

Research Article

Toeplitz Operators on Dirichlet-Type Space of Unit Ball

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We construct a function u in $L^2(\mathbb{B}_n, dV)$ which is unbounded on any neighborhood of each boundary point of \mathbb{B}_n such that Toeplitz operator T_u is a Schatten p -class ($0 < p < \infty$) operator on Dirichlet-type space $\mathcal{D}(\mathbb{B}_n, dV)$. Then, we discuss some algebraic properties of Toeplitz operators with radial symbols on the Dirichlet-type space $\mathcal{D}(\mathbb{B}_n, dV)$. We determine when the product of two Toeplitz operators with radial symbols is a Toeplitz operator. We investigate the zero-product problem for several Toeplitz operators with radial symbols. Furthermore, the corresponding commuting problem of Toeplitz operators whose symbols are of the form $\xi^k u$ is studied, where $k \in \mathbb{Z}^n$, $\xi \in \partial\mathbb{B}_n$, and u is a radial function.

1. Introduction

Let \mathbb{B}_n represent the open unit ball in several complex spaces \mathbb{C}^n . The Sobolev space $L^{2,1}(\mathbb{B}_n, dV)$ consists of the functions which satisfy

$$\|f\|_{2,1} = \left(\sum_{i=1}^n \int_{\mathbb{B}_n} \left(\left| \frac{\partial f}{\partial z_i}(z) \right|^2 + \left| \frac{\partial f}{\partial \bar{z}_i}(z) \right|^2 \right) dV \right)^{1/2} < \infty, \quad (1)$$

where dA denotes the normalized Lebesgue volume measure on \mathbb{B}_n . $L^{2,1}(\mathbb{B}_n)$ is a Hilbert space with inner product

$$\langle f, g \rangle_{2,1} = \sum_{i=1}^n \left\langle \frac{\partial f}{\partial z_i}, \frac{\partial g}{\partial z_i} \right\rangle + \sum_{i=1}^n \left\langle \frac{\partial f}{\partial \bar{z}_i}, \frac{\partial g}{\partial \bar{z}_i} \right\rangle, \quad (2)$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2(\mathbb{B}_n, dV)$. The Dirichlet-type space \mathcal{D} is the subspace of all analytic functions g in $L^{2,1}(\mathbb{B}_n)$ with $g(0) = 0$. Then, \mathcal{D} is a closed subspace of the $L^{2,1}(\mathbb{B}_n)$. Let P be the orthogonal projection from $L^{2,1}(\mathbb{B}_n)$ onto \mathcal{D} . P is an integral operator represented by

$$P(f)(w) = \langle f, K(z, w) \rangle_{2,1} = \sum_{i=1}^n \int_{\mathbb{B}_n} \frac{\partial f}{\partial z_i} \overline{\frac{\partial K_w}{\partial z_i}} dV(z), \quad (3)$$

where $K_w(z) = K(z, w)$ is the reproducing kernel of \mathcal{D} . By computation, we know

$$K(z, w) = \sum_{\alpha \in \mathbb{N}^n - \{(0, \dots, 0)\}} \frac{(|\alpha| + n - 1)!}{|\alpha| n! \alpha!} z^\alpha \bar{w}^\alpha, \quad (4)$$

where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, $\alpha! = \alpha_1! \cdots \alpha_n!$, $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$, and $|\alpha| = \sum_{i=1}^n \alpha_i$.

In recent years, the Dirichlet-type space has received a lot of attention from mathematicians in the areas of modern analysis and probability and statistical analysis. Many mathematicians are interested in function theory and operator theory on the Dirichlet-type space (See [1, 2]). In [3, 4], for the Dirichlet-type space of one complex variable, that is, $n = 1$, Rochberg and Wu defined the Toeplitz operator with nonnegative measure μ on \mathbb{B}_1 as follows: $T_\mu : \mathcal{D} \rightarrow \mathcal{D}$ by

$$T_\mu(f)(w) = \int_{\mathbb{B}_1} f(z) \overline{K(z, w)} d\mu(z). \quad (5)$$

Rochberg and Wu discussed the boundedness and compactness. Lu and Sun define Toeplitz operators on a Dirichlet-type space of several variables in [5].

Definition 1. Suppose that μ is a finite measure on \mathbb{B}_n . Toeplitz operators on the Dirichlet-type space with symbol μ are defined as follows:

$$T_\mu(f)(w) = \int_{\mathbb{B}_n} f(z) \overline{K(z,w)} d\mu(z), \quad f \in \mathcal{D}. \quad (6)$$

If $d\mu = \varphi dA$, $\varphi \in L^1(\mathbb{B}_n, dA)$, we write $T_\mu = T_\varphi$ and T_φ is called the Toeplitz operator with symbol φ .

2. Schatten p -Class Toeplitz Operators with Unbounded Symbols

In general, if $u \notin L^\infty(\mathbb{B}_n)$, the space of essentially bounded functions on \mathbb{B}_n and then T_u is densely defined only. In the case of Hardy space, it is well known that T_u is bounded if and only if u is essentially bounded, and T_u is compact if and only if $u = 0$ (see Douglas [6] and Davie and Jewell [7]). However, there are indeed bounded and compact Toeplitz operators with unbounded symbols on Bergman spaces of one complex variable; in fact, Miao and Zheng [8] have introduced a class of functions, called BT, which contains L^∞ , for $u \in \text{BT}$; T_u is compact on L^2_a if and only if the Berezin transform of T_u vanishes on the unit circle. Zorboska [9] has proved that if u belongs to the hyperbolic BMO space, the T_u is compact if and only if the Berezin transform of u vanishes on the unit circle. Cima and Cuckovic [10] construct a class of unbounded functions over a Cantor set; the Toeplitz operators with these functions are compact. Essentially, if the values of the function T_u vanish rapidly near the unit circle in the sense of measure dV , then T_u will be compact. Cao [11] also construct compact Toeplitz operators and trace class Toeplitz operators with unbounded symbols on Bergman space of several complex variables.

In this section, we construct the compact Toeplitz operators and the Schatten p -class Toeplitz operators for $0 < p < \infty$ with unbounded symbols on Dirichlet-type space of several complex variables.

For preparation, we introduce a special set in \mathbb{B}_n , which is important for building our main results in this section.

For $a > 0$ and $\xi \in \partial\mathbb{B}_n$, the boundary of \mathbb{B}_n , let

$$\begin{aligned} D(\xi, a) &= \left\{ z \in \mathbb{B}_n : [1 - (1 - |z|)^a]^{1/2} \cdot |z - \xi| \right. \\ &\leq |\text{Re} \langle \xi, z - \xi \rangle|, \\ &\left. \text{Re} \langle z, \xi \rangle > 0 \right\}; \end{aligned} \quad (7)$$

then it is obvious that $D(\xi, a)$ is a domain in \mathbb{B}_n , which is called circular-like cone with vertex ξ , because it looks like a circular cone. For $0 < r < 1$, set $\mathbb{B}_n(r) = \{z \in \mathbb{C}_n : |z| < r\}$ to be the ball with radius r . We use dS_r to denote the area measure on $\partial\mathbb{B}_n(r)$, for $r = 1$; $dS_1 = dS$ is the normalized area measure on $\partial\mathbb{B}_n$; then there holds $S_r(\partial\mathbb{B}_n(r)) = O(r^{2n-1})$ and $S_1(\partial\mathbb{B}_n) = 1$. For $b > 0$, because of the property of $D(\xi, a)$, there exists proper $a(b) > 0$ such that, for any $0 < r < 1$,

$$S_r(D(\xi, a(b)) \cap \partial\mathbb{B}_n(r)) < \lambda(1 - r^2)^b, \quad (8)$$

where λ is a constant number that is independent of ξ and r . For convenience, we write $D_b(\xi) = D(\xi, a(b))$.

Lemma 2 (Poincaré's inequality). *Suppose that $1 \leq p < \infty$, Ω is a bounded domain in \mathbb{R}^n and u belongs to Sobolev space $W_0^{1,p}(\Omega)$; then*

$$\int_{\Omega} |u|^p dV \leq C \int_{\Omega} |\nabla u|^p dV, \quad (9)$$

where C is a constant that depends on Ω .

Lemma 3. *Suppose that $c > 0$, $U_c(z) = (1 - |z|^2)^{-c}$, and $z \in \mathbb{B}_n$. For any $\xi \in \mathbb{B}_n$, let $b \geq n + 2c + 2$; we use $\chi_{D_b(\xi)}(z)$ to represent the characteristic function of the set $D_b(\xi)$. Then, $u(z) = \chi_{D_b(\xi)}(z)U_c(z)$ introduces a compact Toeplitz operator T_φ on \mathcal{D} .*

Proof. Take a sequence $\{f_k(z)\}_k$ from \mathcal{D} such that $f_k \rightarrow 0$ weakly and $\|f_k\|_{2,1} \leq 1$; it is enough to obtain that

$$\|T_u f_k\|_{2,1} \rightarrow 0. \quad (10)$$

From Definition 1, we find that

$$\begin{aligned} &\|T_u f_k(z)\|_{2,1}^2 \\ &= \sum_{i=1}^n \int_{\mathbb{B}_n} \left| \int_{\mathbb{B}_n} u(w) f_k(w) \frac{\partial K(z,w)}{\partial z_i} dV(w) \right|^2 dV(z) \\ &= \sum_{i=1}^n \int_{\mathbb{B}_n} \left| \int_{D_b(\xi)} \frac{f_k(w)}{(1 - |w|^2)^c} \frac{\partial K(z,w)}{\partial z_i} dV(w) \right|^2 dV(z). \end{aligned} \quad (11)$$

For $m \in (0, 1)$, set $D_b(\xi, m) = \{z \in D_b(\xi) : |z| > m\}$; then

$$\begin{aligned} &\sum_{i=1}^n \int_{\mathbb{B}_n} \left| \int_{D_b(\xi)} \frac{f_k(w)}{(1 - |w|^2)^c} \frac{\partial K(z,w)}{\partial z_i} dV(w) \right|^2 dV(z) \\ &\leq 2 \sum_{i=1}^n \int_{\mathbb{B}_n} \left| \int_{D_b(\xi, m)} \frac{f_k(w)}{(1 - |w|^2)^c} \frac{\partial K(z,w)}{\partial z_i} dV(w) \right|^2 dV(z) \\ &\quad + 2 \sum_{i=1}^n \int_{\mathbb{B}_n} \left| \int_{D_b(\xi) - D_b(\xi, m)} \frac{f_k(w)}{(1 - |w|^2)^c} \frac{\partial K(z,w)}{\partial z_i} dV(w) \right|^2 dV(z). \end{aligned} \quad (12)$$

By Hölder's and Poincaré's inequality,

$$\begin{aligned}
 & \left| \int_{D_b(\xi, m)} \frac{f_k(w)}{(1-|w|^2)^c} \frac{\partial K(z, w)}{\partial z_i} dV(w) \right|^2 \\
 & \leq \int_{D_b(\xi, m)} |f_k(w)|^2 dV(w) \\
 & \quad \times \int_{D_b(\xi, m)} \frac{1}{(1-|w|^2)^{2c}} \left| \frac{\partial K(z, w)}{\partial z_i} \right|^2 dV(w) \\
 & \leq C \int_{D_b(\xi, m)} \frac{1}{(1-|w|^2)^{2c}} \left| \frac{\partial K(z, w)}{\partial z_i} \right|^2 dV(w),
 \end{aligned} \tag{13}$$

where C is constant. Then

$$\begin{aligned}
 & \sum_{i=1}^n \int_{\mathbb{B}_n} \left| \int_{D_b(\xi, m)} \frac{f_k(w)}{(1-|w|^2)^c} \frac{\partial K(z, w)}{\partial z_i} dV(w) \right|^2 dV(z) \\
 & \leq C \sum_{i=1}^n \int_{\mathbb{B}_n} \int_{D_b(\xi, m)} \frac{1}{(1-|w|^2)^{2c}} \left| \frac{\partial K(z, w)}{\partial z_i} \right|^2 dV(w) dV(z) \\
 & = C \int_{D_b(\xi, m)} \frac{1}{(1-|w|^2)^{2c}} dV(w) \\
 & \quad \times \int_{\mathbb{B}_n} \sum_{i=1}^n \left| \frac{\partial K(z, w)}{\partial z_i} \right|^2 dV(z) \\
 & = C \int_{D_b(\xi, m)} \frac{K(w, w)}{(1-|w|^2)^{2c}} dV(w) \\
 & = C \int_{D_b(\xi, m)} \left(\left(\sum_{\alpha \in \mathbb{N}^n - \{(0, \dots, 0)\}} \right. \right. \\
 & \quad \times \left. \left. \frac{(|\alpha| + n - 1)!}{|\alpha| n! \alpha!} |w|^{2\alpha} \right) \right. \\
 & \quad \times \left. \left. \left((1-|w|^2)^{2c} \right)^{-1} \right) dV(w) \\
 & \leq C \int_{D_b(\xi, m)} \frac{\sum_{\alpha \in \mathbb{N}^n} ((|\alpha| + n)! / n! \alpha!) |w|^{2\alpha}}{(1-|w|^2)^{2c}} dV(w) \\
 & = C \int_{D_b(\xi, m)} \frac{1}{(1-|w|^2)^{n+1+2c}} dV(w) \\
 & = 2nC \int_0^1 r^{2n-1} dr \\
 & \quad \times \int_{\partial \mathbb{B}_n} \chi_{D_b(\xi, m)}(r\eta) \frac{1}{(1-r^2)^{n+1+2c}} dS(\eta) \\
 & = c_0 \int_m^1 r^{2n-1} dr
 \end{aligned}$$

$$\begin{aligned}
 & \times \int_{\partial \mathbb{B}_n(r) \cap D_b(\xi, m)} \frac{1}{(1-r^2)^{n+1+2c}} dS(\eta) \\
 & = c_0 \int_0^1 dr \int_{\partial \mathbb{B}_n(r) \cap D_b(\xi, m)} \frac{1}{(1-r^2)^{n+1+2c}} dS_r(\eta) \\
 & \leq \frac{c_0}{m} \int_m^1 \int_{\partial \mathbb{B}_n(r) \cap D_b(\xi, m)} \frac{1}{(1-r^2)^{n+1+2c}} d(r^2) \\
 & \leq \frac{c_1}{m} \int_m^1 \frac{(1-r^2)^b}{(1-r^2)^{n+1+2c}} d(r^2) \\
 & = \frac{c_1}{m} \frac{(1-m^2)^{b-n-2c}}{b-n-2c} \leq \frac{c_1}{m} (1-m^2)^2,
 \end{aligned} \tag{14}$$

where c_0, c_1 are constants. Thus, for any $\varepsilon > 0$, there exists a $m_0 \in (0, 1)$ such that

$$\sum_{i=1}^n \int_{\mathbb{B}_n} \left| \int_{D_b(\xi, m)} \frac{f_k(w)}{(1-|w|^2)^c} \frac{\partial K(z, w)}{\partial z_i} dV(w) \right|^2 dV(z) \leq \varepsilon, \tag{15}$$

for $m \in [m_0, 1)$.

On the other hand, because $D_b(\xi) - D_b(\xi, m) \subset \overline{\mathbb{B}_n(m_0)}$, the sequence f_k converges uniformly to 0 in $\overline{\mathbb{B}_n(m_0)}$ by its weak convergence (see Zhu [12]). Then, for any $\varepsilon > 0$, there is a K_0 such that, for any $k > K_0$, we find that $|f_k(w)| < \varepsilon$, $w \in D_b(\xi) - D_b(\xi, m)$. Thus

$$\begin{aligned}
 & \left| \int_{D_b(\xi) - D_b(\xi, m_0)} \frac{f_k(w)}{(1-|w|^2)^c} \frac{\partial K(z, w)}{\partial z_i} dV(w) \right| \\
 & \leq \varepsilon \int_{D_b(\xi) - D_b(\xi, m_0)} \frac{1}{(1-|w|^2)^c} \left| \frac{\partial K(z, w)}{\partial z_i} \right| dV(w) \\
 & \leq \varepsilon \frac{1}{(1-|m_0|^2)^c} \sum_{\alpha \in \mathbb{Z}^{+n}} \frac{(|\alpha| + n - 1)!}{n! \alpha!} m_0^\alpha.
 \end{aligned} \tag{16}$$

This shows that $\|T_u f_k\|_{2,1} \rightarrow 0$. □

Using Lemma 3, we can construct a compact Toeplitz operator with a symbol that is unbounded on any neighborhood of every point in unit surface.

Theorem 4. *There exists a function $u \in L^1(\mathbb{B}_n, dA)$ which is unbounded on any neighborhood of every point in unit surface (i.e., for any $\xi \in \partial \mathbb{B}_n$ and $r > 0$,*

$$\operatorname{esssup}_{z \in \mathbb{B}_n \cap \mathbb{B}(\xi, r)} |u(z)| = \infty, \tag{17}$$

where $\mathbb{B}(\xi, r) = \{z : |z - \xi| < r\}$; T_u is a compact operator on \mathcal{D} .

Proof. Suppose that $c > 0$, $b \geq n + 2c + 2$, and $U_c(z)$ is the function defined in Lemma 3. Let $\{\xi_i\}_{i=1}^\infty$ be a countable dense

subset of $\partial\mathbb{B}_n$; then, for every ξ_i , the $u_i = \chi_{D_b(\xi_i)}U_c$ introduces a compact Toeplitz operator T_{u_i} by Lemma 3. Furthermore, for any $f \in \mathcal{D}$, by Hölder's and Poincaré's inequality,

$$\begin{aligned} & \|T_{u_i}f\|_{2,1}^2 \\ &= \sum_{i=1}^n \int_{\mathbb{B}_n} \left| \int_{\mathbb{B}_n} u_i(w) f(w) \frac{\partial K(z,w)}{\partial z_i} dV(w) \right|^2 dV(z) \\ &= \sum_{i=1}^n \int_{\mathbb{B}_n} \left| \int_{D_b(\xi_i)} \frac{f(w)}{(1-|w|^2)^c} \frac{\partial K(z,w)}{\partial z_i} dV(w) \right|^2 dV(z) \\ &\leq \|f\|_2^2 \sum_{i=1}^n \int_{\mathbb{B}_n} \int_{D_b(\xi_i)} \frac{|\partial K(z,w)/\partial z_i|^2}{(1-|w|^2)^{2c}} dV(w) dV(z) \\ &\leq C \|f\|_{2,1}^2 \int_{D_b(\xi_i)} \frac{1}{(1-|w|^2)^{2c}} dV(w) \\ &\quad \times \sum_{i=1}^n \int_{\mathbb{B}_n} \left| \frac{\partial K(z,w)}{\partial z_i} \right|^2 dV(z) \\ &\leq C \|f\|_{2,1}^2 \int_{D_b(\xi_i)} \frac{1}{(1-|w|^2)^{2c}} K(w,w) dV(w) \\ &\leq C \|f\|_{2,1}^2 \int_{D_b(\xi_i)} \frac{1}{(1-|w|^2)^{n+1+2c}} dV(w) \\ &\leq C \|f\|_{2,1}^2 \int_0^1 \int_{\partial\mathbb{B}_n(r) \cap D_b(\xi_i)} \frac{1}{(1-r^2)^{n+2c+1}} dS_r \\ &\leq C\lambda \|f\|_{2,1}^2 \int_0^1 (1-r^2)^{b-n-2c-1} dr \leq C\lambda \|f\|_{2,1}^2, \end{aligned} \tag{18}$$

where C is a constant and λ is defined in (8); thus $\|T_{u_i}\| \leq \sqrt{C\lambda}$. Set $T_N = \sum_{i=1}^N (1/2^i)T_{u_i}$; then T_N is compact, and, for any $M, N \in \mathbb{N}$ and $f \in \mathcal{D}$, we find

$$\left\| \sum_{i=N}^M \frac{1}{2^i} T_{u_i} f \right\|_{2,1} \leq \sqrt{C\lambda} \|f\|_{2,1} \sum_{i=N}^M \frac{1}{2^i}. \tag{19}$$

Equation (19) shows that $\|\sum_{i=N}^M (1/2^i)T_{u_i}\|_{2,1} \leq \sqrt{C\lambda} \sum_{i=N}^M (1/2^i)$. Then $T_N = \sum_{i=1}^N (1/2^i)T_{u_i}$ converges to a compact operator $T = \sum_{i=1}^\infty (1/2^i)T_{u_i}$ in norm. It is obvious that $u_i \in L^2(\mathbb{B}_n)$ and $\|u_i\|_2 \leq 1$; thus $u_N = \sum_{i=1}^N u_i$ converges to function $u = \sum_{i=1}^\infty (1/2^i)u_i \in L^2(\mathbb{B}_n)$ in norm. For any holomorphic polynomial p ,

$$\begin{aligned} \|(T_u - T_N)p\|_{2,1} &= \left\| \left(\sum_{i=N+1}^\infty \frac{1}{2^i} T_{u_i} \right) p \right\|_{2,1} \\ &\leq \|p\|_{2,1} \sqrt{C\lambda} \sum_{i=N+1}^\infty \frac{1}{2^i} \rightarrow 0, \end{aligned} \tag{20}$$

namely, $T = T_u$ which is the Toeplitz operator with symbol u . Hence, for any $\xi \in \partial\mathbb{B}_n$ and $r > 0$,

$$\operatorname{esssup}_{z \in \mathbb{B}_n \cap \mathbb{B}(\xi,r)} |u(z)| = \infty, \tag{21}$$

where $\mathbb{B}(\xi, r) = \{z : |z - \xi| < r\}$, because of the density of set $\{\xi_i\}$ in unit surface. \square

Following the above theorem, we construct Schatten p -class Toeplitz operators for all $0 < p < \infty$ whose symbols are also unbounded on any neighborhood of every point in unit surface.

Lemma 5 (See [5]). *Let $f, g \in \mathcal{D}$, $d\mu = \varphi dA$, $\varphi \in L^1(\mathbb{B}_n, dA)$. Then*

$$\langle T_\mu f, g \rangle_{2,1} = \int_{\mathbb{B}_n} \varphi f \bar{g} dA. \tag{22}$$

Theorem 6. *There is a function $u \in L^1(\mathbb{B}_n)$ which is unbounded on any neighborhood of each boundary point of \mathbb{B}_n (i.e., for any $\xi \in \partial\mathbb{B}_n$ and $r > 0$,*

$$\operatorname{esssup}_{z \in \mathbb{B}_n \cap \mathbb{B}(\xi,r)} |u(z)| = \infty, \tag{23}$$

where $\mathbb{B}(\xi, r) = \{z : |z - \xi| < r\}$; T_u is a Schatten p -class operator on Dirichlet-type space \mathcal{D} for $0 < p < \infty$.

Proof. For every multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^{+n} - \{(0, \dots, 0)\}$, we have $\|z^\alpha\|_{2,1}^2 = n! \alpha! / (|\alpha| + n - 1)!$ (see Rudin [13]). Thus $\{e_\alpha\} = \{z^\alpha / \|z^\alpha\|_{2,1}\} = \{\sqrt{|\alpha| n! \alpha! / (|\alpha| + n - 1)!} z^\alpha\}$ forms the orthonormal base in \mathcal{D} . Choose a countable dense subset $\{\xi_i\}_{i=1}^\infty$ of $\partial\mathbb{B}_n$; let $U_c(z) = (1 - |z|^2)^{-c}$ ($c > 0$), and set

$$u_i = \chi_{D_b(\xi_i,m)}(z) U_c(z), \tag{24}$$

for any $m \in (0, 1)$, where $b = 2c + ((n + 2)/p)$ ($0 < p \leq 1$). Then

$$\begin{aligned} & \left| \langle T_{u_i} e_\alpha, e_\alpha \rangle_{2,1} \right| \\ &= \frac{(|\alpha| + n - 1)!}{|\alpha| n! \alpha!} \left| \int_{\mathbb{B}_n} u_i(w) w^\alpha \bar{w}^\alpha dV(w) \right| \\ &= \frac{(|\alpha| + n - 1)!}{|\alpha| n! \alpha!} \int_{D_b(\xi_i,m)} (1 - |w|^2)^{-c} w^\alpha \bar{w}^\alpha dV(w) \\ &= C \frac{(|\alpha| + n - 1)!}{|\alpha| n! \alpha!} \int_m^1 \int_{\partial\mathbb{B}_n(r) \cap D_b(\xi_i,m)} (1 - r^2)^{-c} w^\alpha \cdot \bar{w}^\alpha dS_r dr \\ &\leq C\lambda \frac{(|\alpha| + n - 1)!}{|\alpha| n! \alpha!} \int_m^1 (1 - r^2)^{c+(n+2)/p} r^{2|\alpha|} dr, \end{aligned} \tag{25}$$

where C is a constant independent of α . Integrating by parts, we find

$$\int_m^1 (1 - r^2)^{c+(n+2)/p} r^{2|\alpha|} dr \leq \frac{d(n)}{(2|\alpha| + 1)^{c+(n+2)/p+1}}, \tag{26}$$

where $d(n)$ is a constant that depends on n . Let

$$T = \sum_{i=1}^{\infty} \frac{1}{2^i} T_{u_i}; \tag{27}$$

then T is compact operator by Theorem 4. Note that T is positive operator; thus, for $0 < p \leq 1$,

$$\begin{aligned} & \sum_{\alpha \in \mathbb{N}^n - \{(0, \dots, 0)\}} |\langle T e_{\alpha}, e_{\alpha} \rangle|^p \\ & \leq \sum_{\alpha \in \mathbb{N}^n - \{(0, \dots, 0)\}} \left(\sum_{i=1}^{\infty} \frac{1}{2^i} \frac{(|\alpha| + n - 1)!}{|\alpha| n! \alpha!} \frac{C \lambda d(n)}{(2|\alpha| + 1)^{c+(n+2)/p+1}} \right)^p \\ & \leq \sum_{\alpha \in \mathbb{N}^n - \{(0, \dots, 0)\}} \sum_{i=1}^{\infty} \frac{1}{2^{ip}} \left(\frac{(|\alpha| + n - 1)!}{|\alpha| n! \alpha!} \right)^p \\ & \quad \times \left(\frac{c \lambda d(n)}{(2|\alpha| + 1)^{c+(n+2)/p+1}} \right)^p \\ & \leq \sum_{i=1}^{\infty} \frac{1}{2^{ip}} \sum_{\alpha \in \mathbb{N}^n - \{(0, \dots, 0)\}} \left(\frac{(|\alpha| + n)!}{n! \alpha!} \right)^p \\ & \quad \times \left(\frac{c \lambda d(n)}{(2|\alpha| + 1)^{c+(n+2)/p+1}} \right)^p \\ & = l \sum_{\alpha \in \mathbb{N}^n - \{(0, \dots, 0)\}} \left(\frac{(|\alpha| + n)!}{n! \alpha!} \right)^p \left(\frac{c \lambda d(n)}{(2|\alpha| + 1)^{c+(n+2)/p+1}} \right)^p, \end{aligned} \tag{28}$$

where $l = \sum_{i=1}^{\infty} (1/2^{ip})$. Changing the order of summation, we have

$$\begin{aligned} (28) & = l \sum_{k=0}^{\infty} \left(\sum_{|\alpha|=k} \left(\frac{(|\alpha| + n)!}{n! \alpha!} \right)^p \right) \left(\frac{c \lambda d(n)}{(2|\alpha| + 1)^{c+(n+2)/p+1}} \right)^p \\ & = l \sum_{k=1}^{\infty} \left(\sum_{|\alpha|=k} \left(\frac{(k + n - 1)!}{|\alpha| n! \alpha!} \right)^p \right) \left(\frac{c \lambda d(n)}{(2k + 1)^{c+(n+2)/p+1}} \right)^p \\ & \leq l \sum_{k=1}^{\infty} \left(\sum_{|\alpha|=k} \frac{(k + n - 1)!}{|\alpha| n! \alpha!} \right) \left(\frac{c \lambda d(n)}{(2k + 1)^{c+(n+2)/p+1}} \right)^p. \end{aligned} \tag{29}$$

Note that, for any continuous function f on \mathbb{B}_n , we have

$$\begin{aligned} & \int_{\mathbb{B}_n} f(z) dV(z) \\ & = \int_0^1 \int_0^{2\pi} \left(\int_{\mathbb{B}_{n-1}(\sqrt{1-r^2})} f(z', r e^{i\theta}) dV(z') \right) r dr d\theta, \end{aligned} \tag{30}$$

where $z' = (z_1, \dots, z_{n-1})$, $z_n = r e^{i\theta}$, $i = \sqrt{-1}$. Then, by induction, we see obviously that

$$\sum_{|\alpha|=k} \frac{(k + n - 1)!}{|\alpha| n! \alpha!} \leq \sum_{|\alpha|=k} \frac{(k + n)!}{n! \alpha!} = O((n + k)^n). \tag{31}$$

Further, there are constants $M_1, M_2 > 0$ such that

$$\begin{aligned} (29) & \leq M_1 \sum_{k=0}^{\infty} \frac{(n + k)^n}{(2k + 1)^{(c+(n+2)/p+1)p}} \\ & \leq M_2 \sum_{k=0}^{\infty} \frac{1}{(2k + 1)^2} < \infty. \end{aligned} \tag{32}$$

Namely, T_u is a S_p -class operator for $0 < p \leq 1$.

For $1 < p < \infty$, let $b = n + 2c + 2$; let T be the operator defined in (27). By the above proof, we have

$$\sum_{\alpha} |\langle T e_{\alpha}, e_{\alpha} \rangle_{2,1}| < \infty. \tag{33}$$

By the convergence in (33), we know that there exists a N_0 such that, for $|\alpha| \geq N_0$, $|\langle T e_{\alpha}, e_{\alpha} \rangle_{2,1}| < 1$; thus, we find $|\langle T e_{\alpha}, e_{\alpha} \rangle_{2,1}|^p < |\langle T e_{\alpha}, e_{\alpha} \rangle_{2,1}|$. Hence,

$$\sum_{\alpha} |\langle T e_{\alpha}, e_{\alpha} \rangle_{2,1}|^p < \infty. \tag{34}$$

That is, T_u is a S_p -class operator for $1 < p < \infty$. □

3. Toeplitz Operators with Radial Symbols

Is the product of two Toeplitz operators equal to a Toeplitz operator? In general, the answer is negative, but Brown and Halmos [14] showed that two bounded Toeplitz operators T_{φ} and T_{ψ} commute on the Hardy space if and only if (I) both φ and ψ are analytic, (II) both φ and ψ are analytic, or (III) one is a linear function of the other. For more details on the same question for Toeplitz operators on the Bergman spaces of one variable, see Cuckovic et al. [15, 16] and Louhichi et al. [17, 18].

For the case of the Bergman spaces of several variable, Zheng [19] studied commuting the Toeplitz operators with pluriharmonic symbols on the unit ball in \mathbb{C}^n . Recently, Quiroga-Barranco and Vasilevski [20, 21] gave the description of many (geometrically defined) classes of commuting Toeplitz operators on the unit ball. Zhou and Dong [22] discussed commuting Toeplitz operators with radial symbols on the unit ball.

In this section, we discuss the same questions for the Dirichlet-type space on the unit ball. The rest of this section is organized as follows. First, we introduce some basic properties of the Mellin transform and Mellin convolution which will be needed later. Second, we discuss when the product of two Toeplitz operators with radial symbols is a Toeplitz operator. Then, the zero-product problem for several Toeplitz operators with radial symbols on the Dirichlet-type space is investigated. Finally, the corresponding commuting problem of Toeplitz operators with quasihomogeneous symbols is studied.

3.1. Mellin Transform and Mellin Convolution. Mellin transform, the most useful tool we use later, is defined as follows.

Definition 7. The Mellin transform \widehat{u} of a function $u \in L^1([0, 1], r dr)$ is

$$\widehat{u}(z) = \int_0^1 u(s) s^{z-1} ds. \tag{35}$$

It is known that \widehat{u} is a bounded analytic function in the half plane $\{z : \operatorname{Re}(z) > 2\}$. It is important and helpful to know that the Mellin transform is uniquely determined by its values on an arithmetic sequence of integers. In fact, we have the following classical theorem (see [23]).

Theorem 8. Let f be a bounded analytic function on $\{z : \operatorname{Re}(z) > 0\}$ which vanishes at the pairwise distinct points z_1, z_2, \dots , where

- (1) $\inf \{|z_n|\} > 0$,
- (2) $\sum_{n \geq 1} \operatorname{Re}(1/z_n) = \infty$.

Then f vanishes identically on $\{z : \operatorname{Re}(z) > 0\}$.

Remark 9. We will often use this theorem to show that if $u \in L^1([0, 1], r dr)$ and if there exists a sequence $\{n_k\}_{k \in \mathbb{Z}^+} \subset \mathbb{N}$ such that

$$\widehat{u}(n_k) = 0, \quad \sum_{k=1}^{\infty} \frac{1}{n_k} = \infty, \tag{36}$$

then $\widehat{u} = 0$ for all $z \in \{z : \operatorname{Re}(z) > 2\}$ and so $u = 0$.

When considering the product of two Toeplitz operators, we need a known fact about the Mellin convolution of their symbols. If f and g are defined on $[0, 1]$, then their Mellin convolution is defined by

$$(f *_M g)(r) = \int_r^1 f\left(\frac{r}{t}\right) g(t) \frac{dt}{t}, \quad 0 \leq r < 1. \tag{37}$$

The Mellin convolution theorem states that

$$\widehat{(f *_M g)}(r) = \widehat{f}(r) \widehat{g}(r) \tag{38}$$

and that if f and g are in $L^1([0, 1], r dr)$, then so is $\widehat{(f *_M g)}(r)$.

3.2. Products of Toeplitz Operators with Radial Symbols. For convenience, we use $\alpha - \beta$ to denote

$$(\alpha_1 - \beta_1, \dots, \alpha_n - \beta_n). \tag{39}$$

The notations $\alpha \geq \beta$ and $\alpha \perp \beta$ mean, respectively,

$$\begin{aligned} \alpha_i \geq \beta_i, \quad i = 1, \dots, n, \\ \sum_{i=1}^n \alpha_i \beta_i = 0. \end{aligned} \tag{40}$$

It is trivial that when $\alpha \geq \beta$,

$$|\alpha - \beta| = |\alpha| - |\beta|. \tag{41}$$

A function u on B_n is called the radial function, if $u(z)$ depends only on $|z|$. It is obvious that u is radial if and only

if $u(Uz) = u(z)$ for any unitary transform U of \mathbb{C}^n . Then, for each radial function u , we define the function \tilde{u} on $[0, 1]$ by $\tilde{u}(r) = u(re)$ where e is a unit vector in \mathbb{C}^n . It is trivial that \tilde{u} is well defined. In the following, we will often identify an integrable radial function u on the unit ball with the corresponding function \tilde{u} defined on the interval $[0, 1]$.

In the following, some basic results concerning Toeplitz operators with radial symbols on the Dirichlet-type space of the unit ball are obtained.

Theorem 10 (see Lu and Sun [5]). Let u be a radial function and $b_\alpha = (2/|\alpha|) \int_0^1 r^{2n+2|\alpha|-1} u(r) dr$. Then,

- (1) T_u is bounded if and only if $\sup_\alpha |b_\alpha| < \infty$;
- (2) T_u is compact if and only if $b_\alpha \rightarrow 0$ ($|\alpha| \rightarrow \infty$).

Theorem 11. Let $u \in L^1(\mathbb{B}_n)$ be a radial function in which T_u is bounded on \mathcal{D} ; then, for any $\alpha \in \mathbb{N}^n - \{(0, \dots, 0)\}$,

$$(T_u z^\alpha)(w) = \frac{2}{|\alpha|} \widehat{u}(2n + 2|\alpha|) w^\alpha. \tag{42}$$

Proof. By the definition of Toeplitz operator, we have

$$\begin{aligned} (T_u z^\alpha)(w) &= \int_{\mathbb{B}_n} u(z) z^\alpha \sum_{\beta \in \mathbb{N}^n - \{(0, \dots, 0)\}} \frac{(|\beta| + n - 1)!}{|\beta| n! \alpha!} \bar{z}^\beta w^\beta dA \\ &= \sum_{\beta \in \mathbb{N}^n - \{(0, \dots, 0)\}} w^\beta \int_{\mathbb{B}_n} u(z) z^\alpha \frac{(|\beta| + n - 1)!}{|\beta| n! \beta!} \bar{z}^\beta dA. \end{aligned} \tag{43}$$

From [13], we know the unique nonzero item in (43) is

$$(T_u z^\alpha)(w) = w^\alpha \int_{\mathbb{B}_n} u(z) z^\alpha \frac{(|\alpha| + n - 1)!}{|\alpha| n! \alpha!} \bar{z}^\alpha dA. \tag{44}$$

A direct computation shows that

$$\begin{aligned} (T_u z^\alpha)(w) &= 2n \frac{(|\alpha| + n - 1)!}{|\alpha| n! \alpha!} w^\alpha \int_0^1 r^{2n-1} dr \int_{\partial \mathbb{B}_n} u(r) (r\xi)^\alpha \overline{(r\xi)^\alpha} dS \\ &= \frac{2}{|\alpha|} w^\alpha \int_0^1 u(r) r^{2n+2|\alpha|-1} dr = \frac{2}{|\alpha|} \widehat{u}(2n + 2|\alpha|) w^\alpha. \end{aligned} \tag{45}$$

□

Theorem 11 shows that the Toeplitz operator with a radial symbol on the Dirichlet-type space of the unit ball acts in a very simple way. On the other hand, the Toeplitz operators of Theorem 11 must be Toeplitz operators with radial symbols.

Theorem 12. Let $u \in L^1(\mathbb{B}_n)$. For each multi-index α , if there exists $\lambda_{|\alpha|} \in \mathbb{C}$ which depends only on $|\alpha|$ such that $T_u z^\alpha = \lambda_{|\alpha|} z^\alpha$, then u is a radial function.

Proof. Suppose $T_u z^\alpha = \lambda_{|\alpha|} z^\alpha$; then, for any unitary transform U of \mathbb{C}^n with

$$U^{-1} = (a_1, \dots, a_n)^T = (a_{ij})_{1 \leq i, j \leq n}, \quad (46)$$

there holds

$$\begin{aligned} & T_u (U^{-1} z)^\alpha \\ &= T_u \left[(a_{11} z_1 + \dots + a_{1n} z_n)^{\alpha_1} \dots (a_{n1} z_1 + \dots + a_{nn} z_n)^{\alpha_n} \right] \\ &= T_u \left[\left(\sum_{|\beta^1|=\alpha_1} \frac{\alpha_1!}{\beta^1!} (a_{11} z_1)^{\beta^1} \dots (a_{1n} z_n)^{\beta^1} \right) \right. \\ &\quad \left. \times \dots \times \left(\sum_{|\beta^n|=\alpha_n} \frac{\alpha_n!}{\beta^n!} (a_{n1} z_1)^{\beta^n} \dots (a_{nn} z_n)^{\beta^n} \right) \right] \\ &= \sum_{|\beta^1|=\alpha_1} \dots \sum_{|\beta^n|=\alpha_n} \frac{\alpha_1!}{\beta^1!} \dots \frac{\alpha_n!}{\beta^n!} (a_1)^{\beta^1} \dots (a_n)^{\beta^n} \\ &\quad \times T_u (z^{\beta^1} \dots z^{\beta^n}) \\ &= \lambda_{|\alpha|} \sum_{|\beta^1|=\alpha_1} \dots \sum_{|\beta^n|=\alpha_n} \frac{\alpha_1!}{\beta^1!} \dots \frac{\alpha_n!}{\beta^n!} (a_1)^{\beta^1} \dots (a_n)^{\beta^n} \\ &\quad \times (z^{\beta^1} \dots z^{\beta^n}) \\ &= \lambda_{|\alpha|} (U^{-1} z)^\alpha, \end{aligned} \quad (47)$$

where β^1, \dots, β^n are multi-indexes. By the definition of Toeplitz operator, note that $\langle z, w \rangle = \sum_{i=1}^n z_i \bar{w}_i$; we have

$$\begin{aligned} & (T_{u \circ U} z^\alpha)(w) \\ &= \int_{\mathbb{B}_n} u(Uz) z^\alpha K(w, z) dA(z) \\ &= \int_{\mathbb{B}_n} u(Uz) z^\alpha \sum_{\beta \in \mathbb{N}^n - \{(0, \dots, 0)\}} \frac{(|\beta| + n - 1)!}{|\beta| n! \beta!} w^\beta \bar{z}^\beta dA(z) \\ &= \int_{\mathbb{B}_n} u(Uz) z^\alpha \sum_{k=1}^{n+k-1} \frac{(n+k-1)!}{kn!k!} \sum_{|\beta|=k} \frac{k!}{\beta!} w^\beta \bar{z}^\beta dA(z) \\ &= \int_{\mathbb{B}_n} u(Uz) z^\alpha \sum_{k=1}^{n+k-1} \frac{(n+k-1)!}{kn!k!} \langle w, z \rangle^k dA(z) \\ &= \int_{\mathbb{B}_n} u(z) (U^{-1} z)^\alpha \sum_{k=1}^{n+k-1} \frac{(n+k-1)!}{kn!k!} \langle w, U^{-1} z \rangle^k dA(z) \\ &= \int_{\mathbb{B}_n} u(z) (U^{-1} z)^\alpha \sum_{k=1}^{n+k-1} \frac{(n+k-1)!}{kn!k!} \langle Uw, z \rangle^k dA(z) \\ &= (T_u (U^{-1} z)^\alpha)(Uw) = \lambda_\alpha w^\alpha = (T_u z^\alpha)(w), \end{aligned} \quad (48)$$

which shows that $T_{u \circ U} = T_u$. Thus, $u \circ U = u$ and u is a radial function. \square

Corollary 13. Let u_1 and u_2 be two radial functions on \mathbb{B}_n such that T_{u_1}, T_{u_2} are bounded. If $T_{u_1} T_{u_2} = T_\nu$, then ν is a radial function and T_ν is a bounded operator.

Proof. Using (42) to calculate $T_{u_1} T_{u_2} z^\alpha$, we obtain

$$T_{u_1} T_{u_2} z^\alpha = \frac{4}{|\alpha|^2} \widehat{u_1} (2n + 2|\alpha|) \widehat{u_2} (2n + 2|\alpha|) z^\alpha. \quad (49)$$

From Theorem 12, ν is a radial function. Moreover, T_ν is obviously a bounded operator. \square

Grudsky et al. [24] gave a particular answer to this question considering the Toeplitz operators with radial symbols. In [22], Zhou and Dong discussed the same question about the Toeplitz operators with radial symbols; they gave a different way to characterize when the product of two Toeplitz operators is equal to a Toeplitz operator. Let u_1 and u_2 be two radial functions on \mathbb{B}_n which induce bounded Toeplitz operators, and let further

$$A_1(t) = u_1(\sqrt{e^{-t}}), \quad A_2(t) = u_2(\sqrt{e^{-t}}). \quad (50)$$

The formal construction (inverse Fourier-Laplace transform)

$$\mathcal{F}^{-1} A(z) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} A(t) e^{izt} dt, \quad z \in \Pi \cup \mathbb{R}, \quad (51)$$

defines a holomorphic function in the upper half plane $\Pi \subset \mathbb{C}$ which coincides on the real axis with the inverse Fourier transform $\mathcal{F}^{-1} A_1(\zeta)$ of the function $A_1(t)$. Theorem 3.7 of [24] shows that if the function

$$\sqrt{2\pi} (n - i\zeta) \mathcal{F}^{-1} A_1(\zeta) \mathcal{F}^{-1} A_2(\zeta), \quad \zeta \in \mathbb{R}, \quad (52)$$

belongs to Wiener ring W_0 of the inverse Fourier transforms of summable functions, then there exists a Toeplitz operator with the radial symbol ν such that $T_{u_1} T_{u_2} = T_\nu$. The following theorem will give the condition for the product of two Toeplitz operators with radial symbols to be a Toeplitz operator on Dirichlet-type space.

Theorem 14. Let u_1 and u_2 be two radial functions on \mathbb{B}_n which induce bounded Toeplitz operators. Then, $T_{u_1} T_{u_2}$ is equal to the Toeplitz operator T_ν if and only if

$$\nu = (4|z|^{-2n}) *_M u_1 *_M u_2. \quad (53)$$

Proof. For any α , it follows from (42) and (49) that

$$T_{u_1} T_{u_2} z^\alpha = T_\nu z^\alpha \quad (54)$$

if and only if

$$\begin{aligned} \widehat{\nu} (2n + 2|\alpha|) &= \frac{2}{|\alpha|} \widehat{u_1} (2n + 2|\alpha|) \widehat{u_2} (2n + 2|\alpha|) \\ &= \frac{2}{|\alpha|} \widehat{u_1 *_M u_2} (2n + 2|\alpha|). \end{aligned} \quad (55)$$

A direct computation gives

$$\begin{aligned} \widehat{v}(2n+2|\alpha|) &= \widehat{u_1 *_{M} u_2}(2n+2|\alpha|) \int_0^1 4s^{-2n} s^{2n+2|\alpha|-1} ds \\ &= 4r^{-2n}(2n+2|\alpha|) \widehat{u_1 *_{M} u_2}(2n+2|\alpha|) \\ &= ((4r^{-2n}) *_{M} u_1 *_{M} u_2)(2n+2|\alpha|), \end{aligned} \tag{56}$$

where $\widehat{4r^{-2n}}(z)$ is well defined if $\text{Re}(z) > 2n$. By Remark 9, (56) is equivalent to (53). \square

For some products of Toeplitz operators, the following fun result is obtained.

Corollary 15. *Let l and m be two real numbers greater than or equal to $-2n$. Then,*

$$\begin{aligned} T_{|z|^l} T_{|z|^m} &= \begin{cases} \frac{4}{(2n+m)(2n+l)} T_{|z|^{-2n}} + \frac{4}{(2n+l)(l-m)} T_{|z|^l} \\ \quad - \frac{4}{(2n+m)(l-m)} T_{|z|^m} \\ \quad \text{if } l \neq m, \\ \frac{4}{2n+l} T_{|z|^l \ln|z|} + \frac{4}{(2n+l)^2} T_{|z|^{-2n}} - \frac{4}{(2n+l)^2} T_{|z|^l} \\ \quad \text{if } l = m. \end{cases} \end{aligned} \tag{57}$$

Proof. By Theorem 14, $T_{|z|^l} T_{|z|^m}$ is equal to the Toeplitz operator T_v if and only if

$$v = (4|z|^{-2n}) *_{M} u_1 *_{M} u_2. \tag{58}$$

A direct calculation shows that

$$v = \begin{cases} \frac{4}{(2n+m)(2n+l)} |z|^{-2n} + \frac{4}{(2n+l)(l-m)} |z|^l \\ \quad - \frac{4}{(2n+m)(l-m)} |z|^m \\ \quad \text{if } l \neq m, \\ \frac{4}{2n+l} |z|^l \ln|z| + \frac{4}{(2n+l)^2} |z|^{-2n} - \frac{4}{(2n+l)^2} |z|^l \\ \quad \text{if } l = m. \end{cases} \tag{59}$$

The desired result follows from (59). \square

Axler and Cuckovic [16] and Choe and Koo [25] study, respectively, the zero-product problem for two Toeplitz operators with harmonic symbols on unit disk and on the Bergman spaces of the unit ball. In [22], Zhou and Dong discuss the same question about the Toeplitz operators with radial symbols. In the following theorem, we will solve the zero-product problem for several Toeplitz operators with radial symbols acting on the Dirichlet-type space of the unit ball.

Theorem 16. *Suppose that u_1, \dots, u_N are radial functions on \mathbb{B}_n that induce bounded Toeplitz operators on Dirichlet-type space. If $T_{u_1} \cdots T_{u_N} = 0$, then $u_i = 0$ for some i .*

Proof. If $T_{u_1} \cdots T_{u_N} = 0$, then, for any multi-index α , by Theorem 11, there holds

$$\widehat{u_1}(2n+2|\alpha|) \cdots \widehat{u_N}(2n+2|\alpha|) = 0. \tag{60}$$

Set

$$E_i = \{|\alpha| \in \mathbb{N} : u_i(2n+2|\alpha|) = 0\}. \tag{61}$$

Since $\cup_{i=1}^n E_i = \mathbb{N}$, there exists an E_i such that

$$\sum_{|\alpha| \in E_i} \frac{1}{2n+2|\alpha|} = \infty; \tag{62}$$

then, by Remark 9, $u_i = 0$. \square

By Theorem 16, we can show that the only idempotent Toeplitz operators with radial symbols are 0 and I .

Corollary 17. *If u is a radial function on \mathbb{B}_n , then $T_u^2 = T_u$ if and only if either $u = 0$ or $u = 1$.*

Proof. If $T_u^2 = T_u$, then $T_u T_{u-1} = 0$, and, by Theorem 16, $u = 0$ or $u = 1$. The converse implication is trivial. \square

3.3. Commuting Toeplitz Operators with Quasihomogeneous Symbols on Dirichlet-Type Space. In this subsection, commuting Toeplitz operators with bounded quasihomogeneous symbols on the Dirichlet-type space of the unit ball are discussed. The definition of the quasihomogeneous function on the unit disk has been given in [17, 26], and the definition on the unit ball has been given in [22].

Definition 18. Let $\lambda \in \mathbb{Z}^n$ and $u \in L^1(\mathbb{B}_n, dA)$. u is called a quasihomogeneous function of quasihomogeneous degree λ if u is of the form $\xi^\lambda \varphi$, where u is a radial function; that is,

$$u(r\xi) = \xi^\lambda \varphi(r) \tag{63}$$

for any ξ in the unit sphere S and $r \in [0, 1)$.

Remark 19. It is obvious that any $\lambda \in \mathbb{Z}^n$ can be uniquely written as $\beta - \gamma$, where β and γ are two multi-indexes such that $\beta \perp \gamma$. Thus, in this paper, we always define the function

$$\xi^\lambda = \xi^\beta \bar{\xi}^\gamma, \quad \xi \in S_n, \tag{64}$$

for any $\lambda \in \mathbb{Z}^n$.

In the following lemma, a result which we will use often is given.

Lemma 20. Let β, γ be two multi-indexes and let u be a bounded radial function on \mathbb{B}_n . Then, for any multi-index α ,

$$T_{\xi^\beta u}(z^\alpha) = \frac{2}{|\alpha| + |\beta|} \widehat{u}(2n + 2|\alpha| + |\beta|) z^{\alpha+\beta}; \quad (65)$$

$$T_{\xi^\gamma u}(z^\alpha) = \begin{cases} 0, & \text{if } \alpha \not\geq \gamma, \\ \frac{2\alpha!(|\alpha| - |\gamma| + n - 1)!}{(|\alpha| + n - 1)!(|\alpha| - |\gamma|)(\alpha - \gamma)!} \\ \quad \times \widehat{u}(2n + 2|\alpha| - |\gamma|) z^{\alpha-\gamma}, & \\ \text{if } \alpha \geq \gamma; \end{cases} \quad (66)$$

$$T_{\xi^{\beta+\gamma} u}(z^\alpha) = \begin{cases} 0, & \text{if } \alpha + \beta \not\geq \gamma, \\ \frac{2(\alpha + \beta)!(|\alpha| + |\beta| - |\gamma| + n - 1)!}{(|\alpha| + |\beta| + n - 1)!(|\alpha| + |\beta| - |\gamma|)(\alpha + \beta - \gamma)!} \\ \quad \times \widehat{u}(2n + 2|\alpha| + |\beta| - |\gamma|) z^{\alpha+\beta-\gamma}, & \\ \text{if } \alpha + \beta \geq \gamma. \end{cases} \quad (67)$$

Proof. We just need to prove (67). For any $\alpha \in \mathbb{N}^n - \{(0, \dots, 0)\}$,

$$\begin{aligned} & (T_{\xi^{\beta+\gamma} u}(z^\alpha))(w) \\ &= \int_{\mathbb{B}_n} \xi^\beta \bar{\xi}^\gamma u(|z|) z^\alpha \\ & \quad \times \sum_{\zeta \in \mathbb{N}^n - \{(0, \dots, 0)\}} \frac{(|\zeta| + n - 1)!}{|\zeta| n! \zeta!} w^\zeta \bar{z}^\zeta dA \\ &= \sum_{\zeta \in \mathbb{N}^n - \{(0, \dots, 0)\}} \frac{(|\zeta| + n - 1)!}{|\zeta| n! \zeta!} w^\zeta 2n \\ & \quad \times \int_0^1 r^{2n-1+|\alpha|+|\zeta|} u(r) dr \int_{\partial \mathbb{B}_n} \xi^{\beta+\alpha} \bar{\xi}^{\gamma+\zeta} dS \\ &= \begin{cases} 0, & \text{if } \alpha + \beta \not\geq \gamma, \\ \sum_{\zeta \in \mathbb{N}^n - \{(0, \dots, 0)\}} \frac{(|\zeta| + n - 1)!}{|\zeta| n! \zeta!} w^\zeta 2n \widehat{u}(2n + |\alpha| + |\zeta|) \\ \quad \times \int_{\partial \mathbb{B}_n} \xi^{\beta+\alpha} \bar{\xi}^{\gamma+\zeta} dS, & \\ \text{if } \alpha + \beta \geq \gamma, \end{cases} \\ &= \begin{cases} 0, & \text{if } \alpha + \beta \not\geq \gamma, \\ 2n \widehat{u}(2n + 2|\alpha| + |\beta| - |\gamma|) \int_{\partial \mathbb{B}_n} \xi^{\beta+\alpha} \bar{\xi}^{\beta+\alpha} dS \\ \quad \times \frac{(|\alpha| + |\beta| - |\gamma| + n - 1)!}{(|\alpha| + |\beta| - |\gamma|) n! (\alpha + \beta - \gamma)!} w^{\alpha+\beta-\gamma}, & \\ \text{if } \alpha + \beta \geq \gamma. \end{cases} \end{aligned} \quad (68)$$

By calculation, we get the desired results. \square

Now, we can discuss the commuting problem of Toeplitz operators with quasihomogeneous symbols on Dirichlet-type space.

Theorem 21. Let β, γ be two multi-indexes and let u and v be two bounded radial functions on \mathbb{B}_n . If u is not equal to 0, then

$$T_u T_{\xi^\beta \bar{\xi}^\gamma v} = T_{\xi^\beta \bar{\xi}^\gamma v} T_u \quad (69)$$

if and only if either $|\beta| = |\gamma|$ or $v = 0$.

Proof. If (69) holds, then for any multi-index α such that $\alpha + \beta \geq \gamma$; by Theorem 11 and Lemma 20, we have

$$\begin{aligned} & \frac{1}{|\alpha| + |\beta| - |\gamma|} \widehat{u}(2n + 2|\alpha| + 2|\beta| - 2|\gamma|) \\ & \quad \times \widehat{v}(2n + 2|\alpha| + |\beta| - |\gamma|) \\ &= \frac{1}{|\alpha|} \widehat{u}(2n + 2|\alpha|) \widehat{v}(2n + 2|\alpha| + |\beta| - |\gamma|). \end{aligned} \quad (70)$$

Assume that $|\beta| \neq |\gamma|$; without loss of generality, we can also assume that $|\beta| > |\gamma|$, for otherwise we could take the adjoints. Set

$$E = \{|\alpha| \in \mathbb{N} : \widehat{v}(2n + 2|\alpha| + |\beta| - |\gamma|) = 0\}. \quad (71)$$

It will show later that $\sum_{\alpha \in E} (1/|\alpha|) = \infty$, which implies that $v = 0$ by Remark 9.

If $\sum_{\alpha \in E} (1/|\alpha|) < \infty$, we will induce a contradiction. Let

$$E' = E^c \cap \{|\alpha| \in \mathbb{N} : \alpha \in \mathbb{Z}^+, \alpha + \beta \geq \gamma\}, \quad (72)$$

where E^c is the complement of E in \mathbb{N} ; then

$$\sum_{\alpha \in E'} \frac{1}{|\alpha|} = \infty. \quad (73)$$

On the other hand, for any $|\alpha| \in E^c$, (70) gives

$$\begin{aligned} & \frac{1}{|\alpha| + |\beta| - |\gamma|} \widehat{u}(2n + 2|\alpha| + 2|\beta| - 2|\gamma|) \\ &= \frac{1}{|\alpha|} \widehat{u}(2n + 2|\alpha|). \end{aligned} \quad (74)$$

Denote

$$F(z) = \widehat{u}(z + 2n + 2|\beta| - 2|\gamma|) - \frac{z + 2|\beta| - 2|\gamma|}{z} \widehat{u}(z + 2n); \quad (75)$$

then F is analytic and bounded on $\{z : \operatorname{Re}(z) > 0\}$ since v is bounded. Moreover, (74) implies that

$$F(2|\alpha|) = 0, \quad \forall |\alpha| \in E'. \quad (76)$$

According to Theorem 8, F must be zero; thus,

$$\begin{aligned} & \frac{1}{z + 2|\beta| - 2|\gamma|} \widehat{u}(z + 2n + 2|\beta| - 2|\gamma|) \\ &= \frac{1}{z} \widehat{u}(z + 2n + 2|\alpha|) \quad z \in \{z : \operatorname{Re}(z) > 0\}. \end{aligned} \quad (77)$$

For any integer n_0 greater than $2n$, the above equation gives that

$$\begin{aligned} & \frac{1}{n_0 - 2n + 2m(|\beta| - |\gamma|)} \widehat{u}(n_0 + 2m(|\beta| - |\gamma|)) \\ &= \frac{1}{n_0 - 2n} \widehat{u}(n_0) \quad \forall m \in \mathbb{N}. \end{aligned} \tag{78}$$

If we denote $(1/(n_0 - 2n))\widehat{u}(n_0)$ by the constant C , we obtain

$$\begin{aligned} & \frac{1}{(n_0 - 2n + 2m(|\beta| - |\gamma|))} \widehat{u}(n_0 + 2m(|\beta| - |\gamma|)) \\ &= \widehat{|z|^{-2n} * u} = C. \end{aligned} \tag{79}$$

Multiply two sides of (79) by $1/(n_0 + 2m(|\beta| - |\gamma|))$, which leads to

$$\begin{aligned} & (\mathbb{1} * \widehat{|z|^{-2n} * u})(n_0 + 2m(|\beta| - |\gamma|)) \\ &= \widehat{C\mathbb{1}}(n_0 + 2m(|\beta| - |\gamma|)), \end{aligned} \tag{80}$$

where $\mathbb{1}$ denotes the constant function with value one, for any $m \in \mathbb{N}$, by Remark 9 again, and, then, clearly, $(\mathbb{1} * |z|^{-2n} * u)$ is constant.

By calculation, we know that

$$\mathbb{1} * |z|^{-2n} * u(r) = \int_r^1 \frac{1}{2n} \left(\frac{t^{2n}}{r^{2n}} - 1 \right) \frac{u(t)}{t} dt = C; \tag{81}$$

then, derive (81) with respect to variable r ; we get

$$\begin{aligned} & \frac{1}{-2n} r^{-2n-1} \int_r^1 t^{2n-1} u(t) dt - r^{-2n} r^{2n-1} u(r) + u(r) r^{-1} \\ &= \frac{1}{2n} r^{-2n-1} \int_r^1 t^{2n-1} u(t) dt = 0, \end{aligned} \tag{82}$$

and, then, clearly, u is 0, which is a contradiction. Thus, we conclude that either $|\beta| = |\gamma|$ or $\nu = 0$.

Conversely, if $|\beta| = |\gamma|$ or $\nu = 0$, then we can easily show that (70) holds, and, consequently,

$$T_u T_{\xi^{\beta} \bar{\xi}^{\gamma}} = T_{\xi^{\beta} \bar{\xi}^{\gamma}} T_u \tag{83}$$

for each multi-index α , which implies $T_{\xi^{\beta} \bar{\xi}^{\gamma}}$ and T_u commute. \square

It was shown in [27] that a Toeplitz operator with a radial symbol on the Bergman spaces of the unit disk D may only commute with another such operator with a radial symbol, but it is not true in higher dimensions by the theorem above. It is known that every function $f \in L^2(D, dA)$ has the decomposition

$$f(re^{ik\theta}) = \sum_{k=-\infty}^{\infty} e^{ik\theta} f_k(r), \tag{84}$$

where $f_k(r)$ are square integrables in $[0, 1]$ with respect to the measure rdr . More details can be found in [27]. Similarly, let

$$R = \left\{ u : \mathbb{B}_n \rightarrow \mathbb{C} \text{ radial} : \int_0^1 r^{2n-1} |u|^2 dr < \infty \right\}. \tag{85}$$

Denote $R_k = \xi^k R$ for $k \in \mathbb{Z}^n$. Each R_k is a subspace of $L^2(\mathbb{B}_n, dA)$ since, for $f \in R_k$, $f(r\xi) = \xi^k u(r)$ and

$$\begin{aligned} & \int_{\mathbb{B}_n} |f(z)|^2 dA(z) \\ &= 2n \int_0^1 r^{2n+1} |u(r)|^2 dr \int_{\partial \mathbb{B}_n} |\xi^k|^2 dS(\xi) < \infty. \end{aligned} \tag{86}$$

It is also clear that $R_k \perp R_l$ if $k \neq l$. Obviously, there exist many square-integrable functions $f(z)$ on \mathbb{B}_n which do not have the decomposition $f(r\xi) = \sum_{k \in \mathbb{Z}^n} \xi^k f_k(r)$, but we still might study a function of that form. In the one-dimensional case, the function $f(r\xi) = \sum_{k \in \mathbb{Z}^n} \xi^k f_k(r)$ is exactly the same as (84). Moreover, if

$$f(r\xi) = \sum_{k \in \mathbb{Z}^n} \xi^k f_k(r) \in L^\infty(\mathbb{B}_n, dA), \tag{87}$$

then, for each $k \in \mathbb{Z}^n$,

$$|f_k(r)| = \frac{1}{\|\xi^k\|^2} \left| \int_{\partial \mathbb{B}_n} f_k(r) \bar{\xi}^k dS(\xi) \right| \leq \sup_{z \in \mathbb{B}_n} |f(z)|, \tag{88}$$

and so the functions f_k are bounded on \mathbb{B}_n .

Lemma 22. Let $\beta \in \mathbb{N}^n$ be a multi-index and let u be a bounded radial function on \mathbb{B}_n . If

$$f(r\xi) = \sum_{k \in \mathbb{Z}^n} \xi^k f_k(r) \in L^\infty(\mathbb{B}_n, dA), \tag{89}$$

then

$$T_f T_{\xi^\beta u} = T_{\xi^\beta u} T_f \iff T_{\xi^k f_k} T_{\xi^\beta u} = T_{\xi^\beta u} T_{\xi^k f_k}, \quad \forall k \in \mathbb{Z}^n. \tag{90}$$

Proof. Suppose that $f(r\xi) = \sum_{k \in \mathbb{Z}^n} \xi^k f_k(r) \in L^\infty(\mathbb{B}_n, dA)$; then, for any multi-index α , a direct calculation by Lemma 20 gives that

$$\begin{aligned} T_f T_{\xi^\beta u} z^\alpha &= \sum_{k+\alpha+\beta \geq 0} T_{\xi^k f_k} T_{\xi^\beta u} z^\alpha, \\ T_{\xi^\beta u} T_f z^\alpha &= \sum_{k+\alpha+\beta \geq 0} T_{\xi^\beta u} T_{\xi^k f_k} z^\alpha. \end{aligned} \tag{91}$$

If T_f and $T_{\xi^\beta u}$ commute, the equality of the above two series implies that

$$T_{\xi^k f_k} T_{\xi^\beta u} z^\alpha = \begin{cases} 0, & \text{if } k + \alpha + \beta \neq 0, k + \alpha \neq 0, \\ T_{\xi^\beta u} T_{\xi^k f_k} z^\alpha, & \text{if } k + \alpha \geq 0. \end{cases} \tag{92}$$

Recall that $T_{\xi^{\beta}u}T_{\xi^k f}z^{\alpha}$ is equal to zero if $k + \alpha \not\geq 0$, then, for each multi-index α and for all $k \in \mathbb{Z}^n$,

$$T_{\xi^k f}T_{\xi^{\beta}u}z^{\alpha} = T_{\xi^{\beta}u}T_{\xi^k f}z^{\alpha}. \tag{93}$$

Thus, $T_{\xi^k f}T_{\xi^{\beta}u} = T_{\xi^{\beta}u}T_{\xi^k f}$, $\forall k \in \mathbb{Z}^n$. The converse implication is clear. \square

Corollary 23. *Let u be a nonzero bounded radial function on \mathbb{B}_n . If*

$$f(r\xi) = \sum_{k \in \mathbb{Z}^n} \xi^k f_k(r) \in L^{\infty}(\mathbb{B}_n, dV), \tag{94}$$

then $T_u T_f = T_f T_u$ if and only if $f(e^{i\theta}z) = f(z)$ for almost all $\theta \in \mathbb{R}$ and $z \in \mathbb{B}_n$.

Proof. It follows from Lemma 22 that T_u commutes with T_f if and only if

$$T_{\xi^k f}T_u = T_u T_{\xi^k f} \tag{95}$$

for all $k \in \mathbb{Z}^n$. Suppose that $k = \beta - \gamma$, where β and γ are two multi-indexes such that $\beta \perp \gamma$; then, by Theorem 21, the above equation is equivalent to

$$|\beta| = |\gamma| \quad \text{or} \quad f_k = 0. \tag{96}$$

Obviously, (96) is equivalent to

$$(\xi^k f_k)(e^{i\theta}z) = (\xi^k f_k)(z), \quad \forall \theta \in \mathbb{R}, z \in \mathbb{B}_n. \tag{97}$$

Therefore, T_u commutes with T_f if and only if $f(e^{i\theta}z) = f(z)$ for almost all $\theta \in \mathbb{R}$ and $z \in \mathbb{B}_n$. \square

Remark 24. In the one-dimensional case, a function satisfying $f(e^{i\theta}z) = f(z)$ is exactly a radial function, so this corollary coincides with Theorem 6 of [27].

There are lots of examples of functions of the form $\xi^{\beta}u$, which are the symbols of commuting Toeplitz operators (see [27]), but the following theorem will show that two Toeplitz operators with quasihomogeneous symbols of degrees β and $-\gamma$, respectively, commute only in the trivial case.

Theorem 25. *Suppose that β and γ are two nonzero multi-indexes, and let u and v be two bounded radial functions on \mathbb{B}_n . If*

$$T_{\xi^{\beta}u}T_{\xi^{\gamma}v} = T_{\xi^{\gamma}v}T_{\xi^{\beta}u}, \tag{98}$$

then, $u = 0$ or $v = 0$.

Proof. For any multi-index α , it follows from Lemma 20 that if $\alpha \geq \gamma$, then

$$\begin{aligned} & T_{\xi^{\beta}u}T_{\xi^{\gamma}v}z^{\alpha} \\ &= (4\alpha! (|\alpha| - |\gamma| + n - 1)! \hat{u}(2n + 2|\alpha| + |\beta| - 2|\gamma|) \\ & \quad \times \hat{v}(2n + 2|\alpha| - |\gamma|) z^{\alpha + \beta - \gamma}) \\ & \quad \times ((|\alpha| + |\beta| - |\gamma|)(|\alpha| - |\gamma|)(|\alpha| + n - 1)! (\alpha - \gamma)!)^{-1}, \end{aligned} \tag{99}$$

and, if $\alpha \not\geq \gamma$, then $T_{\xi^{\beta}u}T_{\xi^{\gamma}v}z^{\alpha} = 0$.

Similarly, if $\alpha + \beta \geq \gamma$, then

$$\begin{aligned} & T_{\xi^{\gamma}v}T_{\xi^{\beta}u}z^{\alpha} \\ &= (4(\alpha + \beta)! (|\alpha| + |\beta| - |\gamma| + n - 1)! \\ & \quad \times \hat{u}(2n + 2|\alpha| + |\beta|) \hat{v}(2n + 2|\alpha| + 2|\beta| - |\gamma|) z^{\alpha + \beta - \gamma}) \\ & \quad \times ((|\alpha| + |\beta|)(|\alpha| + |\beta| - |\gamma|) \\ & \quad \times (|\alpha| + |\beta| + n - 1)! (\alpha + \beta - \gamma)!)^{-1}, \end{aligned} \tag{100}$$

and, if $\alpha + \beta \not\geq \gamma$, then $T_{\xi^{\gamma}v}T_{\xi^{\beta}u}z^{\alpha} = 0$.

For two nonzero multi-indexes β and γ , to prove this theorem, we need to consider two cases.

Case 1. Suppose that $\beta_{i_0} \neq 0$ and $\gamma \neq 0$ for some $i_0 \in \{1, \dots, n\}$. If $T_{\xi^{\beta}u}$ and $T_{\xi^{\gamma}v}$ commute, then

- (a) $\hat{u}(2n + 2|\alpha| + |\beta|) \hat{v}(2n + 2|\alpha| + 2|\beta| - |\gamma|) = 0$, if $\alpha + \beta \geq \gamma$ and $\alpha \not\geq \gamma$;
- (b) $\hat{u}(2n + 2|\alpha| + |\beta|) \hat{v}(2n + 2|\alpha| + 2|\beta| - |\gamma|) = C_{\alpha} \hat{u}(2n + 2|\alpha| + |\beta| - 2|\gamma|) \hat{v}(2n + 2|\alpha| - |\gamma|)$, if $\alpha \geq \gamma$,

where

$$\begin{aligned} C_{\alpha} &= (\alpha! (|\alpha - |\gamma| + n - 1)! (|\alpha| + |\beta| + n - 1)! (\alpha + \beta - \gamma)!) \\ & \quad \times ((\alpha + \beta)! (|\alpha| + |\beta| - |\gamma| + n - 1)! \\ & \quad \times (|\alpha| + n - 1)! (|\alpha| - |\gamma|) (\alpha - \gamma)!)^{-1}. \end{aligned} \tag{101}$$

Consider the multi-index $\alpha' = (\gamma_1, \dots, \gamma_{i_0-1}, \gamma_{i_0} - 1, \gamma_{i_0+1}, \dots, \gamma_n)$; then $\alpha'_{i_0} < \gamma_{i_0}$ and $\alpha' + \beta \geq \gamma$, since $\beta_{i_0} \neq 0$ and $\gamma_{i_0} \neq 0$. Denote $a_0 = |\alpha'|$; then, it follows from (a) that

$$\hat{u}(2n + 2|a_0| + |\beta|) \hat{v}(2n + 2|a_0| + 2|\beta| - |\gamma|) = 0. \tag{102}$$

If $\hat{u}(2n + 2a_0 + |\beta|) = 0$, we will let $a_1 = a_0 + |\gamma|$; then, a direct calculation from (b) shows that

$$\hat{u}(2n + 2|a_1| + |\beta|) \hat{v}(2n + 2|a_1| + 2|\beta| - |\gamma|) = 0. \tag{103}$$

So we can find a sequence $\{a_m\}_{m \in \mathbb{N}}$, which is defined by $a_m + 1 = a_m + |s|$ or $a_m + |p|$, such that

$$\hat{u}(2n + 2|a_m| + |\beta|) \hat{v}(2n + 2|a_m| + 2|\beta| - |\gamma|) = 0. \tag{104}$$

It is clear that $\sum_{m \in \mathbb{N}} (1/a_m) = \infty$. Let $E_1 = \{m \in \mathbb{N} : \hat{u}(2n + 2a_m + |\beta|) = 0\}$ and $E_2 = \{m \in \mathbb{N} : \hat{v}(2n + 2|a_m| + 2|\beta| - |\gamma|) = 0\}$. Since

$$\sum_{m \in \mathbb{N}} \frac{1}{a_m} \leq \sum_{m \in E_1} \frac{1}{a_m} + \sum_{m \in E_2} \frac{1}{a_m}, \tag{105}$$

we know that at least one of the series $\sum_{m \in E_1} (1/a_m)$ and $\sum_{m \in E_2} (1/a_m)$ diverges; then it follows from Remark 9 that $u = 0$ or $v = 0$.

Case 2. Suppose that either $\beta_i = 0$ or $\gamma_i = 0$ for all $i \in \{1, \dots, n\}$. Without loss of generality, we can also assume that

$|\gamma| \geq |\beta| > 0$. Obviously, for any multi-index $\alpha \geq \gamma$, the fact that either $\beta_i = 0$ or $\gamma_i = 0$ for all $i \in \{1, \dots, n\}$ implies

$$\alpha! (\alpha + \beta - \gamma)! = (\alpha + \beta)! (\alpha - \gamma)! \tag{106}$$

Thus, if $T_{\xi^{\beta_u}}$ and $T_{\xi^{\gamma_v}}$ commute, it follows from (b) that

$$\begin{aligned} & (\widehat{u}(2n + 2|\alpha| + |\beta| - 2|\gamma|) \widehat{v}(2n + 2|\alpha| - |\gamma|)) \\ & \times ((|\alpha| + |\beta| - |\gamma|)(|\alpha| - |\gamma|)) \\ & \times (|\alpha| + n - |\gamma|) \cdots (|\alpha| + n - |\gamma| + |\gamma| - 1)^{-1} \\ & = (\widehat{u}(2n + 2|\alpha| + |\beta|) \widehat{v}(2n + 2|\alpha| + 2|\beta| - |\gamma|)) \\ & \times ((|\alpha| + |\beta| - |\gamma|)(|\alpha| + |\beta|)) \\ & \times (|\alpha| + |\beta| + n - |\gamma|) \cdots \\ & (|\alpha| + |\beta| + n - |\gamma| + |\gamma| - 1)^{-1}. \end{aligned} \tag{107}$$

As in the proof of Theorem 21, the above equation implies that

$$\begin{aligned} & \frac{\widehat{u}(2z + |\beta|) \widehat{v}(2z + |\gamma|)}{(z - n)(z + |\beta| - n)z \cdots (z + |\gamma| - 1)} \\ & = (\widehat{u}(2z + |\beta| + 2|\gamma|) \widehat{v}(2z + 2|\beta| + |\gamma|)) \\ & \times ((z + |\beta| - n)(z - n + |\beta| + |\gamma|)) \\ & \times (z + |\beta|) \cdots (z + |\beta| + |\gamma| - 1)^{-1} \end{aligned} \tag{108}$$

for $z \in \{z : \text{Re}(z) > n\}$. Let

$$f(z) = \frac{\widehat{v}(2z + |\gamma|)}{(z - n)z \cdots (z + |\gamma| - 1)}, \quad g(z) = \frac{\widehat{u}(2z + |\beta|)}{(z + |\beta| - n)}; \tag{109}$$

thus, the above equation can be written as

$$f(z)g(z) = f(z + |\beta|)g(z + |\gamma|). \tag{110}$$

Next, denote

$$\begin{aligned} H(z) & = f(z)f(z + 1) \cdots f(z + |\beta| - 1) \\ & \times g(z)g(z + 1) \cdots g(z + |\gamma| - 1). \end{aligned} \tag{111}$$

Obviously, $H(z)$ is analytic on $\{z : \text{Re}(z) > n\}$. It follows from (110) and $|\gamma| \geq |\beta| > 0$ that

$$\begin{aligned} H(z) & = f(z + |\beta|)g(z + |\gamma|) \cdots f(z + |\beta| - 1 + |\beta|) \\ & \times g(z + |\beta| - 1 + |\gamma|) \\ & \times g(z + |\beta|) \cdots g(z + |\gamma| - 1) \\ & = f(z + |\beta|)f(z + |\beta| + 1) \cdots f(z + 2|\beta| - 1) \\ & \times g(z + |\beta|)g(z + |\beta| + 1) \cdots g(z + |\beta| + |\gamma| - 1) \\ & = H(z + |\beta|), \end{aligned} \tag{112}$$

which implies that $H(z)$ is a periodic function with period $|\beta|$ on $\{z : \text{Re}(z) > n\}$. Thus, the function $H(z)$ can be extended to whole plane \mathbb{C} , so we can think of the function $H(z)$ as an entire function. By the definition of the Mellin transform, we get

$$\begin{aligned} |H(z)| & = |f(z)f(z + 1) \cdots f(z + |\beta| - 1)g(z) \\ & \times g(z + 1) \cdots g(z + |\gamma| - 1)| \\ & \leq \|u\|_{\infty}^{|\beta|} \|v\|_{\infty}^{|\gamma|} \times \left| \frac{1}{(z - n) \cdots (z + |\beta| - n - 1)} \right| \\ & \times \left| \frac{1}{(z + |\beta| - n - 1) \cdots (z + |\beta| + |\gamma| - n - 1)} \right| \\ & \times \left| \int_0^1 t^{2z + |\gamma| - 1} dt \cdots \int_0^1 t^{2z + 2(|\beta| - 1) + |\gamma| - 1} dt \right. \\ & \quad \left. \times \int_0^1 t^{2z + |\beta| - 1} dt \cdots \int_0^1 t^{2z + 2(|\gamma| - 1) + |\beta| - 1} dt \right| \\ & \times \frac{1}{z \cdots (z + |\gamma| - 1)} \cdots \\ & \quad \frac{1}{(z + |\beta| - 1) \cdots (z + |\beta| + |\gamma| - 2)} \\ & = O\left(z^{-|\beta||\gamma| - 2(|\beta| + |\gamma|)}\right). \end{aligned} \tag{113}$$

Noting that $-|\beta||\gamma| - 2(|\beta| + |\gamma|) < 0$, we obtain $H(z) = 0$, which implies $u = 0$ or $v = 0$. \square

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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