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Research Article

T-S Fuzzy Model-Based Approximation and Filter Design for Stochastic Time-Delay Systems with Hankel Norm Criterion

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This paper investigates the Hankel norm filter design problem for stochastic time-delay systems, which are represented by Takagi-Sugeno (T-S) fuzzy model. Motivated by the parallel distributed compensation (PDC) technique, a novel filtering error system is established. The objective is to design a suitable filter that guarantees the corresponding filtering error system to be mean-square asymptotically stable and to have a specified Hankel norm performance level γ. Based on the Lyapunov stability theory and the Itô differential rule, the Hankel norm criterion is first established by adopting the integral inequality method, which can make some useful efforts in reducing conservativeness. The Hankel norm filtering problem is casted into a convex optimization problem with a convex linearization approach, which expresses all the conditions for the existence of admissible Hankel norm filter as standard linear matrix inequalities (LMIs). The effectiveness of the proposed method is demonstrated via a numerical example.

1. Introduction

The filtering problem can be briefly described as the design of an estimator from the measured output to estimate the state of the given systems and plays an important role in control fields and signal processing. During the last decades, various methodologies have been developed for the filter designs, such as Kalman filter [1, 2], H_∞ filter [3, 4], and H_2 or H_2/H_∞ filter [5, 6]. To mention a few, the earlier appeared Kalman filter is based on the precise noise statistics, while H_∞ filter can be designed without the statistical assumption on the noise signals. With the continuous development of filtering technology, research on the above filtering methods has made a lot of achievements. In recent years, more and more scholars pay their attentions to other performance index, such as L_1, L_2–L_∞, and Hankel norm, where the analysis of Hankel norm takes the effects of past inputs on the future outputs into account. Since the inputs and outputs of the plants for actual control systems change over time, environment and any other factors, the past inputs will affect the future outputs, which is one issue need to consider in the filtering analysis. Therefore, the study on Hankel norm filter has significance of theoretical guidance and engineering application.

On another research frontline, a great number of results on stochastic systems have been reported since stochastic modeling has come to be a key part in many branches of science and engineering. As far as we know, the study of stochastic systems mainly focusses on the stability analysis [7, 8], controller design [9, 10], filtering [11], model reduction [12] and fault detection [13], and so forth. Among them, the literature [8] proposed some sufficient conditions to ensure that the stochastic interval delay system is exponentially stable by using the Razumikhin-type theorem, and the robust H_∞ control and filtering problem for a class of uncertain stochastic time-delay systems were discussed in [9, 11], respectively. In the literature [12], the Hankel norm gain criterion of model reduction was established for neutral stochastic time-delay systems by using the projection lemma. For the existence of nonlinearity and unknown measured noise as well as stochastic perturbation, researchers have proposed different methods as data-driven approach [14, 15] and fault tolerant control with an iterative optimization scheme [16]. It is noted that the research on the filtering problem for stochastic time-delay systems has great significance and the major works are obtained with H_∞ performance, while being relatively less with other performance constraints, especially Hankel norm.

As well known, a significant body of research on the aforementioned filter design problem has been investigated up to now and the closely related results of nonlinear systems are also fruitful with the T-S fuzzy model approach. Over the past few years, the T-S fuzzy model has been recognised as a powerful tool in approximating complex nonlinear systems to a number of linear subsystems by employing piecewise smooth membership functions. It has been proved that some stability analysis and synthesis methods in the linear systems can be effectively extended to the T-S fuzzy systems [17, 18]. Through the T-S fuzzy model approach, the filtering problem for nonlinear systems has undergone a fast development in recent year. Some results are cited in the study [19, 20], where the literature [19] considered both continuous and differential uniformly bounded time-varying delays and proposed some novel delay-dependent H_∞ filtering criteria for nonlinear systems via a T-S fuzzy model approach, and [20] is concerned with the design problem of H_∞ filter for continuous T-S fuzzy systems based on the delay partitioning idea. However, it should be pointed out that the mentioned results are mostly established with the induced norms, such as H_2 and H_∞, while more and more researchers have switched their interests to Hankel norm very recently. Different from other norms, the analysis of the Hankel norm included both the past inputs and the future outputs. By estimating the effect of the system past inputs on the system future outputs, the Hankel norm can be used to achieve the system performance analysis more efficiently. So far, the applications of the Hankel norm is mainly in system model reduction [12, 21, 22]. To the best of the authors' knowledge, the Hankel norm filtering problem for T-S model-based stochastic time-delay systems has not been investigated, which motivates the current research.

The goal of this paper is to design a robust Hankel norm filter for stochastic time-delay systems. Firstly, based on the T-S fuzzy model approximation and the parallel distributed compensation (PDC) technique, a novel filtering error system is established. Then, two appropriate Lyapunov-Krasovskii functions are chosen for the stability and Hankel norm performance analysis. By using the Itô differential rule and the integral inequality method, the Hankel norm criterion is first proposed for the existence of admissible filter that guarantees the mean-square asymptotic stability and Hankel norm performance of the corresponding filtering error system. Finally, the existence conditions of the admissible Hankel norm filter can be expressed as LMIs and the filter parameters are obtained by using standard numerical software. An example is illustrated to show the efficiency of the proposed filter design methods.

The notation used in this paper is standard. $\mathbb{R}^n$ denotes the n-dimensional real Euclidean space, $\mathbb{R}^{n\times m}$ is the set of $n \times m$ real matrices. $\mathbb{N}$ denotes the natural numbers set. The notation X^T and X^{-1} denote its transpose and inverse when it exists, respectively. Given a symmetric matrix $X = X^T$, the notation $X > 0$ ($X \geq 0$) means that the matrix X is real positive definiteness (semidefiniteness). By *diag* we denote a block diagonal matrix with its input arguments on the diagonal. I denotes the identity matrix. The symbol $*$ within a matrix represents the symmetric entries. $L_2[0,\infty)$ denotes the space of square integrable functions over $[0,\infty)$. The notation $\mathscr{E}\{\cdot\}$ stands for the expectation operator.

2. Problem Statement

Consider a stochastic time-delay system which could be approximated by a T-S fuzzy model with r plant rules.

Plant Rule i. If $\theta_1(t)$ is W_{i1}, $\theta_2(t)$ is W_{i2} and...and $\theta_g(t)$ is W_{ig}, then

$$\begin{aligned} dx(t) &= [A_i x(t) + A_{di} x(t-\tau) + B_i v(t)]\,dt \\ &\quad + [M_i x(t) + M_{di} x(t-\tau) + N_i v(t)]\,d\omega(t), \\ dy(t) &= [C_i x(t) + C_{di} x(t-\tau) + D_i v(t)]\,dt \\ &\quad + [E_i x(t) + E_{di} x(t-\tau) + F_i v(t)]\,d\omega(t), \\ z(t) &= L_i x(t), \\ x(t) &= 0, \quad t \in [-\tau, 0], \quad i = 1,2,\ldots,r, \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $y(t) \in \mathbb{R}^m$ is the measured output signal, and $v(t) \in \mathbb{R}^p$ is the exogenous disturbance that is assumed to be an arbitrary signal belonging to $L_2[0,\infty)$. $z(t) \in \mathbb{R}^q$ is the signal to be estimated. $\omega(t)$ is a zero-mean real scalar Wiener process on $(\Omega, \mathscr{F}, \mathscr{P})$. And $\mathscr{E}\{d\omega(t)\} = 0$, $\mathscr{E}\{d\omega^2(t)\} = 0$. τ is the time delay and is assumed to be constant in the whole dynamic process. $\theta(t) = [\theta_1(t), \theta_2(t), \ldots, \theta_g(t)]$ is the premise variables vector, W_{ij} ($i = 1,2,\ldots,r$, $j = 1,2,\ldots,g$) is the fuzzy set, and r is the number of IF-THEN rules. A_i, A_{di}, B_i, M_i, M_{di}, N_i, C_i, C_{di}, D_i, E_i, E_{di}, F_i, and L_i are known constant matrices with appropriate dimensions.

The fuzzy system (1) is supposed to have singleton fuzzifier, product inference, and centroid difuzzifier. The final output of the fuzzy system is inferred as follows:

$$\begin{aligned} dx(t) &= \sum_{i=1}^{r} h_i(\theta(t)) \\ &\quad \times \{[A_i x(t) + A_{di} x(t-\tau) + B_i v(t)]\,dt \\ &\quad + [M_i x(t) + M_{di} x(t-\tau) + N_i v(t)]\,d\omega(t)\}, \\ dy(t) &= \sum_{i=1}^{r} h_i(\theta(t)) \\ &\quad \times \{[C_i x(t) + C_{di} x(t-\tau) + D_i v(t)]\,dt \\ &\quad + [E_i x(t) + E_{di} x(t-\tau) + F_i v(t)]\,d\omega(t)\}, \\ z(t) &= \sum_{i=1}^{r} h_i(\theta(t))\, L_i x(t), \\ x(t) &= 0, \quad t \in [-\tau, 0], \quad i = 1,2,\ldots,r, \end{aligned} \tag{2}$$

where

$$h_i(\theta(t)) = \frac{\mu_i(\theta(t))}{\sum_{i=1}^{r} \mu_i(\theta(t))}, \quad \mu_i(\theta(t)) = \prod_{j=1}^{g} W_{ij}(\theta_j(t)), \tag{3}$$

and $W_{ij}(\theta_j(t))$ representing the grade of membership of $\theta_j(t)$ in W_{ij}. Here, for all t, $h_i(\theta(t)) \geq 0$ and $\sum_{i=1}^{r} h_i(\theta(t)) = 1$.

In this paper, we will design the following Hankel norm filter by employing the parallel distributed compensation technique.

Filter Rule i. If $\theta_1(t)$ is W_{i1}, $\theta_2(t)$ is W_{i2} and...and $\theta_g(t)$ is W_{ig}, then

$$d\hat{x}(t) = A_{fi}\hat{x}(t) + B_{fi}dy(t), \quad \hat{z}(t) = C_{fi}\hat{x}(t), \tag{4}$$

where $\hat{x}(t) \in \mathbb{R}^n$ and $\hat{z}(t) \in \mathbb{R}^q$ are the state and output of the filter, respectively. The matrices A_{fi}, B_{fi}, and C_{fi} are filter parameters to be determined.

The defuzzified output of (4) is referred by

$$d\hat{x}(t) = \sum_{i=1}^{r} h_i(\theta(t)) \{A_{fi}\hat{x}(t) + B_{fi}dy(t)\}$$
$$\hat{z}(t) = \sum_{i=1}^{r} h_i(\theta(t)) C_{fi}\hat{x}(t). \tag{5}$$

Defining the augmented state vector $\xi^T(t) = [x^T(t) \ \hat{x}^T(t)]$ and $e(t) = z(t) - \hat{z}(t)$, then the filtering error system can be written in the following form:

$$\begin{aligned} d\xi(t) &= [\overline{A}(t)\xi(t) + \overline{A}_d(t)G\xi(t-\tau) + \overline{B}(t)v(t)]\,dt \\ &\quad + [\overline{M}(t)\xi(t) + \overline{M}_d(t)G\xi(t-\tau) + \overline{N}(t)v(t)]\,d\omega(t), \\ e(t) &= \overline{L}(t)\xi(t), \end{aligned} \tag{6}$$

where $G = [I \ \ 0]$,

$$\begin{aligned}
\overline{A}(t) &= \sum_{i=1}^{r} h_i(\theta(t)) \sum_{j=1}^{r} h_j(\theta(t)) \begin{bmatrix} A_j & 0 \\ B_{fi}C_j & A_{fi} \end{bmatrix} = \begin{bmatrix} A(t) & 0 \\ B_f(t)C(t) & A_f(t) \end{bmatrix}, \\
\overline{A}_d(t) &= \sum_{i=1}^{r} h_i(\theta(t)) \sum_{j=1}^{r} h_j(\theta(t)) \begin{bmatrix} A_{dj} \\ B_{fi}C_{dj} \end{bmatrix} = \begin{bmatrix} A_d(t) \\ B_f(t)C_d(t) \end{bmatrix}, \\
\overline{B}(t) &= \sum_{i=1}^{r} h_i(\theta(t)) \sum_{j=1}^{r} h_j(\theta(t)) \begin{bmatrix} B_j \\ B_{fi}D_j \end{bmatrix} = \begin{bmatrix} B(t) \\ B_f(t)D(t) \end{bmatrix}, \\
\overline{M}(t) &= \sum_{i=1}^{r} h_i(\theta(t)) \sum_{j=1}^{r} h_j(\theta(t)) \begin{bmatrix} M_j & 0 \\ B_{fi}E_j & 0 \end{bmatrix} = \begin{bmatrix} M(t) & 0 \\ B_f(t)E(t) & 0 \end{bmatrix}, \\
\overline{M}_d(t) &= \sum_{i=1}^{r} h_i(\theta(t)) \sum_{j=1}^{r} h_j(\theta(t)) \begin{bmatrix} M_{dj} \\ B_{fi}E_{dj} \end{bmatrix} = \begin{bmatrix} M_d(t) \\ B_f(t)E_d(t) \end{bmatrix}, \\
\overline{N}(t) &= \sum_{i=1}^{r} h_i(\theta(t)) \sum_{j=1}^{r} h_j(\theta(t)) \begin{bmatrix} N_j \\ B_{fi}F_j \end{bmatrix} = \begin{bmatrix} N(t) \\ B_f(t)F(t) \end{bmatrix}, \\
\overline{L}(t) &= \sum_{i=1}^{r} h_i(\theta(t)) \sum_{j=1}^{r} h_j(\theta(t)) [L_j \ \ -C_{fi}] = [L(t) \ \ -C_f(t)].
\end{aligned} \tag{7}$$

The Hankel norm filtering problem addressed in this paper can be expressed as follows.

Given a scalar $\gamma > 0$, determine the matrices A_{fi}, B_{fi}, and C_{fi} to find a suitable filter in the form of (5) such that

(i) the filtering error system (6) with $v(t) = 0$ is mean-square asymptotically stable;

(ii) subjected to the zero initial condition ($\xi(t) = 0$, for all $t \leq 0$)

$$\mathcal{E}\left\{\int_{\mathcal{T}}^{\infty} e^T(t)e(t)\,dt\right\} < \gamma^2 \int_0^{\mathcal{T}} v^T(t)v(t)\,dt; \tag{8}$$

for all $v(t) \in L_2[0,\infty)$ with $v(t) = 0$, for all $t \geq \mathcal{T}$.

Then, the filtering error system (6) is said to be mean-square asymptotically stable with a Hankel norm performance level γ.

Lemma 1. *Given matrix $R = R^T \geq 0$ and scalar $\tau > 0$, $\overline{y}(t)$ is a vector function which satisfies $\overline{y}(t)dt = d\xi(t)$, then*

$$\begin{aligned} &-\tau \int_{t-\tau}^{t} \overline{y}^T(s) R \overline{y}(s)\,ds \\ &\leq [\xi^T(t) \ \ \xi^T(t-\tau)] \begin{bmatrix} -R & R \\ R & -R \end{bmatrix} \begin{bmatrix} \xi(t) \\ \xi(t-\tau) \end{bmatrix}. \end{aligned} \tag{9}$$

Remark 2. $\overline{y}(t)$ in Lemma 1 is not equivalent to $\dot{\xi}(t)$ in deterministic time-delay systems and cannot be expressed by the known system parameters for the existence of the stochastic perturbation $d\omega(t)$. If $d\omega(t) = 0$, $\overline{y}(t) = \dot{\xi}(t)$.

3. Hankel Norm Performance Analysis

In this subsection, we will derive a sufficient condition for the existence of the Hankel norm filter that guarantees the filtering error system (6) to be mean-square asymptotically stable with a specified Hankel norm performance level. By making use of the Itô differential rule, the stochastic differentials of Lyapunov functions along the solution of system (6) are obtained and the integral inequality method is also used during the derivation. Based on these, the Hankel norm criterion of filtering problem is first established. Now, we will first give the following theorem which will play a key role in the derivation of our main results.

Theorem 3. *The filtering error system (6) is mean-square asymptotically stable and has a guaranteed Hankel norm performance γ if there exist $P_1 > 0$, $P_2 > 0$, $Q_1 > 0$, $Q_2 > 0$, $R_1 > 0$, $R_2 > 0$, and S_1, S_2 satisfying*

$$\begin{bmatrix} -P_1 & P_1\overline{M}(t) & P_1\overline{M}_d(t) & 0 & P_1\overline{N}(t) \\ * & P_1\overline{A}(t) + \overline{A}^T(t)P_1 + G^T\left(Q_1 - \frac{R_1}{\tau}\right)G & P_1\overline{A}_d(t) + G^T\frac{R_1}{\tau} & \overline{A}^T(t)G^TS_1 & P_1\overline{B}(t) \\ * & * & -Q_1 - \frac{R_1}{\tau} & \overline{A}_d^T(t)G^TS_1 & 0 \\ * & * & * & \tau R_1 - S_1 - S_1^T & S_1^TG\overline{B}(t) \\ * & * & * & * & -\gamma^2 I \end{bmatrix} < 0, \quad (10)$$

$$\begin{bmatrix} -P_2 & P_2\overline{M}(t) & P_2\overline{M}_d(t) & 0 & 0 \\ * & P_2\overline{A}(t) + \overline{A}^T(t)P_2 + G^T\left(Q_2 - \frac{R_2}{\tau}\right)G & P_2\overline{A}_d(t) + G^T\frac{R_2}{\tau} & \overline{A}^T(t)G^TS_2 & \overline{L}^T(t) \\ * & * & -Q_2 - \frac{R_2}{\tau} & \overline{A}_d^T(t)G^TS_2 & 0 \\ * & * & * & \tau R_2 - S_2 - S_2^T & 0 \\ * & * & * & * & -I \end{bmatrix} < 0, \quad (11)$$

$$P_1 - P_2 \geq 0, \quad (12)$$

$$G^T(Q_1 - Q_2)G \geq 0, \quad (13)$$

$$G^T(R_1 - R_2)G \geq 0. \quad (14)$$

Proof. Choose the Lyapunov-Krasovskii functionals as

$$\begin{aligned} V_1(\xi_t, t) = {} & \xi^T(t)P_1\xi(t) + \int_{t-\tau}^{t}\xi^T(\alpha)G^TQ_1G\xi(\alpha)\,d\alpha \\ & + \int_{-\tau}^{0}\int_{t+\beta}^{t}\overline{y}^T(\alpha)G^TR_1G\overline{y}(\alpha)\,d\alpha\,d\beta, \end{aligned} \quad (15)$$

$$\begin{aligned} V_2(\xi_t, t) = {} & \xi^T(t)P_2\xi(t) + \int_{t-\tau}^{t}\xi^T(\alpha)G^TQ_2G\xi(\alpha)\,d\alpha \\ & + \int_{-\tau}^{0}\int_{t+\beta}^{t}\overline{y}^T(\alpha)G^TR_2G\overline{y}(\alpha)\,d\alpha\,d\beta, \end{aligned} \quad (16)$$

where P_1, P_2, Q_1, Q_2, R_1, and R_2 are real symmetric positive definite matrices to be determined, $\xi_t = \xi(t + \iota)$, $-\tau \leq \iota \leq 0$. $\overline{y}(t)$ is defined as $\overline{y}(t)dt = d\xi(t)$, and according to the Newton-Leibniz formula, we have

$$\int_{t-\tau}^{t}\overline{y}(\alpha)\,d\alpha = \xi(t) - \xi(t-\tau). \quad (17)$$

Then by making use of the Itô differential rule, the stochastic differential $dV_1(\xi_t, t)$ along the solution of system (6) can be obtained as

$$\begin{aligned} dV_1(\xi_t, t) & = \mathscr{L}V_1(\xi_t, t)\,dt + 2\xi^T(t)P_1 \\ & \quad \times \left[\overline{M}(t)\xi(t) + \overline{M}_d(t)G\xi(t-\tau) + \overline{N}(t)v(t)\right]d\omega(t), \end{aligned} \quad (18)$$

where

$$\begin{aligned}
&\mathscr{L}V_1(\xi_t,t)\\
&= 2\xi^T(t)P_1\left[\overline{A}(t)\xi(t)+\overline{A}_d(t)G\xi(t-\tau)+\overline{B}(t)\nu(t)\right]\\
&\quad+\left[\overline{M}(t)\xi(t)+\overline{M}_d(t)G\xi(t-\tau)+\overline{N}(t)\nu(t)\right]^T\\
&\quad\times P_1\left[\overline{M}(t)\xi(t)+\overline{M}_d(t)G\xi(t-\tau)+\overline{N}(t)\nu(t)\right]\\
&\quad+\xi^T(t)G^TQ_1G\xi(t)\\
&\quad-\xi^T(t-\tau)G^TQ_1G\xi(t-\tau)+\tau\overline{y}^T(t)G^TR_1G\overline{y}(t)\\
&\quad-\int_{t-\tau}^{t}\overline{y}^T(s)G^TR_1G\overline{y}(s)\,ds.
\end{aligned}\tag{19}$$

Applying Lemma 1 to $\mathscr{L}V_1(\xi_t,t)$, we have

$$\begin{aligned}
&\mathscr{L}V_1(\xi_t,t)\\
&\le\xi^T(t)\left[2P_1\overline{A}(t)+\overline{M}^T(t)P_1\overline{M}(t)+G^T\left(Q_1-\frac{R_1}{\tau}\right)G\right]\\
&\quad\times\xi(t)+\xi^T(t)\left[2P_1\overline{A}_d(t)+\overline{M}^T(t)P_1\overline{M}_d(t)+G^T\frac{R_1}{\tau}\right]\\
&\quad\times G\xi(t-\tau)\\
&\quad+\xi^T(t)\left[2P_1\overline{B}(t)+\overline{M}^T(t)P_1\overline{N}(t)\right]\nu(t)\\
&\quad+\xi^T(t-\tau)G^T\left[\overline{M}_d^T(t)P_1\overline{M}(t)+\frac{R_1}{\tau}G\right]\xi(t)\\
&\quad+\xi^T(t-\tau)G^T\\
&\quad\times\left[\overline{M}_d^T(t)P_1\overline{M}_d(t)-\left(Q_1+\frac{R_1}{\tau}\right)\right]G\xi(t-\tau)\\
&\quad+\xi^T(t-\tau)G^T\\
&\quad\times\left[\overline{M}_d^T(t)P_1\overline{N}(t)\right]\nu(t)+\nu^T(t)\left[\overline{N}^T(t)P_1\overline{M}(t)\right]\xi(t)\\
&\quad+\nu^T(t)\left[\overline{N}^T(t)P_1\overline{M}_d(t)\right]G\xi(t-\tau)\\
&\quad+\nu^T(t)\left[\overline{N}^T(t)P_1\overline{N}(t)\right]\nu(t)\\
&\quad+\tau\overline{y}^T(t)G^TR_1G\overline{y}(t).
\end{aligned}\tag{20}$$

Noting that $\overline{y}(t)dt = d\xi(t)$ and system (6), for arbitrary matrix $S_1\in\mathbb{R}^{n\times n}$ it can be seen that

$$\begin{aligned}
0&=2\overline{y}^T(t)G^TS_1^TG\\
&\quad\times\left\{\left[\overline{M}(t)\xi(t)\right.\right.\\
&\qquad\left.+\overline{M}_d(t)G\xi(t-\tau)+\overline{N}(t)\nu(t)\right]d\omega(t)\\
&\qquad+\left[\overline{A}(t)\xi(t)+\overline{A}_d(t)G\xi(t-\tau)\right.\\
&\qquad\left.\left.+\overline{B}(t)\nu(t)-\overline{y}(t)\right]dt\right\}.
\end{aligned}\tag{21}$$

Thus, it follows from (18) and (21) that

$$\begin{aligned}
dV_1(\xi_t,t)&=\mathscr{L}\widetilde{V}_1(\xi_t,t)\,dt\\
&\quad+2\left[\xi^T(t)P_1+\overline{y}^T(t)G^TS_1^TG\right]\\
&\quad\times\left[\overline{M}(t)\xi(t)+\overline{M}_d(t)G\xi(t-\tau)+\overline{N}(t)\nu(t)\right]\\
&\quad\times d\omega(t),
\end{aligned}\tag{22}$$

where

$$\begin{aligned}
&\mathscr{L}\widetilde{V}_1(\xi_t,t)\\
&=\mathscr{L}V_1(\xi_t,t)+2\overline{y}^T(t)G^TS_1^TG\\
&\quad\times\left[\overline{A}(t)\xi(t)+\overline{A}_d(t)G\xi(t-\tau)\right.\\
&\qquad\left.+\overline{B}(t)\nu(t)-\overline{y}(t)\right]\\
&\le\xi^T(t)\left[2P_1\overline{A}(t)+\overline{M}^T(t)P_1\overline{M}(t)\right.\\
&\qquad\left.+G^T\left(Q_1-\frac{R_1}{\tau}\right)G\right]\\
&\quad\times\xi(t)+2\xi^T(t)\left[P_1\overline{A}_d(t)+\overline{M}^T(t)P_1\overline{M}_d(t)\right.\\
&\qquad\left.+G^T\frac{R_1}{\tau}\right]G\xi(t-\tau)\\
&\quad+2\xi^T(t)\left[\overline{A}(t)^TG^TS_1\right]G\overline{y}(t)\\
&\quad+2\xi^T(t)\left[P_1\overline{B}(t)+\overline{M}^T(t)P_1\overline{N}(t)\right]\nu(t)\\
&\quad+\xi^T(t-\tau)G^T\\
&\quad\times\left[\overline{M}_d^T(t)P_1\overline{M}_d(t)-Q_1-\frac{R_1}{\tau}\right]\\
&\quad\times G\xi(t-\tau)+2\xi^T(t-\tau)G^T\\
&\quad\times\left[\overline{A}_d^T(t)G^TS_1\right]G\overline{y}(t)\\
&\quad+2\xi^T(t-\tau)G^T\left[\overline{M}_d^T(t)P_1\overline{N}(t)\right]\nu(t)\\
&\quad+\overline{y}^T(t)G^T\left[\tau R_1-S_1-S_1^T\right]G\overline{y}(t)\\
&\quad+2\overline{y}^T(t)G^T\left[S_1^TG\overline{B}(t)\right]\nu(t)\\
&\quad+\nu^T(t)\left[\overline{N}^T(t)P_1\overline{N}(t)\right]\nu(t).
\end{aligned}\tag{23}$$

Therefore, when assuming zero input $\nu(t)=0$, it follows that

$$\mathscr{L}\widetilde{V}_1(\xi_t,t)\le\eta^T(t)\Theta_1\eta(t),\tag{24}$$

where

$$\eta^T(t) = \left[\xi^T(t) \;\; \xi^T(t-\tau)G^T \;\; \overline{y}^T(t)G^T\right],$$

$$\Theta_1 = \begin{bmatrix} \Pi_{11} & \Pi_{12} & \overline{A}^T(t)G^TS_1 \\ * & \Pi_{22} & \overline{A}_d^T(t)G^TS_1 \\ * & * & \tau R_1 - S_1 - S_1^T \end{bmatrix},$$

$$\Pi_{11} = P_1\overline{A}(t) + \overline{A}^T(t)P_1 + \overline{M}^T(t)P_1\overline{M}(t) + G^T\left(Q_1 - \frac{R_1}{\tau}\right)G, \tag{25}$$

$$\Pi_{12} = P_1\overline{A}_d(t) + \overline{M}^T(t)P_1\overline{M}_d(t) + G^T\frac{R_1}{\tau},$$

$$\Pi_{22} = \overline{M}_d^T(t)P_1\overline{M}_d(t) - Q_1 - \frac{R_1}{\tau}.$$

By using the Schur complement lemma, the inequality (10) implies the negative definiteness of Θ_1. Then, we have $\mathscr{L}\widetilde{V}_1(\xi_t,t) < 0$, and the filtering error system (6) with $v(t) = 0$ is guaranteed to be mean-square asymptotically stable. And the next step is to establish the Hankel norm performance:

$$\mathscr{E}\left\{\int_{\mathscr{T}}^{\infty} e^T(t)e(t)\,dt\right\} < \gamma^2\int_0^{\mathscr{T}} v^T(t)v(t)\,dt, \tag{26}$$

under zero initial condition and $v(t) \in L_2[0,\infty)$ with $v(t) = 0$, for all $t \geq \mathscr{T}$.

For any nonzero $v(t) \in L_2[0,\infty)$ with $v(t) = 0$, for all $t \geq \mathscr{T}$, the inequality of (23) can be rewritten in the following quadratic form:

$$\mathscr{L}\widetilde{V}_1(\xi_t,t) \leq \zeta^T(t)\widetilde{\Theta}_1\zeta(t), \tag{27}$$

where

$$\zeta^T(t) = \left[\xi^T(t) \;\; \xi^T(t-\tau)G^T \;\; \overline{y}^T(t)G^T \;\; v^T(t)\right],$$

$$\widetilde{\Theta}_1 = \begin{bmatrix} \Pi_{11} & \Pi_{12} & \overline{A}^T(t)G^TS_1 & P_1\overline{B}(t) + \overline{M}^T(t)P_1\overline{N}(t) \\ * & \Pi_{22} & \overline{A}_d^T(t)G^TS_1 & M_d^T(t)P_1\overline{N}(t) \\ * & * & \tau R_1 - S_1 - S_1^T & S_1^TG\overline{B}(t) \\ * & * & * & \overline{N}^T(t)P_1\overline{N}(t) \end{bmatrix}. \tag{28}$$

The inequalities (10) and (27) imply

$$\mathscr{L}\widetilde{V}_1(\xi_t,t) - \gamma^2v^T(t)v(t) \leq \zeta^T(t)\widetilde{\Theta}_1\zeta(t) - \gamma^2v^T(t)v(t) < 0. \tag{29}$$

Integrating both sides of (22) and (29), respectively, from 0 to $\mathscr{T}$ and then taking expectation, we have

$$\mathscr{E}\{V_1(\xi_{\mathscr{T}},\mathscr{T})\} = \mathscr{E}\left\{\int_0^{\mathscr{T}} \mathscr{L}\widetilde{V}_1(\xi_t,t)\,dt\right\} < \gamma^2\int_0^{\mathscr{T}} v^T(t)v(t)\,dt, \tag{30}$$

where zero initial condition is used.

Second, introduce $V_2(\xi_t,t)$ in (16). By following similar lines as above, it is not difficult to obtain the stochastic differential $dV_2(\xi_t,t)$ as

$$dV_2(\xi_t,t) = \mathscr{L}\widetilde{V}_2(\xi_t,t)\,dt + 2\left[\xi^T(t)P_2 + \overline{y}^T(t)G^TS_2^TG\right] \times \left[\overline{M}(t)\xi(t) + \overline{M}_d(t)G\xi(t-\tau) + \overline{N}(t)v(t)\right] \times d\omega(t), \tag{31}$$

where

$$\mathscr{L}\widetilde{V}_2(\xi_t,t) \leq \zeta^T(t)\widetilde{\Theta}_2\zeta(t),$$

$$\widetilde{\Theta}_2 = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \overline{A}^T(t)G^TS_2 & P_2\overline{B}(t) + \overline{M}^T(t)P_2\overline{N}(t) \\ * & \Gamma_{22} & \overline{A}_d^T(t)G^TS_2 & M_d^T(t)P_2\overline{N}(t) \\ * & * & \tau R_2 - S_2 - S_2^T & S_2^TG\overline{B}(t) \\ * & * & * & \overline{N}^T(t)P_2\overline{N}(t) \end{bmatrix},$$

$$\Gamma_{11} = P_2\overline{A}(t) + \overline{A}^T(t)P_2 + \overline{M}^T(t)P_2\overline{M}(t) + G^T\left(Q_2 - \frac{R_2}{\tau}\right)G,$$

$$\Gamma_{12} = P_2\overline{A}_d(t) + \overline{M}^T(t)P_2\overline{M}_d(t) + G^T\frac{R_2}{\tau},$$

$$\Gamma_{22} = \overline{M}_d^T(t)P_2\overline{M}_d(t) - Q_2 - \frac{R_2}{\tau}, \tag{32}$$

By Schur complement lemma, the inequality (11) is equivalent to

$$\begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \overline{A}^T(t)G^TS_2 \\ * & \Gamma_{22} & \overline{A}_d^T(t)G^TS_2 \\ * & * & \tau R_2 - S_2 - S_2^T \end{bmatrix} + \begin{bmatrix} \overline{L}^T(t) \\ 0 \\ 0 \end{bmatrix}\left[\overline{L}(t) \;\; 0 \;\; 0\right] < 0. \tag{33}$$

Thus, we have

$$\begin{aligned}\Upsilon = {} & \xi^T(t)\Big[P_2\overline{A}(t) + \overline{A}^T(t)P_2 + \overline{M}^T(t)P_2\overline{M}(t) + G^T\left(Q_2 - \frac{R_2}{\tau}\right)G\Big]\xi(t) \\ & + 2\xi^T(t)\left[P_2\overline{A}_d(t) + \overline{M}^T(t)P_2\overline{M}_d(t) + G^T\frac{R_2}{\tau}\right] \times G\xi(t-\tau) \\ & + 2\xi^T(t)\left[\overline{A}(t)^TG^TS_2\right]G\overline{y}(t) \\ & + \xi^T(t-\tau)G^T\left[\overline{M}_d^T(t)P_2\overline{M}_d(t) - Q_2 - \frac{R_2}{\tau}\right] \times G\xi(t-\tau) \\ & + 2\xi^T(t-\tau)G^T\left[\overline{A}_d^T(t)G^TS_2\right]G\overline{y}(t)\end{aligned}$$

$$
\begin{aligned}
&+\bar{y}^T(t)G^T\left[\tau R_2 - S_2 - S_2^T\right]G\bar{y}(t)\\
&+\left[\bar{L}(t)\xi(t)\right]^T\left[\bar{L}(t)\xi(t)\right]\\
&< 0.
\end{aligned} \tag{34}
$$

By considering $v(t) = 0$, for all $t \geq \mathcal{T}$ and (32), for any $t \geq \mathcal{T}$, inequalities (32) and (34) guarantee

$$\mathscr{L}\widetilde{V}_2(\xi_t, t) + e^T(t)e(t) < 0, \quad \forall t \geq \mathcal{T}. \tag{35}$$

Integrating both sides of (31) and (35), respectively, from $\mathcal{T}$ to ∞ and then taking expectation, we have

$$\mathscr{E}\left\{\int_{\mathcal{T}}^{\infty}\mathscr{L}\widetilde{V}_2(\xi_t, t)\,dt\right\} + \mathscr{E}\left\{\int_{\mathcal{T}}^{\infty}e^T(t)e(t)\,dt\right\} < 0. \tag{36}$$

Due to $\mathscr{E}\{\int_{\mathcal{T}}^{\infty}\mathscr{L}\widetilde{V}_2(\xi_t, t)dt\} = \mathscr{E}\{V_2(\xi_\infty, \infty)\} - \mathscr{E}\{V_2(\xi_{\mathcal{T}}, \mathcal{T})\}$ and $\mathscr{E}\{V_2(\xi_\infty, \infty)\} \geq 0$, then

$$\mathscr{E}\left\{\int_{\mathcal{T}}^{\infty}e^T(t)e(t)\,dt\right\} < \mathscr{E}\{V_2(\xi_{\mathcal{T}}, \mathcal{T})\}. \tag{37}$$

By considering (12), (13), (14), (30), and (37), we obtain (26), the proof is concluded. □

Remark 4. For general continuous time stochastic time-delay systems, the delay-independent results can be obtained by choosing the following form of Lyapunov functional:

$$V(t) = \xi^T(t)P\xi(t) + \int_{t-\tau}^{t}\xi^T(s)Q\xi(s)\,ds. \tag{38}$$

However, the presence of stochastic perturbation (Wiener process) in the stochastic time-delay systems makes $\dot{\xi}(t)$ undefined and the above function is not suitable for its large conservative. Thus, we adopt the Lyapunov functionals in the form of (15) and (16) in the original version and obtain delay-dependent criterion of filtering problem for stochastic time-delay systems. It should be pointed out that the Lyapunov functions are chosen with constant delay τ in this paper, and the proposed method can be also extended to the case of time-varying delay $\tau(t)$, which can have more conservative results.

Remark 5. Theorem 3 provides a delay-dependent sufficient condition of the robustly mean-square asymptotic stability with a Hankel norm performance level γ for the filtering error system (6). By introducing the assistant vector $\bar{y}(t)$ and free-weighting matrices S_i, the derivation of the above theorem is completed without using any model transformations and cross terms bounding techniques. The introduction of S_i helps establishing the contact of $\xi(t)$, $\bar{y}(t)$, and $\xi(t-\tau)$ and then the delay-dependent results are obtained. This approach has been proved to be less conservative.

4. Hankel Norm Filter Design

In this section, we will provide the solution to Hankel norm filtering problem for stochastic time-delay systems. As mentioned above, Theorem 3 gives a sufficient condition for the existence of a filter that guarantees the filtering error system mean-square asymptotically stable with Hankel norm performance. However, the inequalities (10) and (11) in Theorem 3 cannot be solved directly for the coupled matrix variables. To solve this problem, we will make decoupling process and adopt the convex linearization approach to transform (10) and (11) into LMI forms, which can be solved easily with the standard numerical software.

Theorem 6. *For the given positive constants $\tau > 0$ and $0 < \alpha \leq 1$, an admissible Hankel norm filter in the form of (5) exists such that the filtering error system (6) is mean-square asymptotically stable and has a guaranteed Hankel norm performance level γ if there exist $X > 0$, $Y > 0$, $Q > 0$, $R > 0$, S, $\overline{A}_f(t)$, $\overline{B}_f(t)$, and $\overline{C}_f(t)$ satisfying*

$$
\begin{bmatrix}
-X & -Y & \Phi_{13} & 0 & \Phi_{15} & 0 & \Phi_{17}\\
* & -Y & \Phi_{23} & 0 & \Phi_{25} & 0 & \Phi_{27}\\
* & * & \Phi_{33} & \Phi_{34} & \Phi_{35} & A^T(t)S & \Phi_{37}\\
* & * & * & \Phi_{44} & \Phi_{45} & 0 & \Phi_{47}\\
* & * & * & * & \Phi_{55} & A_d^T(t)S & 0\\
* & * & * & * & * & \Phi_{66} & S^TB(t)\\
* & * & * & * & * & * & -\gamma^2 I
\end{bmatrix} < 0, \tag{39}
$$

$$
\begin{bmatrix}
-\alpha X & -\alpha Y & \Psi_{13} & 0 & \Psi_{15} & 0 & 0\\
* & -\alpha Y & \Psi_{23} & 0 & \Psi_{25} & 0 & 0\\
* & * & \Psi_{33} & \Psi_{34} & \Psi_{35} & \alpha A^T(t)S & L^T(t)\\
* & * & * & \Psi_{44} & \Psi_{45} & 0 & -\overline{C}_f^T(t)\\
* & * & * & * & \Psi_{55} & \alpha A_d^T(t)S & 0\\
* & * & * & * & * & \Psi_{66} & 0\\
* & * & * & * & * & * & -I
\end{bmatrix} < 0, \tag{40}
$$

where

$$
\begin{aligned}
\Phi_{13} &= XM(t) + \overline{B}_f(t)E(t),\\
\Phi_{15} &= XM_d(t) + \overline{B}_f(t)E_d(t),\\
\Phi_{17} &= XN(t) + \overline{B}_f(t)F(t),\\
\Phi_{23} &= YM(t) + \overline{B}_f(t)E(t),\\
\Phi_{25} &= YM_d(t) + \overline{B}_f(t)E_d(t),\\
\Phi_{27} &= YN(t) + \overline{B}_f(t)F(t),\\
\Phi_{33} &= XA(t) + A^T(t)X + \overline{B}_f(t)C(t)\\
&\quad + C^T(t)\overline{B}_f^T(t) + Q - \frac{R}{\tau},
\end{aligned}
$$

$$\Phi_{34} = \overline{A}_f(t) + A^T(t)Y + C^T(t)\overline{B}_f^T(t),$$
$$\Phi_{35} = XA_d(t) + \overline{B}_f(t)C_d(t) + \frac{R}{\tau},$$
$$\Phi_{37} = XB(t) + \overline{B}_f(t)D(t),$$
$$\Phi_{44} = \overline{A}_f(t) + \overline{A}_f^T(t),$$
$$\Phi_{45} = YA_d(t) + \overline{B}_f(t)C_d(t),$$
$$\Phi_{47} = YB(t) + \overline{B}_f(t)D(t),$$
$$\Phi_{55} = -Q - \frac{R}{\tau},$$
$$\Phi_{66} = \tau R - S - S^T,$$
$$\Psi_{13} = \alpha\left(XM(t) + \overline{B}_f(t)E(t)\right),$$
$$\Psi_{15} = \alpha\left(XM_d(t) + \overline{B}_f(t)E_d(t)\right),$$
$$\Psi_{23} = \alpha\left(YM(t) + \overline{B}_f(t)E(t)\right),$$
$$\Psi_{25} = \alpha\left(YM_d(t) + \overline{B}_f(t)E_d(t)\right),$$
$$\Psi_{33} = \alpha\left(XA(t) + A^T(t)X + \overline{B}_f(t)C(t) + C^T(t)\overline{B}_f^T(t) + Q - \frac{R}{\tau}\right),$$
$$\Psi_{34} = \alpha\left(\overline{A}_f(t) + A^T(t)Y + C^T(t)\overline{B}_f^T(t)\right),$$
$$\Psi_{35} = \alpha\left(XA_d(t) + \overline{B}_f(t)C_d(t) + \frac{R}{\tau}\right),$$
$$\Psi_{44} = \alpha\left(\overline{A}_f(t) + \overline{A}_f^T(t)\right),$$
$$\Psi_{45} = \alpha\left(YA_d(t) + \overline{B}_f(t)C_d(t)\right),$$
$$\Psi_{55} = -\alpha\left(Q + \frac{R}{\tau}\right),$$
$$\Psi_{66} = \alpha\left(\tau R - S - S^T\right). \quad (41)$$

Proof. Inequality (39) implies $X > 0$ and $Y > 0$. For arbitrary symmetric positive definite matrix Y, one can always find a nonsingular matrix V and symmetric positive definite matrix W satisfying $Y = VW^{-1}V^T$. Now we introduce, respectively, the following matrix variables

$$P = \begin{bmatrix} X & V \\ V^T & W \end{bmatrix}, \qquad J_1 = \begin{bmatrix} I & 0 \\ 0 & W^{-1}V^T \end{bmatrix}. \quad (42)$$

By Schur complement lemma, we can infer from (39) that $X - VW^{-1}V^T = X - Y > 0$, and then $P > 0$.

Defining $P_1 = P$, $P_2 = \alpha P$, $Q_1 = Q$, $Q_2 = \alpha Q$, $R_1 = R$, $R_2 = \alpha R$, and applying the congruence transformation by matrix $\widehat{\Delta} = \operatorname{diag}\{J_1, J_1, I, I, I\}$ to (10) and (11), respectively, we can easily infer the following inequalities:

$$\begin{bmatrix} -X & -Y & \widetilde{\Phi}_{13} & 0 & \widetilde{\Phi}_{15} & 0 & \widetilde{\Phi}_{17} \\ * & -Y & \widetilde{\Phi}_{23} & 0 & \widetilde{\Phi}_{25} & 0 & \widetilde{\Phi}_{27} \\ * & * & \widetilde{\Phi}_{33} & \widetilde{\Phi}_{34} & \widetilde{\Phi}_{35} & A^T(t)S & \widetilde{\Phi}_{37} \\ * & * & * & \widetilde{\Phi}_{44} & \widetilde{\Phi}_{45} & 0 & \widetilde{\Phi}_{47} \\ * & * & * & * & -Q - \dfrac{R}{\tau} & A_d^T(t)S & 0 \\ * & * & * & * & * & \tau R - S - S^T & S^T B(t) \\ * & * & * & * & * & * & -\gamma^2 I \end{bmatrix} < 0,$$

$$\begin{bmatrix} -\alpha X & -\alpha Y & \widetilde{\Psi}_{13} & 0 & \widetilde{\Psi}_{15} & 0 & 0 \\ * & -\alpha Y & \widetilde{\Psi}_{23} & 0 & \widetilde{\Psi}_{25} & 0 & 0 \\ * & * & \widetilde{\Psi}_{33} & \widetilde{\Psi}_{34} & \widetilde{\Psi}_{35} & \alpha A^T(t)S & L^T(t) \\ * & * & * & \widetilde{\Psi}_{44} & \widetilde{\Psi}_{45} & 0 & -\overline{C}_f^T(t) \\ * & * & * & * & -\alpha\left(Q + \dfrac{R}{\tau}\right) & \alpha A_d^T(t)S & 0 \\ * & * & * & * & * & \alpha\left(\tau R - S - S^T\right) & 0 \\ * & * & * & * & * & * & -I \end{bmatrix} < 0, \quad (43)$$

where

$$\widetilde{\Phi}_{13} = XM(t) + VB_f(t)E(t),$$

$$\widetilde{\Phi}_{15} = XM_d(t) + VB_f(t)E_d(t),$$

$$\widetilde{\Phi}_{17} = XN(t) + VB_f(t)F(t),$$

$$\widetilde{\Phi}_{23} = YM(t) + VB_f(t)E(t),$$

$$\widetilde{\Phi}_{25} = YM_d(t) + VB_f(t)E_d(t),$$

$$\widetilde{\Phi}_{27} = YN(t) + VB_f(t)F(t),$$

$$\widetilde{\Phi}_{33} = XA(t) + A^T(t)X + VB_f(t)C(t) + C^T(t)B_f^T(t)V^T + Q - \frac{R}{\tau},$$

$$\widetilde{\Phi}_{34} = VA_f(t)W^{-1}V^T + A^T(t)Y + C^T(t)B_f^T(t)V^T,$$

$$\widetilde{\Phi}_{35} = XA_d(t) + VB_f(t)C_d(t) + \frac{R}{\tau},$$

$$\widetilde{\Phi}_{37} = XB(t) + VB_f(t)D(t),$$

$$\widetilde{\Phi}_{44} = VA_f(t)W^{-1}V^T + VW^{-1}A_f^T(t)V^T,$$

$$\widetilde{\Phi}_{45} = YA_d(t) + VB_f(t)C_d(t),$$

$$\widetilde{\Phi}_{47} = YB(t) + VB_f(t)D(t),$$

$$\widetilde{\Psi}_{13} = \alpha\left(XM(t) + VB_f(t)E(t)\right),$$

$$\widetilde{\Psi}_{15} = \alpha\left(XM_d(t) + VB_f(t)E_d(t)\right),$$

$$\widetilde{\Psi}_{23} = \alpha\left(YM(t) + VB_f(t)E(t)\right),$$

$$\widetilde{\Psi}_{25} = \alpha\left(YM_d(t) + VB_f(t)E_d(t)\right),$$

$$\widetilde{\Psi}_{33} = \alpha\left(XA(t) + A^T(t)X + VB_f(t)C(t) + C^T(t)B_f^T(t)V^T + Q - \frac{R}{\tau}\right),$$

$$\widetilde{\Psi}_{34} = \alpha\left(VA_f(t)W^{-1}V^T + A^T(t)Y + C^T(t)B_f^T(t)V^T\right),$$

$$\widetilde{\Psi}_{35} = \alpha\left(XA_d(t) + VB_f(t)C_d(t) + \frac{R}{\tau}\right),$$

$$\widetilde{\Psi}_{44} = \alpha\left(VA_f(t)W^{-1}V^T + VW^{-1}A_f^T(t)V^T\right),$$

$$\widetilde{\Psi}_{45} = \alpha\left(YA_d(t) + VB_f(t)C_d(t)\right). \tag{44}$$

Letting $\overline{A}_f(t) = VA_f(t)W^{-1}V^T$, $\overline{B}_f(t) = VB_f(t)$, $\overline{C}_f(t) = C_f(t)W^{-1}V^T$, we readily obtain (39) and (40). The proof is completed. □

Remark 7. It is noted that there exist different approaches to solve the Hankel norm filtering problem as mentioned above, such as the well-known projection lemma and the convex linearization approach. In this paper, the later approach is employed to solve the Hankel norm filtering problem. Compared with the projection lemma, the convex linearization approach has been proved to be less conservative. The contrast analysis of the two methods can be referred in the literature [19].

Remark 8. Although Theorem 6 overcome the coupled problem in Theorem 3, the inequalities (39) and (40) still cannot be used to solve the filter parameters in (5) directly. Therefore, the next step of using $\Delta(t) = \sum_{i=1}^{r} h_i(\theta(t))\Delta_i$ to substitute the matrix functions in Theorem 6 is necessary, where Δ denotes system matrices A, A_d, B, M, M_d, N, C, C_d, D, E, E_d, F, L, S and corresponding parameters $\overline{A}_f$, $\overline{B}_f$, $\overline{C}_f$. By this way, the following theorem is obtained to present the final results.

Theorem 9. *For the given positive constants $\tau > 0$ and $0 < \alpha \leq 1$, the filtering error system (6) is mean-square asymptotically stable and has a guaranteed Hankel norm performance level γ if there exist $X > 0$, $Y > 0$, $Q > 0$, $R > 0$, S_i, $\overline{A}_{fi}$, $\overline{B}_{fi}$, and $\overline{C}_{fi}$ $(i = 1, 2, \ldots, r)$ satisfying*

$$\Omega_1^{ij} + \Omega_1^{ji} < 0,$$
$$\Omega_2^{ij} + \Omega_2^{ji} < 0, \quad i \leq j, \tag{45}$$

where Ω_1^{ij} and Ω_2^{ij} are given as

$$\Omega_1^{ij} = \begin{bmatrix} -X & -Y & XM_j + \overline{B}_{fi}E_j & 0 & XM_{dj} + \overline{B}_{fi}E_{dj} & 0 & XN_j + \overline{B}_{fi}F_j \\ * & -Y & YM_j + \overline{B}_{fi}E_j & 0 & YM_{dj} + \overline{B}_{fi}E_{dj} & 0 & YN_j + \overline{B}_{fi}F_j \\ * & * & \Lambda_{33} & \Lambda_{34} & XA_{dj} + \overline{B}_{fi}C_{dj} + \dfrac{R}{\tau} & A_j^T S_i & XB_j + \overline{B}_{fi}D_j \\ * & * & * & \Lambda_{44} & YA_{dj} + \overline{B}_{fi}C_{dj} & 0 & YB_j + \overline{B}_{fi}D_j \\ * & * & * & * & -Q - \dfrac{R}{\tau} & A_{dj}^T S_i & 0 \\ * & * & * & * & * & \tau R - S_i - S_i^T & S_i^T B_j \\ * & * & * & * & * & * & -\gamma^2 I \end{bmatrix},$$

$$\Omega_2^{ij} = \begin{bmatrix} -\alpha X & -\alpha Y & \alpha\left(XM_j + \overline{B}_{fi}E_j\right) & 0 & \alpha\left(XM_{dj} + \overline{B}_{fi}E_{dj}\right) & 0 & 0 \\ * & -\alpha Y & \alpha\left(YM_j + \overline{B}_{fi}E_j\right) & 0 & \alpha\left(YM_{dj} + \overline{B}_{fi}E_{dj}\right) & 0 & 0 \\ * & * & \widetilde{\Lambda}_{33} & \widetilde{\Lambda}_{34} & \alpha\left(XA_{dj} + \overline{B}_{fi}C_{dj} + \dfrac{R}{\tau}\right) & \alpha A_j^T S_i & L^T(t) \\ * & * & * & \widetilde{\Lambda}_{44} & \alpha\left(YA_{dj} + \overline{B}_{fi}C_{dj}\right) & 0 & -\overline{C}_{fi}^T \\ * & * & * & * & -\alpha\left(Q + \dfrac{R}{\tau}\right) & \alpha A_{dj}^T S_i & 0 \\ * & * & * & * & * & \alpha\left(\tau R - S_i - S_i^T\right) & 0 \\ * & * & * & * & * & * & -I \end{bmatrix},$$

$$\begin{aligned} \Lambda_{33} &= XA_j + A_j^T X + \overline{B}_{fi}C_j + C_j^T\overline{B}_{fi}^T + Q - \frac{R}{\tau}, \\ \Lambda_{34} &= \overline{A}_{fi} + A_j^T Y + C_j^T\overline{B}_{fi}^T, \qquad \Lambda_{44} = \overline{A}_{fi} + \overline{A}_{fi}^T, \\ \widetilde{\Lambda}_{33} &= \alpha\left(XA_j + A_j^T X + \overline{B}_{fi}C_j + C_j^T\overline{B}_{fi}^T + Q - \frac{R}{\tau}\right), \\ \widetilde{\Lambda}_{34} &= \alpha\left(\overline{A}_{fi} + A_j^T Y + C_j^T\overline{B}_{fi}^T\right), \qquad \widetilde{\Lambda}_{44} = \alpha\left(\overline{A}_{fi} + \overline{A}_{fi}^T\right). \end{aligned} \tag{46}$$

In this case, the filter parameters in (5) *are given by*

$$\begin{aligned} A_{fi} &= Y^{-1}\overline{A}_{fi}, \qquad B_{fi} = Y^{-1}\overline{B}_{fi}, \\ C_{fi} &= \overline{C}_{fi}, \quad i = 1, 2, \ldots, r. \end{aligned} \tag{47}$$

Proof. Based on Theorems 3 and 6, we set

$$\begin{aligned} \overline{A}_f(t) &= \sum_{i=1}^{r} h_i(\theta(t))\overline{A}_{\text{fi}}, \\ \overline{B}_f(t) &= \sum_{i=1}^{r} h_i(\theta(t))\overline{B}_{\text{fi}}, \\ \overline{C}_f(t) &= \sum_{i=1}^{r} h_i(\theta(t))\overline{C}_{\text{fi}}, \\ S(t) &= \sum_{i=1}^{r} h_i(\theta(t)) S_i. \end{aligned} \tag{48}$$

From (39) and (40), we have

$$\begin{aligned} \Omega_1(t) &= \sum_{i=1}^{r} h_i^2(\theta(t))\Omega_1^{ii} \\ &\quad + \sum_{i=1}^{r} h_i(\theta(t)) \sum_{j=1}^{r} h_j(\theta(t))\left(\Omega_1^{ij} + \Omega_1^{ji}\right) < 0, \\ \Omega_2(t) &= \sum_{i=1}^{r} h_i^2(\theta(t))\Omega_2^{ii} \\ &\quad + \sum_{i=1}^{r} h_i(\theta(t)) \sum_{j=1}^{r} h_j(\theta(t))\left(\Omega_2^{ij} + \Omega_2^{ji}\right) < 0. \end{aligned} \tag{49}$$

By virtue of Theorems 3 and 6, the Hankel norm filter design problem is solvable and the filter parameters are given by

$$\begin{aligned} A_f(t) &= V^{-1}\overline{A}_f(t)V^{-T}W, \qquad B_f(t) = V^{-1}\overline{B}_f(t), \\ C_f(t) &= \overline{C}_f(t)V^{-T}W, \end{aligned} \tag{50}$$

where matrices $W > 0$ and V are such that $Y = VW^{-1}V^T$. Or equivalently under transformation $V^{-T}W\widehat{x}(t)$, the filter parameters can be obtained as (47). The proof is completed. □

Remark 10. Notice that the obtained conditions in Theorem 9 are all in LMI forms and the Hankel norm filtering problem can be solved by the following convex optimization problem with LMI Toolbox in MATLAB:

$$\min_{X>0, Y>0, Q>0, R>0, S_i, \overline{A}_{\text{fi}}, \overline{B}_{\text{fi}}, \overline{C}_{\text{fi}}} \lambda \quad \text{Subject to } (45), \tag{51}$$

where $\lambda = \gamma^2$, and the admissible filter parameters can be determined by (47).

5. Numerical Example

In this section, we will present a numerical example to demonstrate the validity of the developed results. Consider a stochastic system of the form (2) with the following parameters ($r = 2$):

$$A_1 = \begin{bmatrix} -1.5 & 0.5 \\ -1 & -3 \end{bmatrix}, \qquad A_{d1} = \begin{bmatrix} -0.8 & 0.2 \\ 0.2 & -0.2 \end{bmatrix}, \tag{52}$$

$$B_1 = \begin{bmatrix} 0.2 \\ -0.2 \end{bmatrix}, \qquad M_1 = \begin{bmatrix} -0.8 & 0.2 \\ 0.5 & -0.5 \end{bmatrix}, \tag{53}$$

$$M_{d1} = \begin{bmatrix} 0.5 & 0.5 \\ 0.2 & 0.3 \end{bmatrix}, \qquad N_1 = \begin{bmatrix} -0.2 \\ 0.5 \end{bmatrix}, \tag{54}$$

$$C_1 = \begin{bmatrix} 0.2 & 0.1 \end{bmatrix}, \qquad C_{d1} = \begin{bmatrix} -0.1 & 0.2 \end{bmatrix}, \qquad D_1 = 0.2, \tag{55}$$

$$E_1 = \begin{bmatrix} -0.2 & 0.2 \end{bmatrix}, \qquad E_{d1} = \begin{bmatrix} 0.2 & -0.5 \end{bmatrix}, \tag{56}$$

$$F_1 = 0.5, \qquad L_1 = \begin{bmatrix} -1 & 0.5 \end{bmatrix}, \tag{57}$$

$$A_2 = \begin{bmatrix} -1 & 0.5 \\ 0.5 & -1.3 \end{bmatrix}, \qquad A_{d2} = \begin{bmatrix} 0.02 & 0.14 \\ 0 & 0.15 \end{bmatrix}, \tag{58}$$

$$B_2 = \begin{bmatrix} 0.3 \\ 0.1 \end{bmatrix}, \qquad M_2 = \begin{bmatrix} -1 & 0 \\ -0.5 & -1.3 \end{bmatrix}, \tag{59}$$

$$M_{d2} = \begin{bmatrix} 0.1 & 0 \\ 0.02 & 0.03 \end{bmatrix}, \qquad N_2 = \begin{bmatrix} 0.2 \\ -0.5 \end{bmatrix}, \tag{60}$$

$$C_2 = \begin{bmatrix} 0.5 & 0.1 \end{bmatrix}, \qquad C_{d2} = \begin{bmatrix} -0.1 & 0.5 \end{bmatrix}, \qquad D_2 = 0.1, \tag{61}$$

$$E_2 = \begin{bmatrix} -0.1 & 0.2 \end{bmatrix}, \qquad E_{d2} = \begin{bmatrix} 0.1 & -0.5 \end{bmatrix}, \tag{62}$$

$$F_2 = 0.2, \qquad L_2 = \begin{bmatrix} 0.5 & -0.1 \end{bmatrix}. \tag{63}$$

According to Theorem 9, we can get the minimum performance level $\gamma = 0.5253$ for $\tau = 0.5$ and $\alpha = 1$, and the solutions of corresponding parameters are as follows:

$$Y = \begin{bmatrix} 0.1901 & -0.1134 \\ -0.1134 & 0.1207 \end{bmatrix}, \qquad \overline{A}_{f1} = \begin{bmatrix} -0.5394 & -0.0057 \\ -0.0057 & -0.3529 \end{bmatrix},$$
$$\overline{B}_{f1} = \begin{bmatrix} 0.2148 \\ -0.1394 \end{bmatrix}, \qquad \overline{C}_{f1} = \begin{bmatrix} -0.6610 & 0.0083 \end{bmatrix}$$
$$\overline{A}_{f2} = \begin{bmatrix} -0.4336 & 0.5944 \\ 0.5944 & -0.9336 \end{bmatrix}, \qquad \overline{B}_{f2} = \begin{bmatrix} -0.0247 \\ -0.1454 \end{bmatrix},$$
$$\overline{C}_{f2} = \begin{bmatrix} 0.6553 & -1.1762 \end{bmatrix}. \tag{64}$$

Then the Hankel norm filter parameter matrices are computed from (47) as

$$A_{f1} = \begin{bmatrix} -6.5154 & -4.0327 \\ -6.1661 & -6.7104 \end{bmatrix}, \qquad B_{f1} = \begin{bmatrix} 1.0035 \\ -0.2123 \end{bmatrix},$$
$$C_{f1} = \begin{bmatrix} -0.6610 & 0.0083 \end{bmatrix}, \qquad A_{f2} = \begin{bmatrix} 0.7727 & -3.3775 \\ 5.6483 & -10.9038 \end{bmatrix},$$
$$B_{f2} = \begin{bmatrix} -1.9283 \\ -3.0147 \end{bmatrix}, \qquad C_{f2} = \begin{bmatrix} 0.6553 & -1.1762 \end{bmatrix}. \tag{65}$$

Table 1: Minimum index γ for different τ.

	$\tau = 0.5$	$\tau = 0.6$	$\tau = 0.8$	$\tau = 1.0$
γ	0.5253	0.5553	0.6387	0.8138

The solvability of the filter parameters indicates that the proposed approach is effective. Furthermore, different value of τ may yield different $\gamma_{\min}$. By selecting several different values of τ, the computation results of minimum γ are obtained in Table 1. Table 1 shows that the results presented in this paper are delay-dependent and less conservative.

6. Conclusions

In this paper, the problem of Hankel norm filter design for stochastic time-delay systems via T-S fuzzy-model-based approach has been investigated. A new filtering error system is established by designing local linear filters for each linear subsystem according to the parallel distributed compensation (PDC) method. Based on the Lyapunov stability theory and LMI techniques, a delay-dependent sufficient condition is developed in terms of LMIs for the mean-square asymptotic stability with Hankel norm performance of the filtering error system. The integral inequality method is adopted and an assistant vector and free matrices are introduced, which helps achieving much less conservative results. The results of numerical example are presented to demonstrate the effectiveness of the proposed approach.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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