$$\begin{aligned} & \left(\int_B \left| T(G(u)) - (T(G(u)))_B \right|^n dx \right)^{1/n} \\ & \quad \leq C_1 (1 + \operatorname{diam}(B)) \left(\int_B |u|^n dx \right)^{1/n} \\ & \quad = C_1 (1 + \operatorname{diam}(B)) |B|^{1/n}. \end{aligned} \tag{51}$$

However, if we choose the (p, q)-type norm inequality to estimate $T(G(u)) - (T(G(u)))_B$ and take $p = n - 1$, $q = n$, then p, q satisfy the condition $0 \leq 1/p - 1/q \leq 1/n$. Hence by using Theorem 10, we obtain

$$\begin{aligned} & \left(\int_B \left| T(G(u)) - (T(G(u)))_B \right|^n dx \right)^{1/n} \\ & \quad \leq C_2 (1 + \operatorname{diam}(B)) \left(\int_B |u|^{n-1} dx \right)^{1/(n-1)} \\ & \quad = C_2 (1 + \operatorname{diam}(B)) |B|^{1/(n-1)}. \end{aligned} \tag{52}$$

Compare (51) and (52), we can easily find that if we choose different (p, q)-type norm inequality to estimate the oscillation $T(G(u)) - (T(G(u)))_B$, we also obtain the different upper bound.

Example 16. In R^2, consider that

$$u(x, y) = \arctan \frac{y}{x-1} - \arctan \frac{y}{x+1}. \tag{53}$$

It is easy to check that $u(x, y)$ is harmonic in the upper half plane. Note that

$$\begin{aligned} du &= \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy, \\ * du &= \frac{\partial u}{\partial x} dy - \frac{\partial u}{\partial y} dx. \end{aligned} \tag{54}$$

Therefore, we have

$$d * du = \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) dx \wedge dy = 0, \tag{55}$$

which implies that $*du$ is a closed form and hence is a solution of nonhomogenous A-harmonic equation. It is easy to see that

$$|*du| = \frac{1}{\sqrt{\left((x-1)^2 + y^2\right)\left((x+1)^2 + y^2\right)}}. \tag{56}$$

Let D denote a bound convex domain in the upper half plane and let $\sigma\overline{B} \subset D$ be a closed ball without the points $(-1, 0)$ and $(1, 0)$. If $\sigma\overline{B}$ and D satisfy that $\operatorname{dist}(\sigma B, \partial D) = M > 0$, then both $|* du|$ and $(\operatorname{dist}(x, \partial D))^{-1}$ have the upper bounds in $\sigma\overline{B}$. Thus, for the term

$$\int_B \left| T(G(u)) - (T(G(u)))_B \right|^p \frac{1}{g(\operatorname{dist}(x, \partial D))} dx, \tag{57}$$

it is usually not easy to be estimated due to the complexity of the compositions $T(G(u))$ and the function g. However, by Theorem 14, (57) can be controlled by the term

$$\int_{\rho B} \frac{|u|^p}{\left(h(\operatorname{dist}(x, \partial D))\right)^\lambda} dx. \tag{58}$$

Thus, we obtain an upper bound of (57).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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