

Research Article

A New Method for Research on the Center-Focus Problem of Differential Systems

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We will introduce Mironenko's method to discuss the Poincaré center-focus problem, and compare the methods of Lyapunov and Mironenko. We apply the Mironenko method to discuss the qualitative behavior of solutions of some planar polynomial differential systems and derive the sufficient conditions for a critical point to be a center.

1. Introduction

As we know [1, 2], for polynomial differential system

$$\begin{aligned} \frac{dx}{dt} &= -y + \sum_{i+j=2}^n p_{ij} x^i y^j, \\ \frac{dy}{dt} &= x + \sum_{i+j=2}^n q_{ij} x^i y^j, \end{aligned} \quad (1)$$

where p_{ij} and q_{ij} are real constants, there has been a long-standing problem, called the Poincaré center-focus problem; for the system (1) find explicit conditions of p_{ij} and q_{ij} under which (1) has a center at the origin $(0, 0)$; that is, all the orbits nearby are closed. The problem is equivalent to an analogue for a corresponding periodic equation

$$\frac{dr}{d\theta} = \frac{\sum_{i=0}^{n-2} A_i(\theta) r^i}{1 + \sum_{i=0}^{n-1} B_i(\theta) r^i} r^2 = R(\theta, r) \quad (2)$$

to have periodic solutions. To see this let us note that the phase curves of (1) near the origin $(0, 0)$ in polar coordinates $x = r \cos \theta$, $y = r \sin \theta$ are determined by (2), where $A_i(\theta)$ and $B_i(\theta)$ are polynomials in $\cos \theta$ and $\sin \theta$.

Since the closed orbits of (1) correspond to 2π -periodic solutions of (2), the planar vector field (1) has a center at $(0, 0)$ if and only if (2) has a center at $r = 0$; that is, all the solutions nearby $r = 0$ are periodic: $r(0) = r(2\pi)$ [1, 2].

To discuss the center-focus problem, there are Lyapunov's method and the others; see the works of Z. Zhang and so forth [2–8]. One of the most commonly used methods is Lyapunov's method. However, sometimes, the use of this method to calculate the focus quantities is very difficult and the amount of calculation is very large.

In this paper, we apply the method of Mironenko [9] (reflecting function method) to study the qualitative behavior of solutions of (2) and give the sufficient conditions for $r = 0$ to be a center. We give a few examples to illustrate that, sometimes, Mironenko's method is better than Lyapunov's method.

In the present section, we introduce the concept of the reflecting function, which will be used throughout the rest of this paper.

Consider differential system

$$x' = X(t, x), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^n, \quad (3)$$

which has a continuously differentiable right-hand side and with a general solution $\phi(t; t_0, x_0)$. For each such system, the *reflecting function* is defined (see Mironenko's [9]) as $F(t, x) := \phi(-t, t, x)$. Therefore, for any solution $x(t)$ of (3), we have $F(t, x(t)) = x(-t)$, $F(0, x) = x$.

If system (3) is 2ω -periodic with respect to t , and $F(t, x)$ is its reflecting function, then $T(x) := F(-\omega, x) = \phi(\omega; -\omega, x)$ is the Poincaré mapping of (3) over the period $[-\omega, \omega]$. Thus, the solution $x = \phi(t; -\omega, x_0)$ of (3) defined on $[-\omega, \omega]$ is 2ω -periodic if and only if x_0 is a fixed point of $T(x)$. The stability

of this periodic solution is equivalent to the stability of the fixed point x_0 .

A differentiable function $F(t, x)$ is a reflecting function of system (3) if and only if it is a solution of the Cauchy problem

$$F'_t + F'_x X(t, x) + X(-t, F) = 0, \quad F(0, x) = x. \quad (4)$$

If $F(t, x)$ is a reflecting function of system (3), then it is also reflecting function of the following systems:

$$x' = X(t, x) + (F'_x)^{-1} G(t, x) - G(-t, F(t, x)), \quad (5)$$

where $G(t, x)$ is an arbitrary vector function. Thus, the shift operators [9, 10] on interval $[-\omega, \omega]$ for such systems coincide, and the qualitative behaviors of the periodic solutions of these systems are the same. See papers [11–18] in which reflecting functions are also applied to study the qualitative behavior of solutions of some nonlinear and nonautonomous differential systems.

In the following, we always assume that all equations in this paper have a continuously differentiable right-hand side and have a unique solution for their initial value problem.

2. Main Results

Now, let us consider differential equation (2).

Lemma 1 ([see [9]). *Suppose that $F(\theta, r)$ is the reflecting function of (2) and $F(\theta + 2\pi, r) = F(\theta, r)$. Then all the solutions of (2) defined on $[-\pi, \pi]$ are 2π -periodic; that is, $r = 0$ is a center.*

Lemma 2 ([see [9]). *If all the solutions of (2) are 2π -periodic, then its reflecting function is 2π -periodic, too.*

First, we introduce the Mironenko's method. Suppose that

$$F = r + \sum_{i=1}^{\infty} a_i(\theta) r^i \quad (6)$$

is the reflecting function of (2), where $a_i(0) = 0$ ($i = 1, 2, \dots$). Substituting it into

$$\frac{\partial F}{\partial \theta} + \frac{\partial F}{\partial r} R(\theta, r) + R(-\theta, F(\theta, r)) = 0 \quad (7)$$

and equating the coefficients of like powers of r , we get the following infinite equations

$$\begin{aligned} a'_1(\theta) + A_0(\theta) + A_0(-\theta) &= 0; \\ a'_2(\theta) + 2a_1(\theta)(A_0(\theta) + A_0(-\theta)) + A_1(\theta) \\ &+ A_1(-\theta) - A_0(\theta)B_1(\theta) \\ &- A_0(-\theta)B_1(-\theta) &= 0; \\ &\vdots \end{aligned} \quad (8)$$

If $a_i(\theta + 2\pi) = a_i(\theta)$ ($i = 1, 2, \dots$), then $F(\theta + 2\pi, r) = F(\theta, r)$ and $r = 0$ is a center. If this is a $a_k(\theta + 2\pi) \neq a_k(\theta)$ and $a_i(\theta + 2\pi) = a_i(\theta)$ ($i = 1, 2, \dots, k-1$), then $a_k(\theta) = \varphi_k(\theta) + h_k$, where $\varphi_k(\theta + 2\pi) = \varphi_k(\theta)$, $h_k \neq 0$ is a constant and when $h_k < 0$, $(0, 0)$ is an unstable focus; when $h_k > 0$, $(0, 0)$ is a stable focus.

By the above description, it seems that Mironenko's method is similar to Lyapunov's method, but do not forget that reflecting function has some good symmetry properties, which make our calculations greatly reduced. The following example will illustrate this advantage.

Example 3. Consider system

$$\begin{aligned} \dot{x} &= -2y + x^2 - xy - 2y^2 - xy^2 - y^3, \\ \dot{y} &= 2x + 3xy - y^2 + xy^2 - y^3. \end{aligned} \quad (9)$$

Taking $x = r \cos \theta$, $y = r \sin \theta$, it is transformed to equation

$$\frac{dr}{d\theta} = \frac{\cos \theta - \sin \theta (1 + r \sin \theta)}{1 + (1 + r \sin \theta)^2} r^2. \quad (10)$$

Now, we use the method of Lyapunov to discuss is the $r = 0$ a center or focus? Suppose that

$$r = c + \sum_{i=1}^{\infty} b_i(\theta) c^i, \quad b_i(0) = 0, \quad i = 1, 2, \dots \quad (11)$$

is a solution of (10). Substituting it into (10), we get

$$\begin{aligned} b'_1(\theta) + b'_2(\theta)c + b'_3(\theta)c^2 + \dots + b'_n(\theta)c^{n-1} + \dots \\ = \frac{1}{2}(\cos \theta - \sin \theta)\psi^2 - \frac{c}{2}\sin \theta \cos \theta \psi^3 \\ + \frac{c^2}{4}\sin^2 \theta (\cos \theta + \sin \theta)\psi^4 \\ - \frac{c^3}{4}\sin^4 \theta \psi^5 - \frac{c^4}{8}\sin^4 \theta (\cos \theta + \sin \theta)\psi^6 \\ + \frac{c^5}{8}\cos \theta \sin^5 \theta \psi^7 + \dots, \end{aligned} \quad (12)$$

where $\psi := 1 + b_1(\theta)c + b_2(\theta)c^2 + \dots + b_n(\theta)c^n + \dots$.

Equating the coefficients of like powers of c implies

$$b'_1(\theta) = \frac{1}{2}(\cos \theta - \sin \theta). \quad (13)$$

Solving it we get

$$\begin{aligned} b_1(\theta) &= \frac{1}{2}(\sin \theta + \cos \theta - 1), \\ b'_2(\theta) &= \frac{1}{2}(\cos 2\theta - \cos \theta + \sin \theta - \sin \theta \cos \theta). \end{aligned} \quad (14)$$

It implies

$$b_2(\theta) = \frac{1}{2} \left(\sin \theta \cos \theta - \sin \theta - \cos \theta + 1 - \frac{1}{2} \sin^2 \theta \right),$$

$$b_3'(\theta) = \frac{1}{2} \sin^3 \theta + \frac{3}{4} \left(\cos \theta - \cos^2 \theta - \sin \theta - 2 \sin^2 \theta \cos \theta + \sin^2 \theta + \sin \theta \cos \theta \right),$$

$$b_3(\theta) = \frac{1}{4} \left(3 \sin \theta + \cos \theta - 3 \sin \theta \cos \theta - 2 \sin^3 \theta + \frac{3}{2} \sin^2 \theta + \frac{2}{3} \cos^3 \theta - \frac{5}{3} \right),$$

$$b_4'(\theta) = -\sin \theta \cos \theta + \frac{3}{4} \cos 2\theta + \frac{1}{4} \sin \theta \cos^2 \theta + \frac{5}{8} \cos \theta \sin^3 \theta + \frac{11}{4} \sin^2 \theta \cos \theta + \frac{3}{8} \sin^4 \theta + \frac{1}{6} \cos^4 \theta - \frac{2}{3} \cos \theta + \frac{2}{3} \sin \theta - \frac{13}{8} \sin^2 \theta \cos^2 \theta + \frac{1}{12} \sin \theta \cos^3 \theta - \frac{1}{4} \cos^3 \theta - \frac{3}{4} \sin^3 \theta,$$

(15)

so

$$b_4 = -\frac{1}{2} \sin^2 \theta - \frac{1}{3} \cos^3 \theta + \frac{5}{32} \sin^4 \theta + \sin^3 \theta - \frac{11}{12} \sin \theta - \frac{1}{48} \cos^4 \theta + \frac{1}{12} \cos \theta + \frac{13}{48} \sin \theta \cos 2\theta + \frac{31}{48} \sin \theta \cos \theta + \frac{13}{48}.$$

(16)

From above, we see that the calculation of the expression of $b_i(\theta)$ ($i = 5, 6, 7, \dots$) will become more difficult. Thus, it is not easy to judge is $r = 0$ a center?

Now, we try to use the method of Mironenko to discuss is $r = 0$ a center? In (7) taking

$$R = \frac{\cos \theta - \sin \theta (1 + r \sin \theta)}{1 + (1 + r \sin \theta)^2} r^2,$$

(17)

we get

$$a_1'(\theta) + a_2'(\theta)r + \dots + a_n'(\theta)r^{n-1} + \dots + (1 + 2a_1(\theta)r + 3a_2(\theta)r^2 + \dots + (n+1)a_n(\theta)r^n + \dots) \times \left(\frac{1}{2} (\cos \theta - \sin \theta) - \frac{r}{2} \sin \theta \cos \theta + \frac{r^2}{4} \sin^2 \theta (\cos \theta + \sin \theta) - \frac{r^3}{4} \sin^4 \theta - \frac{r^4}{8} \sin^4 \theta (\cos \theta - \sin \theta) + \frac{r^5}{8} \cos \theta \sin^4 \theta + \dots \right) + \frac{1}{2} (\cos \theta + \sin \theta) \varphi^2 + \frac{r}{2} \cos \theta \sin \theta \varphi^3$$

$$+ \frac{r^2}{4} \sin^2 \theta (\cos \theta - \sin \theta) \varphi^4 - \frac{r^3}{4} \sin^4 \theta \varphi^5 - \frac{r^4}{8} \sin^4 \theta (\cos \theta + \sin \theta) \varphi^6 - \frac{r^5}{8} \cos \theta \sin^5 \theta \varphi^7 + \dots = 0.$$

(18)

Equating the coefficients of like powers of r implies

$$a_1'(\theta) = -\cos \theta, \quad a_1(\theta) = -\sin \theta; \\ a_2'(\theta) = 2 \sin \theta \cos \theta, \quad a_2(\theta) = \sin^2 \theta; \\ a_3'(\theta) = -3 \sin^2 \theta \cos \theta, \quad a_3(\theta) = -\sin^3 \theta; \\ a_4'(\theta) = 4 \sin^3 \theta \cos \theta, \quad a_4(\theta) = \sin^4 \theta;$$

(19)

and so forth. Now we can conjecture that $a_i(\theta) = (-\sin \theta)^i$, $i = 1, 2, 3, \dots$; that is, $F(\theta, r) = r/(1 + r \sin \theta)$. It is not difficult to check that $F(\theta, r) = r/(1 + r \sin \theta)$ is a solution of (7), so it is the reflecting function of (10). As $F(\theta + 2\pi, r) = F(\theta, r)$, so the origin point $(0, 0)$ of system (9) is a center.

Such example shows us that, sometimes, the method of Mironenko is better than Lyapunov's method. Just this advantage is not enough. By the equivalence, we know that $F(\theta, r) = r/(1 + r \sin \theta)$ also is the reflecting function of the following equations:

$$\frac{dr}{d\theta} = \frac{\cos \theta}{1 + (1 + r \sin \theta)^2} r^2 + (1 + r \sin \theta)^2 G(\theta, r) - G\left(-\theta, \frac{r}{1 + r \sin \theta}\right),$$

(20)

where $G(\theta, r)$ is an arbitrary function. If we take

$$G(\theta, r) = \frac{r^{2k-1}}{(1 + r \sin \theta) (1 + (1 + r \sin \theta)^2)^k},$$

(21)

where k is a positive integer, then (20) becomes

$$\frac{dr}{d\theta} = \frac{r^2 \cos \theta (1 + (1 + r \sin \theta)^2)^{k-1} + r^{2k-1} (1 + r \sin \theta - r^2)}{(1 + (1 + r \sin \theta)^2)^k}.$$

(22)

Taking $x = r \cos \theta$, $y = r \sin \theta$, then (22) is transformed to

$$\frac{dx}{dt} = (x^2 - 2y - 2y^2 - y^3) (2 + 2y + y^2)^{k-1} + x(x^2 + y^2)^{k-1} (1 + y - x^2 - y^2), \\ \frac{dy}{dt} = x(2 + 3y + y^2) (2 + 2y + y^2)^{k-1} + y(x^2 + y^2)^{k-1} (1 + y - x^2 - y^2)$$

(23)

and its origin point $(0, 0)$ is a center too.

Thus, by using the method of Mironenko we not only solve a center-focus problem, but at the same time open a

class of differential equations with the same character of point $r = 0$. Therefore, we can say, sometimes, the method of Mironenko is more effective than Lyapunov's method.

Theorem 4. *If $F(\theta, r)$ is a solution of equation $R(\theta, r) = R(-\theta, F)$ such that $F(0, r) = r$ and satisfies (7), then all the solutions of (2) determined on $[-\pi, \pi]$ are 2π -periodic; that is, $r = 0$ is a center.*

Proof. By the assumptions, we see that $F(\theta, r)$ is a reflecting function of (2) and 2π -periodic; thus by Lemma 1, the present conclusion is correct. \square

Theorem 5. *If there is a function $(\theta, r)(\partial u/\partial r \neq 0)$, $u(\theta + 2\pi, r) = u(\theta, r)$ which satisfies*

$$\frac{\partial u}{\partial \theta} + \frac{\partial u}{\partial r} R(\theta, r) = f(\theta, u), \tag{24}$$

where $f(\theta, u) + f(-\theta, u) = 0$. Then all the solutions of (2) determined on $[-\pi, \pi]$ are 2π -periodic; that is, $r = 0$ is a center.

Proof. Let $F(\theta, r)$ be the reflecting function of (2). Now, we prove that it satisfies $u(\theta, r) = u(-\theta, F)$.

In fact, by relation (24), for any solution $r(\theta)$ of (2), we have

$$\frac{du(\theta, r(\theta))}{d\theta} = f(\theta, u(\theta, r(\theta))). \tag{25}$$

In view of $f(\theta, u) + f(-\theta, u) = 0$, we have [9, page 65]

$$u(\theta, r(\theta)) = u(-\theta, r(-\theta)); \tag{26}$$

that is,

$$u(\theta, r) = u(-\theta, F), \tag{27}$$

which implies that $F(\theta, r)$ is a 2π -periodic function with respect to θ . By Lemma 1, the proof of the present theorem is finished. \square

Corollary 6. *If there is a differentiable 2π -periodic function $a(\theta)$ such that*

$$\alpha_0 := a'(\theta) + \sum_{j=0}^n b_j(\theta) (-a(\theta))^j, \tag{28}$$

$$\alpha_k := \sum_{j=k}^n b_j(\theta) C_j^k (-a(\theta))^{k-j}, \quad (k = 1, 2, \dots, n)$$

($C_n^k = n!/k!(n-k)!$) are 2π -periodic and odd functions, then all the solutions defined on $[-\pi, \pi]$ of equation

$$\frac{dr}{d\theta} = \sum_{j=0}^n b_j(\theta) r^j := Y(\theta, r) \tag{29}$$

are 2π -periodic.

This result is implied by Theorem 5 if we take $u = a(\theta) + r$ and put

$$\frac{\partial u}{\partial \theta} + \frac{\partial u}{\partial r} Y(\theta, r) = \sum_{j=0}^n \alpha_j u^j. \tag{30}$$

Example 7. Taking $a(\theta) = \sin \theta$, we get that all the solutions determined on $[-\pi, \pi]$ of the Riccati equation

$$\frac{dr}{d\theta} = \sin^2 \theta \cos \theta - \cos \theta + (\sin \theta \cos \theta + 2\sin^4 \theta) r + \sin^3 \theta r^2 \tag{31}$$

are 2π -periodic.

Theorem 8. *For (2), suppose that*

$$\sum_{i=0}^k (A_i(\theta) B_{k-i}(-\theta) + A_i(-\theta) B_{k-i}(\theta)) = 0, \tag{32}$$

$$k = 1, 2, \dots, 2n - 3,$$

$A_i(\theta) = 0$ ($i > n - 2$), $B_j(\theta) = 0$ ($j > n - 1$), $B_0(\theta) = 1$. Then $r = 0$ is a center.

Proof. By the assumptions, it is not difficult to check that $F(\theta, r) = r$ is the reflecting function of (2), so $r = 0$ is a center. \square

Theorem 9. *For (2), suppose that $\int_0^\pi (A_0 + \bar{A}_0) d\theta = 0$ and*

$$\begin{aligned} \sum_{j=0}^k (B_j(A_0 + \bar{A}_0) - A_j) \sum_{i=0}^{k-j} \bar{B}_i C_{n-1-i}^{k-j-i} \alpha^{k-j-i} \\ = \sum_{j=0}^k (B_j + B_{j-1} \alpha) \sum_{i=0}^{k-j} \bar{A}_i C_{n-2-i}^{k-j-i} \alpha^{k-j-i}, \end{aligned} \tag{33}$$

$$k = 1, 2, \dots, 2n - 2,$$

$$B_0 = 1, \quad A_i = 0, \quad (i > n - 2),$$

$$B_j = 0, \quad (j > n - 1),$$

where $\alpha = \int_0^\theta (A_0 + \bar{A}_0) ds$, $\bar{A}_i = A_i(-\theta)$, $\bar{B}_i = B_i(-\theta)$ ($i = 0, 1, 2, \dots, 2n - 2$). Then all the solutions of (2) and the following equations

$$\frac{dr}{d\theta} = R(\theta, r) + (1 + r\alpha)^2 G(\theta, r) - G\left(-\theta, \frac{r}{1 + r\alpha}\right) \tag{34}$$

determined on $[-\pi, \pi]$ are 2π -periodic; that is, $r = 0$ is a center, where $G(\theta, r)$ is a continuously differentiable 2π -periodic function with respect to θ and $G(\theta, 0) = G(-\theta, 0)$.

Proof. By condition (33), we can check that $F = r/(1 + r\alpha)$ is the reflecting function of (2). As $\int_0^\pi (A_0 + \bar{A}_0) d\theta = 0$, so α and $F = r/(1 + r\alpha)$ are 2π -periodic functions with respect to θ . Then by Lemma 1, $r = 0$ is a center. \square

Theorem 10. *Suppose that for (2) $r = 0$ is a center and functions $\Delta_k(\theta, r)$ are 2π -periodic with respect to θ and satisfy*

$$\Delta_\theta + \Delta_r R(\theta, r) = R_r(\theta, r) \Delta(\theta, r), \quad \Delta(\theta, 0) = 0. \tag{35}$$

Then for the following equations

$$\frac{dr}{d\theta} = R(\theta, r) + \sum_{k=0}^{\infty} \alpha_k(\theta) \Delta_k(\theta, r), \quad (36)$$

$r = 0$ is a center, too. Here $\alpha_k(\theta)$ ($k = 0, 1, 2, \dots$) are arbitrary continuously differentiable 2π -periodic odd functions.

Proof. By the assumptions and [9, 14], we know that the reflecting functions of (2) and (36) coincide and all the solutions of (2) determined on $[-\pi, \pi]$ are 2π -periodic. By Lemma 2, we know that the common reflecting function is 2π -periodic; thus all the solutions of (36) determined on $[-\pi, \pi]$ are 2π -periodic; that is, $r = 0$ is a center, too. \square

Remark 11. Theorem 10 shows us that if the center-focus problem for (2) is already solved, then the same problem for (36) is solved, too.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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