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Research Article

Convexity of Certain q-Integral Operators of p-Valent Functions

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By applying the concept (and theory) of fractional q-calculus, we first define and introduce two new q-integral operators for certain analytic functions defined in the unit disc $\mathscr{U}$. Convexity properties of these q-integral operators on some classes of analytic functions defined by a linear multiplier fractional q-differintegral operator are studied. Special cases of the main results are also mentioned.

1. Introduction and Preliminaries

The subject of fractional calculus has gained noticeable importance and popularity due to its established applications in many fields of science and engineering during the past three decades or so. Much of the theory of fractional calculus is based upon the familiar Riemann-Liouville fractional derivative (or integral). The fractional q-calculus is the extension of the ordinary fractional calculus in the q-theory. Recently, there was a significant increase of activity in the area of the q-calculus due to applications of the q-calculus in mathematics, statistics, and physics. For more details, one may refer to the books [1–4] on the subject. Recently, Purohit and Raina [5–7] have added one more dimension to this study by introducing certain subclasses of functions which are analytic in the open disk $\mathscr{U}$, by using fractional q-calculus. Purohit [8] also studied similar work and considered new classes of multivalently analytic functions in the open unit disk.

The aim of this paper is to consider a linear multiplier fractional q-differintegral operator and to define certain new subclasses of functions which are p-valent and analytic in the open unit disk. The results derived include convexity properties of these q-integral operators on some classes of analytic functions. Special cases of the main results are also mentioned.

Let $\mathscr{A}_p$ denote the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \quad (p \in \mathbb{N} = \{1, 2, 3, \ldots\}), \tag{1}$$

which are analytic and p-valent in the open unit disk $\mathscr{U} = \{z \in \mathbb{C} : |z| < 1\}$. A function $f \in \mathscr{A}_p$ is said to be p-valently starlike of order α $(0 \le \alpha < p)$ if and only if

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \quad (z \in \mathscr{U}). \tag{2}$$

We denote by $\mathscr{S}_p^*(\alpha)$ the class of all such functions. On the other hand, a function $f \in \mathscr{A}_p$ is said to be in the class $\mathscr{C}_p(\alpha)$ of p-valently convex of order α $(0 \le \alpha < p)$ if and only if

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, \quad (z \in \mathscr{U}). \tag{3}$$

Note that $\mathscr{S}_p^*(0) = \mathscr{S}_p^*$ and $\mathscr{C}_p(0) = C_p$ are, respectively, the classes of p-valently starlike and p-valently convex functions in $\mathscr{U}$. Also, we note that $\mathscr{S}_1^*(0) = \mathscr{S}^*$ and $\mathscr{C}_1(0) = \mathscr{C}$ are, respectively, the usual classes of starlike and convex functions in $\mathscr{U}$. A function $f \in \mathscr{A}_p$ is said to be in the class $\mathscr{U}\mathscr{S}_p(\alpha, k)$

of k-uniformly p-valent starlike of order α $(-1 \le \alpha < p)$ if it satisfies

$$\Re\left\{\frac{zf'(z)}{f(z)} - \alpha\right\} \ge k\left|\frac{zf'(z)}{f(z)} - p\right|, \quad (k \ge 0,\ z \in \mathscr{U}). \tag{4}$$

Furthermore, a function $f \in \mathscr{A}_p$ is said to be in the class $\mathscr{UC}_p(\alpha, k)$ of k-uniformly p-valent convex of order α $(-1 \le \alpha < p)$ if it satisfies

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)} - \alpha\right\} \ge k\left|1 + \frac{zf''(z)}{f'(z)} - p\right|, \quad (k \ge 0,\ z \in \mathscr{U}). \tag{5}$$

For uniformly starlike and uniformly convex functions we refer to the papers [9–11]. Note that $\mathscr{US}_1(\alpha, k) = \mathscr{UST}(\alpha, k)$ and $\mathscr{UC}_1(\alpha, k) = \mathscr{UCV}(\alpha, k)$, where the classes $\mathscr{UST}(\alpha, k)$ and $\mathscr{UCV}(\alpha, k)$ are, respectively, the classes of k-uniformly starlike of order α $(0 \le \alpha < 1)$ and k-uniformly convex of order α $(0 \le \alpha < 1)$ studied in [12].

For the convenience of the reader, we now give some basic definitions and related details of q-calculus which are used in the sequel.

For any complex number α the q-shifted factorials are defined as

$$(\alpha; q)_0 = 1, \qquad (\alpha; q)_n = \prod_{k=0}^{n-1}\left(1 - \alpha q^k\right), \quad n \in \mathbb{N}, \tag{6}$$

and in terms of the basic analogue of the gamma function

$$(q^\alpha; q)_n = \frac{\Gamma_q(\alpha + n)(1 - q)^n}{\Gamma_q(\alpha)}, \quad (n > 0), \tag{7}$$

where the q-gamma function is defined by

$$\Gamma_q(x) = \frac{(q, q)_\infty (1 - q)^{1-x}}{(q^x; q)_\infty}, \quad (0 < q < 1). \tag{8}$$

If $|q| < 1$, the definition (6) remains meaningful for $n = \infty$ as a convergent infinite product:

$$(\alpha; q)_\infty = \prod_{j=0}^{\infty}\left(1 - \alpha q^j\right). \tag{9}$$

In view of the relation

$$\lim_{q \to 1^-} \frac{(q^\alpha; q)_n}{(1 - q)^n} = (\alpha)_n, \tag{10}$$

we observe that the q-shifted factorial (6) reduces to the familiar Pochhammer symbol $(\alpha)_n$, where $(\alpha)_n = \alpha(\alpha + 1)\cdots(\alpha + n - 1)$. Also, the q-derivative and q-integral of a function on a subset of $\mathbb{C}$ are, respectively, given by (see [2] pp. 19–22)

$$\begin{gathered} D_q f(z) = \frac{f(z) - f(zq)}{(1 - q)z}, \quad (z \ne 0,\ q \ne 0), \\ \int_0^z f(t)\, d_q t = z(1 - q)\sum_{k=0}^{\infty} q^k f\left(zq^k\right). \end{gathered} \tag{11}$$

Therefore, the q-derivative of $f(z) = z^n$, where n is a positive integer, is given by

$$D_q z^n = \frac{z^n - (zq)^n}{(1 - q)z} = [n]_q\, z^{n-1}, \tag{12}$$

where

$$[n]_q = \frac{1 - q^n}{1 - q} = q^{n-1} + \cdots + 1 \tag{13}$$

and is called the q-analogue of n. As $q \to 1$, we have $[n]_q = q^{n-1} + \cdots + 1 \to 1 + \cdots + 1 = n$.

The q-analogues to the function classes $\mathscr{S}_p^*(\alpha)$, $\mathscr{C}_p(\alpha)$, $\mathscr{US}_p(\alpha, k)$, and $\mathscr{UC}_p(\alpha, k)$ are given as follows.

A function $f \in \mathscr{A}_p$ is said to be in the class $\mathscr{S}_{q,p}^*(\alpha)$ of p-valently starlike with respect to q-differentiation of order α $(0 \le \alpha < p)$ if it satisfies

$$\Re\left\{\frac{zD_q(f(z))}{f(z)}\right\} > \alpha, \quad (z \in \mathscr{U}). \tag{14}$$

Also, a function $f \in \mathscr{A}_p$ is said to be in the class $\mathscr{C}_{q,p}(\alpha)$ of p-valently convex with respect to q-differentiation of order α $(0 \le \alpha < p)$ if it satisfies

$$\Re\left\{1 + \frac{zD_q^2(f(z))}{D_q(f(z))}\right\} > \alpha, \quad (z \in \mathscr{U}). \tag{15}$$

On the other hand, a function $f \in \mathscr{A}_p$ is said to be in the class $\mathscr{US}_{q,p}(\alpha, k)$ of k-uniformly p-valent starlike with respect to q-differentiation of order α $(-1 \le \alpha < p)$ if it satisfies

$$\Re\left\{\frac{zD_q(f(z))}{f(z)} - \alpha\right\} \ge k\left|\frac{zD_q(f(z))}{f(z)} - [p]_q\right|, \quad (k \ge 0,\ z \in \mathscr{U}). \tag{16}$$

Furthermore, a function $f \in \mathscr{A}_p$ is said to be in the class $\mathscr{UC}_{q,p}(\alpha, k)$ of k-uniformly p-valent convex with respect to q-differentiation of order α $(-1 \le \alpha < p)$ if it satisfies

$$\Re\left\{1 + \frac{zD_q^2(f(z))}{D_q(f(z))} - \alpha\right\} \ge k\left|1 + \frac{zD_q^2(f(z))}{D_q(f(z))} - [p]_q\right|, \quad (k \ge 0,\ z \in \mathscr{U}). \tag{17}$$

In the following, we define the fractional q-calculus operators of a complex-valued function $f(z)$, which were recently studied by Purohit and Raina [5].

Definition 1 (fractional q-integral operator). The fractional q-integral operator $I_{q,z}^\delta$ of a function $f(z)$ of order δ is defined by

$$\begin{aligned} I_{q,z}^\delta f(z) &\equiv D_{q,z}^{-\delta} f(z) \\ &= \frac{1}{\Gamma_q(\delta)}\int_0^z (z - tq)_{\delta-1} f(t)\, d_q t, \quad (\delta > 0), \end{aligned} \tag{18}$$

where $f(z)$ is analytic in a simply connected region of the z-plane containing the origin and the q-binomial function $(z-tq)_{\delta-1}$ is given by

$$(z-tq)_{\delta-1}=z^{\delta-1}\,{}_1\Phi_0\left[q^{-\delta+1};-;q,\frac{tq^{\delta}}{z}\right]. \tag{19}$$

The series ${}_1\Phi_0[\delta;-;q,z]$ is single valued when $|\arg(z)|<\pi$ and $|z|<1$ (see for details [2], pp. 104–106); therefore, the function $(z-tq)_{\delta-1}$ in (18) is single valued when $|\arg(-tq^{\delta}/z)|<\pi$, $|tq^{\delta}/z|<1$, and $|\arg(z)|<\pi$.

Definition 2 (fractional q-derivative operator). The fractional q-derivative operator $D_{q,z}^{\delta}$ of a function $f(z)$ of order δ is defined by

$$D_{q,z}^{\delta}f(z)\equiv D_{q,z}\,I_{q,z}^{1-\delta}f(z)=\frac{1}{\Gamma_q(1-\delta)}\times D_{q,z}\int_0^z(z-tq)_{-\delta}f(t)\,d_qt,\quad(0\le\delta<1), \tag{20}$$

where $f(z)$ is suitably constrained and the multiplicity of $(z-tq)_{-\delta}$ is removed as in Definition 1.

Definition 3 (extended fractional q-derivative operator). Under the hypotheses of Definition 2, the fractional q-derivative for a function $f(z)$ of order δ is defined by

$$D_{q,z}^{\delta}f(z)=D_{q,z}^{m}I_{q,z}^{m-\delta}f(z), \tag{21}$$

where $m-1\le\delta<1$, $m\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$, and $\mathbb{N}$ denotes the set of natural numbers.

Remark 4. It follows from Definition 2 that

$$D_{q,z}^{\delta}z^{n}=\frac{\Gamma_q(n+1)}{\Gamma_q(n+1-\delta)}z^{n-\delta}\quad(\delta\ge0,\ n>-1). \tag{22}$$

2. The Operator $\mathscr{D}_{q,p,\lambda}^{\delta,m}$

Using $D_{q,z}^{\delta}$, we define a q-differintegral operator $\Omega_{q,p}^{\delta}:\mathscr{A}_p\to\mathscr{A}_p$ as follows:

$$\begin{aligned}\Omega_{q,p}^{\delta}f(z)&=\frac{\Gamma_q(p+1-\delta)}{\Gamma_q(p+1)}z^{\delta}D_{q,z}^{\delta}f(z)\\&=z^{p}+\sum_{n=p+1}^{\infty}\frac{\Gamma_q(p+1-\delta)\Gamma_q(n+1)}{\Gamma_q(p+1)\Gamma_q(n+1-\delta)}a_nz^{n},\\&\qquad(\delta<p+1;n\in\mathbb{N};0<q<1;z\in\mathscr{U}),\end{aligned} \tag{23}$$

where $D_{q,z}^{\delta}f(z)$ in (23) represents, respectively, a fractional q-integral of $f(z)$ of order δ when $-\infty<\delta<0$ and a fractional q-derivative of $f(z)$ of order δ when $0\le\delta<p+1$. Here we note that $\Omega_{q,p}^{0}f(z)=f(z)$.

We now define a linear multiplier fractional q-differintegral operator $\mathscr{D}_{q,p,\lambda}^{\delta,m}$ as follows:

$$\begin{aligned}\mathscr{D}_{q,p,\lambda}^{\delta,0}f(z)&=f(z),\\\mathscr{D}_{q,p,\lambda}^{\delta,1}f(z)&=(1-\lambda)\Omega_{q,p}^{\delta}f(z)\\&\quad+\frac{\lambda z}{[p]_q}D_q\left(\Omega_{q,p}^{\delta}f(z)\right),\quad(\lambda\ge0),\\\mathscr{D}_{q,p,\lambda}^{\delta,2}f(z)&=\mathscr{D}_{q,p,\lambda}^{\delta,1}\left(\mathscr{D}_{q,p,\lambda}^{\delta,1}f(z)\right),\\&\ \ \vdots\\\mathscr{D}_{q,p,\lambda}^{\delta,m}f(z)&=\mathscr{D}_{q,p,\lambda}^{\delta,1}\left(\mathscr{D}_{q,p,\lambda}^{\delta,m-1}f(z)\right),\qquad m\in\mathbb{N}.\end{aligned} \tag{24}$$

If $f(z)\in\mathscr{A}_p$ is given by (1), then by (24) we have

$$\begin{aligned}&\mathscr{D}_{q,p,\lambda}^{\delta,m}f(z)\\&=z^{p}+\sum_{n=p+1}^{\infty}\left(\frac{\Gamma_q(p+1-\delta)\Gamma_q(n+1)}{\Gamma_q(p+1)\Gamma_q(n+1-\delta)}\left[1-\lambda+\frac{[n]_q}{[p]_q}\lambda\right]\right)^{m}\\&\quad\times a_nz^{n}.\end{aligned} \tag{25}$$

It can be seen that, by specializing the parameters, the operator $\mathscr{D}_{q,p,\lambda}^{\delta,m}$ reduces to many known and new integral and differential operators. In particular, when $\delta=0$, $p=1$, and $q\to1$ the operator $\mathscr{D}_{q,p,\lambda}^{\delta,m}$ reduces to the operator introduced by AL-Oboudi [13] and if $\delta=0$, $p=1$, $\lambda=1$, and $q\to1$ it reduces to the operator introduced by Sălăgean [14].

By using the operator $\mathscr{D}_{q,p,\lambda}^{\delta,m}f(z)$ defined by (24) and q-differentiation, we introduce two new subclasses of analytic functions $\mathscr{US}_{q,p,\lambda}^{\delta,m}(\alpha,k)$ and $\mathscr{UC}_{q,p,\lambda}^{\delta,m}(\alpha,k)$ as follows.

A function $f\in\mathscr{A}_p$ is said to be in the class $\mathscr{US}_{q,p,\lambda}^{\delta,m}(\alpha,k)$ if and only if

$$\begin{aligned}&\Re\left\{\frac{zD_q\left(\mathscr{D}_{q,p,\lambda}^{\delta,m}f(z)\right)}{\mathscr{D}_{q,p,\lambda}^{\delta,m}f(z)}-\alpha\right\}\\&\ge k\left|\frac{zD_q\left(\mathscr{D}_{q,p,\lambda}^{\delta,m}f(z)\right)}{\mathscr{D}_{q,p,\lambda}^{\delta,m}f(z)}-[p]_q\right|,\quad(-1\le\alpha<p,\ k\ge0).\end{aligned} \tag{26}$$

Furthermore, a function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{UC}^{\delta,m}_{q,p,\lambda}(\alpha,k)$ if and only if

$$\Re\left\{1+\frac{zD_q^2\left(\mathcal{D}^{\delta,m}_{q,p,\lambda}f(z)\right)}{D_q\left(\mathcal{D}^{\delta,m}_{q,p,\lambda}f(z)\right)}-\alpha\right\} \geq k\left|1+\frac{zD_q^2\left(\mathcal{D}^{\delta,m}_{q,p,\lambda}f(z)\right)}{D_q\left(\mathcal{D}^{\delta,m}_{q,p,\lambda}f(z)\right)}-[p]_q\right|, \quad (-1\leq\alpha<p, k\geq 0). \tag{27}$$

It is interesting to note that the classes $\mathcal{US}^{\delta,m}_{q,p,\lambda}(\alpha,k)$ and $\mathcal{UC}^{\delta,m}_{q,p,\lambda}(\alpha,k)$ generalize several well-known subclasses of analytic functions. For instance, if $q\to 1$, then

(1) $\mathcal{US}^{\delta,0}_{1,p,\lambda}(\alpha,k)=\mathcal{US}_p(\alpha,k)$,

(2) $\mathcal{US}^{0,1}_{1,p,1}(\alpha,k)=\mathcal{UC}^{\delta,0}_{1,p,\lambda}(\alpha,k)=\mathcal{UC}_p(\alpha,k)$,

(3) $\mathcal{US}^{\delta,0}_{1,p,\lambda}(\alpha,0)=\mathcal{S}^*_p(\alpha)$,

(4) $\mathcal{US}^{0,1}_{1,p,1}(\alpha,0)=\mathcal{UC}^{\delta,0}_{1,p,\lambda}(\alpha,0)=\mathcal{C}_p(\alpha)$.

3. The p-Valent q-Integral Operators F_q and G_q

We now introduce two new p-valent q-integral operators as follows.

Definition 5. Let $m=(m_1,m_2,\ldots,m_n)\in\mathbb{N}_0^n$, $\gamma=(\gamma_1,\gamma_2,\ldots,\gamma_n)\in\mathbb{R}_+^n$ and $f_i\in\mathcal{A}_p$ for all $i=\{1,2,\ldots,n\}$, $n\in\mathbb{N}$. Then $F_q(z):\mathcal{A}_p^n\to\mathcal{A}_p$ is defined as

$$\begin{aligned}F_q(z)&=\mathcal{F}^{\delta,\gamma,m}_{q,p,\lambda}(f_1,f_2,\ldots,f_n)\\&=\int_0^z[p]_q\,t^{p-1}\prod_{i=1}^n\left(\frac{\mathcal{D}^{\delta,m_i}_{q,p,\lambda}f_i(t)}{t^p}\right)^{\gamma_i}d_qt,\end{aligned}\tag{28}$$

and $G_q(z):\mathcal{A}_p^n\to\mathcal{A}_p$ is defined as

$$\begin{aligned}G_q(z)&=\mathcal{G}^{\delta,\gamma,m}_{q,p,\lambda}(f_1,f_2,\ldots,f_n)\\&=\int_0^z[p]_q t^{p-1}\prod_{i=1}^n\left(\frac{D_q\left(\mathcal{D}^{\delta,m_i}_{q,p,\lambda}f_i(t)\right)}{[p]_q\,t^{p-1}}\right)^{\gamma_i}d_qt,\end{aligned}\tag{29}$$

where $\mathcal{D}^{\delta,m_i}_{q,p,\lambda}f_i(t)$ is given by (24).

It is interesting to observe that several well-known and new integral operators are special cases of the operators $F_q(z)$ and $G_q(z)$. We list a few of them in the following remarks.

Remark 6. Letting $m_i=0$ for all $i=\{1,2,\ldots,n\}$ and $q\to 1$, the q-integral operator $F_q(z)$ reduces to the operator $F_p(z)$ studied by Frasin in [15]. Upon setting $p=1$, $\delta=0$, $\lambda=1$, and $q\to 1$, we obtain the integral operator $D^kF(z)$ studied by Breaz et al. in [16]. For $p=1$, $m_1=m_2=\cdots=m_n=0$, and $q\to 1$, the operator $F_q(z)$ reduces to the operator $F_n(z)$ which was studied by D. Breaz and N. Breaz in [17]. Observe that when $p=n=1$, $m_1=0$, $\gamma_1=\gamma$, and $q\to 1$, we obtain the integral operator $I_\gamma(f)(z)$ studied by Pescar and Owa in [18]. Also, for $p=n=1$, $m_1=0$, $\gamma_1=1$, and $q\to 1$, the q-integral operator $F_q(z)$ reduces to the Alexander integral operator $I(f)(z)$ studied in [19].

Remark 7. Letting $m_i=0$ for all $i=\{1,2,\ldots,n\}$ and $q\to 1$, the q-integral operator $G_q(z)$ reduces to the operator $G_p(z)$ studied by Frasin in [15]. For $p=1$, $m_1=m_2=\cdots=m_n=0$ and $q\to 1$, the operator $G_q(z)$ reduces to the operator $G_{\gamma_1,\gamma_2,\ldots,\gamma_n}(z)$ which was studied by Breaz et al. (see [20]). Also, for $p=n=1$, $m_1=0$, $\gamma_1=1$, and $q\to 1$, the q-integral operator $G_q(z)$ reduces to the integral operator $G(z)$ introduced and studied by Pfaltzgra (see [21]).

In this paper, we obtain the order of convexity with respect to q-differentiation of the q-integral operators $F_q(z)$ and $G_q(z)$ on the classes $\mathcal{US}^{\delta,m}_{q,p,\lambda}(\alpha,k)$ and $\mathcal{UC}^{\delta,m}_{q,p,\lambda}(\alpha,k)$. As special cases, the order of convexity of the operators $\int_0^z(f(t)/t)^\gamma dt$ and $\int_0^z(f'(t))^\gamma dt$ is also given.

4. Convexity of the Operator F_q

First, we prove the following convexity result with respect to q-differentiation of the operator F_q.

Theorem 8. *Let $m=(m_1,m_2,\ldots,m_n)\in\mathbb{N}_0^n$, $\gamma=(\gamma_1,\gamma_2,\ldots,\gamma_n)\in\mathbb{R}_+^n$, $-1\leq\alpha_i<p$, $k_i>0$, and $f_i\in\mathcal{US}^{\delta,m_i}_{q,p,\lambda}(\alpha_i,k_i)$ for all $i=\{1,2,\ldots,n\}$, $n\in\mathbb{N}$. If*

$$0\leq p+\sum_{i=1}^n\gamma_i(\alpha_i-p)<p,\tag{30}$$

then the q-integral operator $F_q(z)$ defined by (28) is p-valently convex with respect to q-differentiation of order $p+\sum_{i=1}^n\gamma_i(\alpha_i-p)$.

Proof. From (28), we observe that $F_q(z)\in\mathcal{A}_p$. On the other hand, it is easy to verify that

$$D_q\left(F_q(z)\right)=[p]_qz^{p-1}\prod_{i=1}^n\left(\frac{\mathcal{D}^{\delta,m_i}_{q,p,\lambda}f_i(z)}{z^p}\right)^{\gamma_i}.\tag{31}$$

Now by logarithmic q-differentiation we have

$$\frac{\ln q}{q-1}\frac{D_q^2\left(F_q(z)\right)}{D_q\left(F_q(z)\right)}=\frac{\ln q}{q-1}\left[\frac{p-1}{z}+\sum_{i=1}^n\gamma_i\left(\frac{D_q\left(\mathcal{D}^{\delta,m_i}_{q,p,\lambda}f_i(z)\right)}{\mathcal{D}^{\delta,m_i}_{q,p,\lambda}f_i(z)}-\frac{p}{z}\right)\right].\tag{32}$$

Therefore,

$$1+\frac{zD_q^2\left(F_q(z)\right)}{D_q\left(F_q(z)\right)}=p+\sum_{i=1}^{n}\gamma_i\left(\frac{zD_q\left(\mathscr{D}_{q,p,\lambda}^{\delta,m_i}f_i(z)\right)}{\mathscr{D}_{q,p,\lambda}^{\delta,m_i}f_i(z)}-\alpha_i\right)+\sum_{i=1}^{n}\gamma_i(\alpha_i-p). \tag{33}$$

Taking the real parts on both sides of the above equation, we have

$$\Re\left\{1+\frac{zD_q^2\left(F_q(z)\right)}{D_q\left(F_q(z)\right)}\right\}=p+\sum_{i=1}^{n}\gamma_i(\alpha_i-p)+\sum_{i=1}^{n}\gamma_i\Re\left(\frac{zD_q\left(\mathscr{D}_{q,p,\lambda}^{\delta,m_i}f_i(z)\right)}{\mathscr{D}_{q,p,\lambda}^{\delta,m_i}f_i(z)}-\alpha_i\right). \tag{34}$$

Since $f_i\in\mathscr{U}\mathscr{S}_{q,p,\lambda}^{\delta,m_i}(\alpha_i,k_i)$ for all $i=\{1,2,\ldots,n\}$, from (26) we get

$$\Re\left\{1+\frac{zD_q^2\left(F_q(z)\right)}{D_q\left(F_q(z)\right)}\right\}\geq p+\sum_{i=1}^{n}\gamma_i(\alpha_i-p)+\sum_{i=1}^{n}\gamma_i\,k_i\left|\frac{zD_q\left(\mathscr{D}_{q,p,\lambda}^{\delta,m_i}f_i(z)\right)}{\mathscr{D}_{q,p,\lambda}^{\delta,m_i}f_i(z)}-[p]_q\right|. \tag{35}$$

As $\sum_{i=1}^{n}\gamma_i\,k_i|(zD_q(\mathscr{D}_{q,p,\lambda}^{\delta,m_i}f_i(z))/\mathscr{D}_{q,p,\lambda}^{\delta,m_i}f_i(z))-[p]_q|>0$, for all $i=\{1,2,\ldots,n\}$, we obtain from the above

$$\Re\left\{1+\frac{zD_q^2\left(F_q(z)\right)}{D_q\left(F_q(z)\right)}\right\}>p+\sum_{i=1}^{n}\gamma_i(\alpha_i-p). \tag{36}$$

This completes the proof. □

Corollary 9. *Let $m=(m_1,m_2,\ldots,m_n)\in\mathbb{N}_0^n$, $\gamma=(\gamma_1,\gamma_2,\ldots,\gamma_n)\in\mathbb{R}_+^n$, $-1\leq\alpha_i<p$, $k_i>0$ and $f_i\in\mathscr{U}\mathscr{S}_{q,p,\lambda}^{\delta,m_i}(\alpha_i,k_i)$ for all $i=\{1,2,\ldots,n\}$, $n\in\mathbb{N}$. If*

$$\left|\frac{zD_q\left(\mathscr{D}_{q,p,\lambda}^{\delta,m_i}f_i(z)\right)}{\mathscr{D}_{q,p,\lambda}^{\delta,m_i}f_i(z)}-[p]_q\right|>-\frac{p+\sum_{i=1}^{n}\gamma_i(\alpha_i-p)}{\sum_{i=1}^{n}\gamma_i k_i}, \tag{37}$$

for all $i=\{1,2,\ldots,n\}$, then the q-integral operator $F_q(z)$ defined by (28) is p-valently convex with respect to q-differentiation in $\mathscr{U}$.

Proof. From (35) and (37) we easily find that $F_q\in\mathscr{C}_{q,p}$. □

Letting $q\to1$, $p=n=1$, $m_1=0$, $\gamma_1=\gamma$, $\alpha_1=\alpha$, $k_1=k$, and $f_1=f$ in Theorem 8, we have the following.

Corollary 10. *Let $\gamma>0$, $-1\leq\alpha<1$, $k>0$, and $f\in\mathscr{U}\mathscr{S}\mathscr{T}(\alpha,k)$. If $0\leq1+\gamma(\alpha-1)<1$, then the integral operator $\int_0^z(f(t)/t)^\gamma dt$ is convex of order $1+\gamma(\alpha-1)$ in $\mathscr{U}$.*

5. Convexity of the Operator G_q

Now, we prove the following convexity result with respect to q-differentiation of the operator G_q.

Theorem 11. *Let $m=(m_1,m_2,\ldots,m_n)\in\mathbb{N}_0^n$, $\gamma=(\gamma_1,\gamma_2,\ldots,\gamma_n)\in\mathbb{R}_+^n$, $-1\leq\alpha_i<p$, $k_i>0$, and $f_i\in\mathscr{U}\mathscr{C}_{q,p,\lambda}^{\delta,m_i}(\alpha_i,k_i)$ for all $i=\{1,2,\ldots,n\}$, $n\in\mathbb{N}$. If*

$$0\leq p+\sum_{i=1}^{n}\gamma_i(\alpha_i-p)<p, \tag{38}$$

then the q-integral operator $G_q(z)$ defined by (29) is p-valently convex with respect to q-differentiation of order $p+\sum_{i=1}^{n}\gamma_i(\alpha_i-p)$.

Proof. From (29), we observe that $G_q(z)\in\mathscr{A}_p$. On the other hand, it is easy to verify that

$$D_q\left(G_q(z)\right)=[p]_q\,z^{p-1}\prod_{i=1}^{n}\left(\frac{D_q\left(\mathscr{D}_{q,p,\lambda}^{\delta,m_i}f_i(z)\right)}{[p]_q\,z^{p-1}}\right)^{\gamma_i}. \tag{39}$$

Now by logarithmic q-differentiation we have

$$\frac{\ln q}{q-1}\frac{D_q^2\left(G_q(z)\right)}{D_q\left(G_q(z)\right)}=\frac{\ln q}{q-1}\left[\frac{p-1}{z}+\sum_{i=1}^{n}\gamma_i\left(\frac{D_q^2\left(\mathscr{D}_{q,p,\lambda}^{\delta,m_i}f_i(z)\right)}{D_q\left(\mathscr{D}_{q,p,\lambda}^{\delta,m_i}f_i(z)\right)}-\frac{p-1}{z}\right)\right]. \tag{40}$$

Therefore,

$$1+\frac{zD_q^2\left(G_q(z)\right)}{D_q\left(G_q(z)\right)}=p+\sum_{i=1}^{n}\gamma_i\left(1+\frac{D_q^2\left(\mathscr{D}_{q,p,\lambda}^{\delta,m_i}f_i(z)\right)}{D_q\left(\mathscr{D}_{q,p,\lambda}^{\delta,m_i}f_i(z)\right)}-\alpha_i\right)+\sum_{i=1}^{n}\gamma_i(\alpha_i-p). \tag{41}$$

Taking the real parts on both sides of the above equation, we have

$$\Re\left\{1+\frac{zD_q^2\left(G_q(z)\right)}{D_q\left(G_q(z)\right)}\right\} = p+\sum_{i=1}^{n}\gamma_i\left(\alpha_i-p\right)+\sum_{i=1}^{n}\gamma_i\Re\left(1+\frac{D_q^2\left(\mathscr{D}_{q,p,\lambda}^{\delta,m_i}f_i(z)\right)}{D_q\left(\mathscr{D}_{q,p,\lambda}^{\delta,m_i}f_i(z)\right)}-\alpha_i\right). \quad (42)$$

Since $f_i \in \mathscr{UC}_{q,p,\lambda}^{\delta,m_i}(\alpha_i,k_i)$ for all $i=\{1,2,\ldots,n\}$, from (27) we get

$$\Re\left\{1+\frac{zD_q^2\left(G_q(z)\right)}{D_q\left(G_q(z)\right)}\right\} \geq p+\sum_{i=1}^{n}\gamma_i\left(\alpha_i-p\right) + \sum_{i=1}^{n}\gamma_i k_i\left|1+\frac{D_q^2\left(\mathscr{D}_{q,p,\lambda}^{\delta,m_i}f_i(z)\right)}{D_q\left(\mathscr{D}_{q,p,\lambda}^{\delta,m_i}f_i(z)\right)}-[p]_q\right|. \quad (43)$$

As $\sum_{i=1}^{n}\gamma_i\, k_i|1+(D_q^2(\mathscr{D}_{q,p,\lambda}^{\delta,m_i}f_i(z))/D_q(\mathscr{D}_{q,p,\lambda}^{\delta,m_i}f_i(z)))-[p]_q|>0$, for all $i=\{1,2,\ldots,n\}$, we obtain from the above

$$\Re\left\{1+\frac{zD_q^2\left(G_q(z)\right)}{D_q\left(G_q(z)\right)}\right\} > p+\sum_{i=1}^{n}\gamma_i\left(\alpha_i-p\right). \quad (44)$$

This completes the proof. □

Corollary 12. *Let* $m=(m_1,m_2,\ldots,m_n)\in\mathbb{N}_0^n$, $\gamma=(\gamma_1,\gamma_2,\ldots,\gamma_n)\in\mathbb{R}_+^n$, $-1\leq\alpha_i<p$, $k_i>0$, *and* $f_i\in\mathscr{UC}_{q,p,\lambda}^{\delta,m_i}(\alpha_i,k_i)$ *for all* $i=\{1,2,\ldots,n\}$, $n\in\mathbb{N}$. *If*

$$\left|1+\frac{D_q^2\left(\mathscr{D}_{q,p,\lambda}^{\delta,m_i}f_i(z)\right)}{D_q\left(\mathscr{D}_{q,p,\lambda}^{\delta,m_i}f_i(z)\right)}-[p]_q\right| > -\frac{p+\sum_{i=1}^{n}\gamma_i\left(\alpha_i-p\right)}{\sum_{i=1}^{n}\gamma_i\,k_i}, \quad (45)$$

for all $i=\{1,2,\ldots,n\}$, *then the q-integral operator* $G_q(z)$ *defined by* (29) *is p-valently convex with respect to q-differentiation in* $\mathscr{U}$.

Proof. From (43) and (45) we easily find that $G_q\in\mathscr{C}_{q,p}$. □

Letting $q\to1$, $p=n=1$, $m_1=0$, $\gamma_1=\gamma$, $\alpha_1=\alpha$, $k_1=k$, and $f_1=f$ in Theorem 11, we have the following.

Corollary 13. *Let* $\gamma>0$, $-1\leq\alpha<1$, $k>0$, *and* $f\in\mathscr{UCV}(\alpha,k)$. *If* $0\leq1+\gamma(\alpha-1)<1$, *then the integral operator* $\int_0^z(f'(t))^\gamma dt$ *is convex of order* $1+\gamma(\alpha-1)$ *in* $\mathscr{U}$.

We remark in conclusion that, by suitably specializing the parameters in Theorems 8 and 11, we can deduce the results obtained in [15, 22, 23].

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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