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Research Article

The Spectrum and Eigenvectors of the Laplacian Matrices of the Brualdi-Li Tournament Digraphs

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Let $m \geq 1$ be an integer, let $\mathscr{B}_{2m}$ denote the Brualdi-Li matrix of order $2m$, and let $\mathscr{L}\mathscr{B}_{2m}$ denote the Laplacian matrices of Brualdi-Li tournament digraphs. We obtain the eigenvalues and eigenvectors of $\mathscr{L}\mathscr{B}_{2m}$.

1. Introduction

The Laplacian spectral theory is currently not only a hot research direction of spectral graph theory but also one of the research directions of combined matrix theory. The Laplacian spectrum of a graph is of importance in graph theory, matrix theory, and the definite solution of partial differential equations. It also has applications in quantum chemistry, biology, and complex network. Therefore, it has important theoretical and practical values to study the Laplacian eigenvalues and eigenvectors of graphs. The Laplacian spectrum of graph has attracted the attention of researchers; see [1–5] and so on. We follow [1, 6] for terminology and notations.

Let G be a digraph of order n with vertex set $V(G)$ and arc set $E(G)$, where $V(G) = \{v_1, v_2, \dots, v_n\}$. The adjacency matrix of G is the $(0,1)$ matrix $A(G) = (a_{ij})$ of order n, where $a_{ij} = 1$ if there is an arc from v_i to v_j and $a_{ij} = 0$ otherwise. The digraph G is called the associated digraph of matrix $A(G)$. Let $D(G) = \operatorname{diag}(d_1, d_2, \dots, d_n)$, the diagonal matrix with vertex outdegrees $d_1, d_2, \dots, d_n$ of $v_1, v_2, \dots, v_n$. $L(G) = D(G) - A(G)$ is called the Laplacian matrix of the digraph G. The characteristic polynomial of the adjacency matrix $A(G)$, that is, $P(G,\lambda) = P(A(G),\lambda) = \det(\lambda I - A(G))$, is called the characteristic polynomial of the digraph G. The equation $\det(\lambda I - A(G)) = 0$ has n complex roots and these roots are called the eigenvalues of $A(G)$. Suppose the distinct eigenvalues of $L(G)$ are denoted by $\lambda_1, \lambda_2, \dots, \lambda_r \in \mathbb{C}$ with corresponding algebraic multiplicities $\mu_{\lambda_1}, \mu_{\lambda_2}, \dots, \mu_{\lambda_r}$, where μ_{λ_i} is a nonnegative integer $i = 1, 2, \dots, r$. $LS = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_r \\ \mu_{\lambda_1} & \mu_{\lambda_2} & \cdots & \mu_{\lambda_r} \end{pmatrix}$ is called the Laplacian spectrum of digraph G. The Laplacian spectral radius of G is the largest modulus of an eigenvalue of $L(G)$, denoted by $\rho(L(G))$. The symbol $\mathbb{C}$ will denote the complex field. Let $\lambda \in \mathbb{C}$ be an eigenvalue of matrix A. There is vector $x \neq 0$ satisfying $Ax = \lambda x$, and then x is called the eigenvectors of matrix A corresponding to λ.

A tournament matrix of order n is a $(0,1)$ matrix T_n satisfying the equation $T_n + T_n^t = J_n - I_n$, where J_n is the all ones matrix, I_n is the identity matrix, and T_n^t is the transpose of T_n. Let

$$\mathscr{B}_{2m} = \begin{pmatrix} U_m & U_m^t \\ I_m + U_m^t & U_m \end{pmatrix}, \tag{1}$$

where U_m is strictly upper triangular tournament matrix (all of whose entries above the main diagonal are equal to one); that is,

$$U_m = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}_{m\times m} \tag{2}$$

is the tournament matrix of order $2m$.

The matrix $\mathscr{B}_{2m}$ has been dubbed by the Brualdi-Li matrix. The associated digraph of matrix $\mathscr{B}_{2m}$ is called the Brualdi-Li tournament digraph. In 1983 Brualdi and Li conjectured that the maximal spectral radius for tournaments of order $2m$ is attained by the Brualdi-Li matrix [7]. This conjecture has been confirmed in [8]. The properties of the Brualdi-Li matrix have been investigated in [9–14].

In this paper, we obtain the spectrum and eigenvectors of the Laplacian matrices of the Brualdi-Li tournament digraphs.

Theorem 1. *Let $m \geq 1$ be an integer, and let $\mathcal{SLB}_{2m}$ be the Laplacian spectrum of the Brualdi-Li tournament digraph. Then*

$$\mathcal{SLB}_{2m} = \begin{pmatrix} 0 & m & \lambda_k & \overline{\lambda_k} \\ 1 & \left\lfloor \frac{m-1}{2} \right\rfloor + \left\lfloor \frac{m+1}{2} \right\rfloor & 1 & 1 \end{pmatrix}, \quad (3)$$

where $\lfloor a \rfloor$ is the floor of number a, $\lambda_k = m - i\cot((\pi + 2k\pi)/2m)$, and $\overline{\lambda_k}$ is the conjugate complex number of λ_k, $k = 0, 1, 2, \ldots, \lfloor m/2 \rfloor - 1$.

Theorem 2. *Let $m \geq 1$ be an integer, and let $\xi = \binom{v}{w} \neq 0$ be the eigenvector of $\mathcal{LB}_{2m}$ corresponding to λ, where $v = (v_1, v_2, \ldots, v_m)^t$, $w = (w_1, w_2, \ldots, w_m)^t$; then*

(1) *if $\lambda = m$, then*

$\xi_m = \binom{0}{1}$ *is the eigenvector of $\mathcal{LB}_2$ corresponding to $\lambda = m = 1$,*

$\xi_m = (1, -1, 1, -1)^t$ *is the eigenvector of $\mathcal{LB}_4$ corresponding to $\lambda = m = 2$,*

$\xi_m = \sum_{k=2}^{m} l_k \xi_{m_k}$ *is the eigenvector of $\mathcal{LB}_{2m}$ corresponding to $\lambda = m > 2$,*

where l_k is an arbitrary constant, $k = 2, 3, \ldots, m$,

$$\left(\xi_{m_2}, \xi_{m_3}, \ldots, \xi_{m_m}\right) = \begin{pmatrix} \mathbf{1}_{m-2}^t & 1 \\ -I_{m-2} & 0 \\ 0 & -1 \\ \mathbf{1}_{m-2}^t & 1 \\ -I_{m-2} & 0 \\ 0 & -1 \end{pmatrix}_{2m \times (m-1)}; \quad (4)$$

(2) *if $\lambda \neq m$, then*

$$v_k = \frac{1}{2(m-\lambda)} \left(1 - \lambda \left(\frac{m-1-\lambda}{m+1-\lambda}\right)^{k-1}\right),$$
$$w_k = \frac{1}{2(m-\lambda)} \left(1 + \lambda \left(\frac{m-1-\lambda}{m+1-\lambda}\right)^{k}\right), \quad (5)$$
$$k = 1, 2, \cdots, m.$$

Corollary 3. *Let $m \geq 1$ be an integer, and let $\rho(\mathcal{LB}_{2m})$ be the Laplacian spectral radius of the Brualdi-Li tournament digraph. Then*

$$\rho\left(\mathcal{LB}_{2m}\right) = \sqrt{m^2 + \cot^2 \frac{\pi}{2m}}. \quad (6)$$

Proof. By Theorem 1,

$$\begin{aligned} \rho\left(\mathcal{LB}_{2m}\right) &= \max_{0 \leq k \leq \lfloor m/2 \rfloor - 1} \left\{0, m, \left| m \pm i\cot \frac{\pi + 2k\pi}{2m} \right| \right\} \\ &= \sqrt{m^2 + \cot^2 \frac{\pi}{2m}}. \end{aligned} \quad (7)$$

□

Corollary 4. *Let $m \geq 1$ be an integer, then*

(1) *if $m > 1$ is odd, then $\mathcal{LB}_{2m}$ is not diagonalizable,*

(2) *if m is even or $m = 1$, then $\mathcal{LB}_{2m}$ is diagonalizable.*

2. Some Lemmas

Fundamental Theorem of Algebra. Every nonzero, single-variable, degree n polynomial with complex coefficients has, counted with multiplicity, exactly n roots.

Complex Conjugate Root Theorem. If P is a polynomial in one variable with real coefficients and $a + bi$ is a root of P with a and b real numbers, then its complex conjugate $a - bi$ is also a root of P, where $i^2 = -1$.

The symbol $\mathcal{LB}_{2m}$ denotes the Laplacian matrix of the Brualdi-Li tournament digraph. Clearly, $\mathcal{LB}_{2m} = \begin{pmatrix} (m-1)I_m - U_m & -U_m^t \\ -I_m - U_m^t & mI_m - U_m \end{pmatrix}$.

Lemma 5. *Let $m > 1$ be an integer, and $X = (1, x, x^2, \ldots, x^{m-1})^t$, where $x \neq 1$ is real variable. Then*

$$\begin{aligned} &(1)\ X^t U_m = -\frac{1}{1-x} X^t + \frac{1}{1-x} \mathbf{1}_m^t, \\ &(2)\ X^t U_m^t = \frac{x}{1-x} X^t - \frac{x^m}{1-x} \mathbf{1}_m^t. \end{aligned} \quad (8)$$

Proof. Consider

(1) $X^t U_m$

$$\begin{aligned} &= \left(1, x, x^2, \ldots, x^{m-1}\right) U_m \\ &= \left(0, \sum_{k=0}^{0} x^k, \sum_{k=0}^{1} x^k, \sum_{k=0}^{2} x^k, \sum_{k=0}^{3} x^k, \ldots, \sum_{k=0}^{m-3} x^k, \sum_{k=0}^{m-2} x^k \right) \\ &= \left(\frac{1-1}{1-x}, \frac{1-x}{1-x}, \frac{1-x^2}{1-x}, \frac{1-x^3}{1-x}, \ldots, \frac{1-x^{m-2}}{1-x}, \frac{1-x^{m-1}}{1-x}\right) \\ &= -\frac{1}{1-x} X^t + \frac{1}{1-x} \mathbf{1}_m^t. \end{aligned}$$

(2) $X^t U_m^t$

$$\begin{aligned}
&= \left(1, x, x^2, \ldots, x^{m-1}\right) U_m^t \\
&= \left(\sum_{k=1}^{m-1} x^k, \sum_{k=2}^{m-1} x^k, \ldots, \sum_{k=m-2}^{m-1} x^k, \sum_{k=m-1}^{m-1} x^k, 0\right) \\
&= \left(\frac{x - x^m}{1-x}, \frac{x^2 - x^m}{1-x}, \ldots, \frac{x^{m-2} - x^m}{1-x}, \right. \\
&\qquad \left. \frac{x^{m-1} - x^m}{1-x}, \frac{x^m - x^m}{1-x}\right) \\
&= \frac{x}{1-x} X^t - \frac{x^m}{1-x} \mathbf{1}_m^t.
\end{aligned} \tag{9}$$

□

Lemma 6. *Let $m > 1$ be an integer, $\lambda \neq m$, let $\lambda \in \mathbb{C}$ be an arbitrary eigenvalue of $\mathscr{LB}_{2m}$, and let $\xi = \binom{v}{w} \neq 0$ be the eigenvector of $\mathscr{LB}_{2m}$ corresponding to λ, where $v = (v_1, v_2, \ldots, v_m)^t$, $w = (w_1, w_2, \ldots, w_m)^t$. Let $X = (1, x, x^2, \ldots, x^{m-1})^t$, $f(x) = \sum_{k=1}^m v_k x^{k-1}$, and $g(x) = \sum_{k=1}^m w_k x^{k-1}$, where x is real variable. Then*

$$\begin{aligned}
&(1)\ f(x) \\
&= \left((m+1-\lambda) a + (a+b)\left(x + x^2 + \cdots + x^{m-1}\right)\right. \\
&\qquad \left. + (a - b(m-\lambda)) x^m\right) \\
&\quad \times ((m-\lambda)((m+1-\lambda) - (m-1-\lambda)x))^{-1}, \\
&(2)\ g(x) \\
&= \left(a + b(m-\lambda) + (a+b)\left(x + x^2 + \cdots + x^{m-1}\right)\right. \\
&\qquad \left. - a(m-1-\lambda) x^m\right) \\
&\quad \times ((m-\lambda)((m+1-\lambda) - (m-1-\lambda)x))^{-1},
\end{aligned} \tag{10}$$

where $a = \mathbf{1}_m^t v$, $b = \mathbf{1}_m^t w$.

Proof. If $\lambda(\neq m)$ is an eigenvalue, with eigenvector $\xi = \binom{v}{w}$, of $\mathscr{LB}_{2m}$, then $\mathscr{LB}_{2m}\xi = \lambda\xi$ expands to

$$\begin{pmatrix} (m-1) I_m - U_m & -U_m^t \\ -I_m - U_m^t & m I_m - U_m \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} = \lambda \begin{pmatrix} v \\ w \end{pmatrix}. \tag{11}$$

Therefore,

$$\begin{aligned}
&((m-1) I_m - U_m) v - U_m^t w = \lambda v, \\
&\left(-I_m - U_m^t\right) v + (m I_m - U_m) w = \lambda w.
\end{aligned} \tag{12}$$

We have

$$\begin{aligned}
&X^t ((m-1) I_m - U_m) v - X^t U_m^t w = \lambda X^t v, \\
&X^t \left(-I_m - U_m^t\right) v + X^t (m I_m - U_m) w = \lambda X^t w.
\end{aligned} \tag{13}$$

According to Lemma 5, we have

$$\begin{aligned}
&((m-1-\lambda)(1-x) + 1) f(x) - x g(x) = a - b x^m \\
&-f(x) + ((m-\lambda)(1-x) + 1) g(x) = b - a x^m.
\end{aligned} \tag{14}$$

Notice that this equation holds for $x = 1$ too. Consider

$$\begin{aligned}
D &= \det \begin{pmatrix} (m-1-\lambda)(1-x) + 1 & -x \\ -1 & (m-\lambda)(1-x) + 1 \end{pmatrix} \\
&= ((m-1-\lambda)(1-x) + 1)((m-\lambda)(1-x) + 1) - x \\
&= (m-\lambda)(1-x)((m+1-\lambda) - (m-1-\lambda)x), \\
D_f &= \det \begin{pmatrix} a - b x^m & -x \\ b - a x^m & (m-\lambda)(1-x) + 1 \end{pmatrix} \\
&= \left(a - b x^m\right)((m-\lambda)(1-x) + 1) + x\left(b - a x^m\right) \\
&= a(m+1-\lambda) + (b - a(m-\lambda)) x \\
&\quad - b(m+1-\lambda) x^m + (b(m-\lambda) - a) x^{m+1} \\
&= (1-x)\left(a(m+1-\lambda) + (a+b)\left(x + \cdots + x^{m-1}\right)\right. \\
&\qquad \left. + (a - b(m-\lambda)) x^m\right), \\
D_g &= \det \begin{pmatrix} (m-1-\lambda)(1-x) + 1 & a - b x^m \\ -1 & b - a x^m \end{pmatrix} \\
&= \left(b - a x^m\right)((m-1-\lambda)(1-x) + 1) + \left(a - b x^m\right) \\
&= a + b(m-\lambda) - b(m-1-\lambda) x \\
&\quad - (b + a(m-\lambda)) x^m + a(m-1-\lambda) x^{m+1} \\
&= (1-x)\left(a + b(m-\lambda) + (a+b)\left(x + \cdots + x^{m-1}\right)\right. \\
&\qquad \left. - a(m-1-\lambda) x^m\right).
\end{aligned} \tag{15}$$

By Cramer's rule,

$$\begin{aligned}
&(1)\ f(x) \\
&= \left((m+1-\lambda) a + (a+b)\left(x + x^2 + \cdots + x^{m-1}\right)\right. \\
&\qquad \left. + (a - b(m-\lambda)) x^m\right) \\
&\quad \times ((m-\lambda)((m+1-\lambda) - (m-1-\lambda)x))^{-1}, \\
&(2)\ g(x) \\
&= \left(a + b(m-\lambda) + (a+b)\left(x + x^2 + \cdots + x^{m-1}\right)\right. \\
&\qquad \left. - a(m-1-\lambda) x^m\right) \\
&\quad \times ((m-\lambda)((m+1-\lambda) - (m-1-\lambda)x))^{-1}.
\end{aligned} \tag{16}$$

We are done. □

Lemma 7. *Under the assumptions and in the notation of Lemma 6,*

$$a+b=\mathbf{1}_m^t v+\mathbf{1}_m^t w\neq 0,\quad a=\mathbf{1}_m^t v\neq 0,\quad b=\mathbf{1}_m^t w\neq 0. \tag{17}$$

Proof. In Lemma 6(1), by setting $x=1$, we have

$$\begin{aligned} a &= f(1) \\ &= \frac{(m+1-\lambda)a+(a+b)(m-1)+(a-b(m-\lambda))}{(m-\lambda)((m+1-\lambda)-(m-1-\lambda))}. \end{aligned} \tag{18}$$

Since $\lambda\neq m$, it follows that

$$(a+b)(1-\lambda)=2a. \tag{19}$$

As we all know that the eigenvalues of a real skew-symmetric matrix are all pure imaginary or nonzero, hence $\det((1-m)I_m+U-U^t)\neq 0$, where $m>1$. Since

$$\begin{aligned} &\det(1I_m-\mathscr{LB}_{2m}) \\ &= \det\begin{pmatrix}(2-m)I_m+U_m & U_m^t \\ I_m+U_m^t & (1-m)I_m+U_m\end{pmatrix} \\ &= \det\begin{pmatrix}(2-m)I_m+U_m & (2-m)I_m+U_m+U_m^t \\ I_m+U_m^t & (2-m)I_m+U_m+U_m^t\end{pmatrix} \\ &= \det\begin{pmatrix}(1-m)I_m+U-U^t & 0 \\ I_m+U_m^t & (1-m)I_m+Jm\end{pmatrix} \\ &= \det\left((1-m)I_m+U-U^t\right)\det\left((1-m)I_m+Jm\right) \\ &= (1-m)^{m-1}\det\left((1-m)I_m+U-U^t\right)\neq 0, \end{aligned} \tag{20}$$

hence $\lambda\neq 1$. Notice that $(a+b)(1-\lambda)=2a$ and if $a+b=0$, then $a=0$ and $b=(a+b)-a=0$. By Lemma 6,

$$\begin{aligned} f(x) &= \sum_{k=1}^{m} v_k x^{k-1}\equiv 0, \\ g(x) &= \sum_{k=1}^{m} w_k x^{k-1}\equiv 0, \end{aligned} \tag{21}$$

for arbitrary real variable x. It is not possible. Hence $a+b\neq 0$. It is easy to see that $a\neq 0$ and $b\neq 0$. □

Lemma 8 (see [15]). *If A is a square matrix, then,*

(1) *for every eigenvalue of A, the geometric multiplicity is less than or equal to the algebraic multiplicity,*

(2) *A is diagonalizable if and only if the geometric multiplicity of every eigenvalue is equal to the algebraic multiplicity.*

3. Proof of Theorem 1

Let $\lambda\in\mathbb{C}$ be an arbitrary eigenvalue of $\mathscr{LB}_{2m}$, and let $\xi=\binom{v}{w}\neq 0$ be the eigenvector of $\mathscr{LB}_{2m}$ corresponding to λ, where $v=(v_1,v_2,\ldots,v_m)^t$, $w=(w_1,w_2,\ldots,w_m)^t$.

For $m=1$, then $\mathscr{B}_2=\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)$, $\mathscr{LB}_2=\left(\begin{smallmatrix}0&0\\-1&1\end{smallmatrix}\right)$. Obviously, $\lambda_1=0$, $\lambda_2=1$ are eigenvalues of $\mathscr{LB}_2$ and $\mu_0=\mu_1=1$. Theorem 1 holds.

For $m>1$, using the previous assumptions and notation, obviously, $P(\mathscr{LB}_{2m},0)=\det(0I_{2m}-\mathscr{LB}_{2m})=0$, by definition, $\lambda=0$ is an eigenvalue of $\mathscr{LB}_{2m}$, $\mu_0\geq 1$.

Note that $P(\mathscr{LB}_{2m},m)=\det(mI_{2m}-\mathscr{LB}_{2m})=0$, and by definition, $\lambda=m$ is an eigenvalue of $\mathscr{LB}_{2m}$. By simple calculation, the rank of matrix $mI_{2m}-\mathscr{LB}_{2m}$ is equal to $m+1$. By Lemma 8(1) $\mu_m\geq 2m-(m+1)=m-1$.

For $\lambda\neq m$, by Lemma 7, we set $a=\mathbf{1}_m^t v$, $a+b=\mathbf{1}_m^t v+\mathbf{1}_m^t w=1$; hence $b=1-\mathbf{1}_m^t v=1-a$. In Lemma 6(1), by setting $x=1$, we have

$$\begin{aligned} a &= \mathbf{1}_m^t v = f(1) \\ &= \frac{(m+1-\lambda)a+m-1+a+(a-1)(m-\lambda)}{(m-\lambda)((m+1-\lambda)-(m-1-\lambda))} \\ &= \frac{(m-\lambda)(2a-1)+m+2a-1}{2(m-\lambda)}. \end{aligned} \tag{22}$$

Hence

$$a=\frac{1-\lambda}{2}. \tag{23}$$

Denoting $f_0=(m+1-\lambda)a$, $f_1=f_2=\cdots=f_{m-1}=1$, $f_m=a+(a-1)(m-\lambda)$, $c=(m-\lambda)(m+1-\lambda)$, and $d=(m-1-\lambda)/(m+1-\lambda)$, we have

$$\begin{aligned} &f(x) \\ &= \sum_{k=1}^{m} v_k x^{k-1} \\ &= \left((m+1-\lambda)a+x+x^2+\cdots+x^{m-1}\right. \\ &\qquad \left.+(a+(a-1)(m-\lambda))x^m\right) \\ &\qquad \times\left((m-\lambda)((m+1-\lambda)-(m-1-\lambda)x)\right)^{-1} \\ &= \frac{1}{c}\frac{1}{(1-dx)}\sum_{k=0}^{m} f_k x^k = \frac{1}{c}\sum_{k=0}^{\infty} d^k x^k \sum_{k=0}^{m} f_k x^k \\ &= \frac{1}{c}\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} f_j d^{k-j}\right)x^k. \end{aligned} \tag{24}$$

It must be that

$$v_k=\frac{1}{c}\sum_{j=0}^{k-1} f_j d^{k-1-j},\quad k=1,2,\ldots,m. \tag{25}$$

Therefore

$$\begin{aligned} a &= \sum_{k=1}^{m} v_k = \frac{1}{c}\sum_{k=1}^{m}\sum_{j=0}^{k-1} f_j d^{k-1-j} \\ &= \frac{1}{c}\sum_{k=1}^{m}\left(f_0 d^{k-1}+\sum_{j=1}^{k-1} d^{k-1-j}\right. \end{aligned}$$

$$= \frac{1}{c}\sum_{k=1}^{m}\left(f_0 d^{k-1} + \frac{1-d^{k-1}}{1-d}\right)$$
$$= \frac{1}{c(1-d)^2} \times \left(f_0\left(1-d^m\right)(1-d) + m(1-d) - 1 + d^m\right)$$
$$= \frac{1}{c(1-d)^2} \times \left(\left(1-f_0(1-d)\right)d^m + \left(f_0+m\right)(1-d) - 1\right). \quad (26)$$

That is, $ac(1-d)^2 = (1-f_0(1-d))d^m + (f_0+m)(1-d) - 1$.

Since $a = (1-\lambda)/2$, $f_0 = (m+1-\lambda)a$, $c = (m-\lambda)(m+1-\lambda)$, and $d = (m-1-\lambda)/(m+1-\lambda)$, then

$$\begin{gathered} \left(1-f_0(1-d)\right)d^m = ac(1-d)^2 - \left(f_0+m\right)(1-d) + 1, \\ \left(1-\frac{2a(m+1-\lambda)}{m+1-\lambda}\right)d^m = \frac{4a(m-\lambda)}{m+1-\lambda} - \frac{2((m+1-\lambda)a+m)}{m+1-\lambda} + 1, \\ \lambda d^m = -\lambda. \end{gathered} \quad (27)$$

Denote $\theta_k = (\pi + 2k\pi)/2m$, $k = 0, 1, 2, m-1$. As $\lambda \neq 0$, $\lambda - m \neq 0$, we have $d = e^{2i\theta_k}$; that is, $(m-1-\lambda)/(m+1-\lambda) = e^{2i\theta_k}$, where $i^2 = -1$.

Therefore,

$$\lambda = \lambda_k = m - \frac{1+e^{2i\theta_k}}{1-e^{2i\theta_k}} = m - \frac{i\sin 2\theta_k}{1-\cos 2\theta_k} = m - i\cot\theta_k, \quad (28)$$

where $1 - e^{2i\theta_k} \neq 0$, $k = 0, 1, 2, \ldots, m-1$.

Note that if m is odd, by $d^m = ((m-1-\lambda)/(m+1-\lambda))^m = -1$, then $(m-1-\lambda)/(m+1-\lambda) = -1$; hence $\lambda = m$. It is an eigenvalue of $\mathscr{LB}_{2m}$.

Furthermore, $\mu_{\lambda_k} \geq 1$. Note that $\overline{\lambda_k} = \lambda_{m-1-k}$, $k = 0, 1, 2, \ldots, \lfloor m/2 \rfloor - 1$.

To sum up, by fundamental theorem of algebra and complex conjugate root theorem, for $m \geq 1$, we obtain the following conclusions.

If m is odd, then $\mu_0 = \mu_{\lambda_k} = \mu_{\overline{\lambda_k}} = 1$ and $\mu_m = m = \lfloor (m-1)/2 \rfloor + \lfloor (m+1)/2 \rfloor$.

If m is even, then $\mu_0 = \mu_{\lambda_k} = \mu_{\overline{\lambda_k}} = 1$ and $\mu_m = m-1 = \lfloor (m-1)/2 \rfloor + \lfloor (m+1)/2 \rfloor$.

Consider $k = 0, 1, 2, \ldots, \lfloor m/2 \rfloor - 1$. We complete the proof of Theorem 1.

4. Proofs of Theorem 2 and Corollary 4

It is easy to see that the distinct eigenvalues of $\mathscr{LB}_{2m}$ are $0, \lambda_k = m - i\cot((\pi+2k\pi)/2m), \overline{\lambda_k}, m$, with corresponding algebraic multiplicities $1, 1, 1, \lfloor (m-1)/2 \rfloor + \lfloor (m+1)/2 \rfloor$, $k = 1, 2, \ldots, \lfloor m/2 \rfloor - 1$.

For $\lambda = m = 1$, $\mathscr{LB}_2 = \left(\begin{smallmatrix} 0 & 0 \\ -1 & 1 \end{smallmatrix}\right)$ and, obviously, $\xi_m = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ is the eigenvector of $\mathscr{LB}_2$ corresponding to $\lambda = m = 1$.

For $\lambda = m = 2$,

$$\mathscr{LB}_4 = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 2 & -1 \\ -1 & -1 & 0 & 2 \end{pmatrix}. \quad (29)$$

By simple calculation, $\xi_m = (1, -1, 1, -1)^t$ is the eigenvector of $\mathscr{LB}_4$ corresponding to $\lambda = m = 2$.

For $\lambda = m > 2$, let $\xi_m = \left(\begin{smallmatrix} v \\ w \end{smallmatrix}\right) \neq 0$ be the eigenvector of $\mathscr{LB}_{2m}$ corresponding to m, and then $\mathscr{LB}_{2m}\xi_m = m\xi_m$ expands to

$$\begin{pmatrix} I_m + U_m & U_m^t \\ I_m + U_m^t & U_m \end{pmatrix}\begin{pmatrix} v \\ w \end{pmatrix} = 0. \quad (30)$$

Equation (30) is equivalent to the following equation:

$$\begin{pmatrix} I_m & \left(I+U_m\right)^{-1}U_m^t \\ 0_m & U_m - \left(I_m - U_m^t\right)\left(I+U_m\right)^{-1}U_m^t \end{pmatrix}\begin{pmatrix} v \\ w \end{pmatrix} = 0. \quad (31)$$

By simple calculation,

$$\begin{gathered} \left(I+U_m\right)^{-1}U_m^t = \begin{pmatrix} -I_{m-1} & 0 \\ \mathbf{1}_{m-1}^t & 0 \end{pmatrix}, \\ U_m - \left(I_m - U_m^t\right)\left(I+U_m\right)^{-1}U_m^t = \begin{pmatrix} J_{m-1} & \mathbf{1}_{m-1} \\ 0 & 0 \end{pmatrix}. \end{gathered} \quad (32)$$

We obtain the solution of (30) as follows:

$$\xi_m = \sum_{k=2}^{m} l_k \xi_{m_k}, \quad (33)$$

where l_k is an arbitrary constant, $k = 2, 3, \ldots, m$, and

$$\left(\xi_{m_2}, \xi_{m_3}, \ldots, \xi_{m_m}\right) = \begin{pmatrix} \mathbf{1}_{m-2}^t & 1 \\ -I_{m-2} & 0 \\ 0 & -1 \\ \mathbf{1}_{m-2}^t & 1 \\ -I_{m-2} & 0 \\ 0 & -1 \end{pmatrix}_{2m\times(m-1)}. \quad (34)$$

For $\lambda \neq m$, let $\xi = \left(\begin{smallmatrix} v \\ w \end{smallmatrix}\right) \neq 0$ be the eigenvector of $\mathscr{LB}_{2m}$ corresponding to $\lambda(\neq m)$, where $v = (v_1, v_2, \ldots, v_m)^t$, $w = (w_1, w_2, \ldots, w_m)^t$; then $\mathscr{LB}_{2m}\xi = \lambda\xi$ expands to

$$\begin{pmatrix} (m-1)I_m - U_m & -U_m^t \\ -I_m - U_m^t & mI_m - U_m \end{pmatrix}\begin{pmatrix} v \\ w \end{pmatrix} = \lambda\begin{pmatrix} v \\ w \end{pmatrix}. \quad (35)$$

By Lemma 7, $\mathbf{1}_m^t v + \mathbf{1}_m^t w \neq 1$; we put $\mathbf{1}_m^t v + \mathbf{1}_m^t w = 1$. By Theorem 1, $a = \mathbf{1}_m^t v = (1-\lambda)/2$, $b = \mathbf{1}_m^t w = 1 - a$.

Denote $g_0 = a + (1-a)(m-\lambda)$, $g_1 = g_2 = \cdots = g_{m-1} = 1$, and $g_m = -a(m-1-\lambda)$. By Lemma 6(2) we have

$$\begin{aligned} g(x) &= \sum_{k=1}^{m} w_k x^{k-1} \\ &= \left(a + (1-a)(m-\lambda) + x + x^2 + \cdots + x^{m-1} \right. \\ &\quad \left. -a(m-1-\lambda)x^m\right) \\ &\quad \times ((m-\lambda)((m+1-\lambda)-(m-1-\lambda)x))^{-1} \\ &= \frac{1}{c}\frac{1}{(1-dx)}\sum_{k=0}^{m} g_k x^k \\ &= \frac{1}{c}\sum_{k=0}^{\infty} d^k x^k \sum_{k=0}^{m} g_k x^k = \frac{1}{c}\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} g_j d^{k-j}\right)x^k. \end{aligned} \tag{36}$$

It must be that

$$w_k = \frac{1}{c}\sum_{j=0}^{k-1} g_j d^{k-1-j}, \quad k = 1, 2, \ldots, m. \tag{37}$$

Hence

$$\begin{aligned} w_k &= \frac{1}{c}\sum_{j=0}^{k-1} g_j d^{k-1-j} \\ &= \frac{1}{c}\left(\frac{1-\lambda+(1+\lambda)(m-\lambda)}{2}d^{k-1} + \sum_{j=0}^{k-2} d^j\right) \\ &= \frac{1}{c}\left(\frac{1-\lambda+(1+\lambda)(m-\lambda)}{2}d^{k-1} + \frac{1-d^{k-1}}{1-d}\right) \\ &= \frac{1}{2(m-\lambda)}\left(1+\lambda\left(\frac{m-1-\lambda}{m+1-\lambda}\right)^k\right), \quad k = 1, 2, \ldots, m. \end{aligned} \tag{38}$$

In the proof of Theorem 1, we have

$$\begin{aligned} v_k &= \frac{1}{c}\sum_{j=0}^{k-1} f_j d^{k-1-j} \\ &= \frac{1}{c}\left(\frac{(m+1-\lambda)(1-\lambda)}{2}d^{k-1} + \sum_{j=0}^{k-2} d^j\right) \\ &= \frac{1}{c}\left(\frac{(m+1-\lambda)(1-\lambda)}{2}d^{k-1} + \frac{1-d^{k-1}}{1-d}\right) \\ &= \frac{1}{2(m-\lambda)}\left(1-\lambda\left(\frac{m-1-\lambda}{m+1-\lambda}\right)^{k-1}\right), \quad k = 1, 2, \ldots, m. \end{aligned} \tag{39}$$

We complete the proof of Theorem 2.

Let λ be an eigenvalue of $\mathscr{LB}_{2m}$, and μ_λ and ν_λ are the algebraic multiplicity and the geometric multiplicity corresponding to λ, respectively.

If $m = 1$, by Theorem 1 and Theorem 2, and $\mu_0 = \nu_0 = \mu_1 = \nu_1 = 1$, then $\mathscr{LB}_2$ is diagonalizable.

If $m > 1$, by Theorem 1, $\mu_0 = 1$, $\mu_m = \lfloor (m-1)/2 \rfloor + \lfloor (m+1)/2 \rfloor$, and $\mu_{\lambda_k} = \mu_{\overline{\lambda_k}} = 1$, where $\lambda_k = m - i\cot((\pi + 2k\pi)/2m)$, $k = 0, 1, 2, \cdots, \lfloor m/2 \rfloor - 1$.

By Theorem 2, $\nu_0 = 1$, $\nu_m = m-1$, and $\nu_{\lambda_k} = \nu_{\overline{\lambda_k}} = 1$. We have the following:

> if $m > 1$ is odd, then $m = \mu_m \neq \nu_m = m-1$;
>
> if m is even, then $\mu_\lambda = \nu_\lambda$, where $\lambda = 0, m, \lambda_k, \overline{\lambda_k}$, $k = 0, 1, 2, \ldots, \lfloor m/2 \rfloor - 1$.

By Lemma 8(2), Corollary 4 holds.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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