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Research Article

The Representations and Continuity of the Metric Projections on Two Classes of Half-Spaces in Banach Spaces

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We show a necessary and sufficient condition for the existence of metric projection on a class of half-space $K_{x_0^*,c} = \{x \in X : x^*(x) \leq c\}$ in Banach space. Two representations of metric projections $P_{K_{x_0^*,c}}$ and $P_{K_{x_0,c}}$ are given, respectively, where $K_{x_0,c}$ stands for dual half-space of $K_{x_0^*,c}$ in dual space X^*. By these representations, a series of continuity results of the metric projections $P_{K_{x_0^*,c}}$ and $P_{K_{x_0,c}}$ are given. We also provide the characterization that a metric projection is a linear bounded operator.

1. Introduction

The metric projection in Banach space is an enduring question for study or discussion. It has been used in many areas of mathematics such as the theories of optimization and approximation, fixed point theory, nonlinear programming, and variational inequalities. On continuity of the metric projection, many mathematicians, for example, Nevesenko [1], Oshman [2], Wang [3], Fang and Wang [4], and Zhang and Shi [5] have done profound research. In practical application, giving the representations of metric projection is very necessary. Generally speaking, this is very difficult. In recent years, Wang and Yu [6] gave a representation of single-valued metric projection on a class of hyperplanes $H_{x_0^*,c} = \{x \in X : x_0^*(x) = c\}$ in reflexive, smooth, and strictly convex Banach space X. Song and Cao [7] gave a representation of metric projection on a class of half-space $K_{x_0^*,c}$ in the reflexive, smooth, and strictly convex Banach space X. Wang [8] and Ni [9] extended the result of Wang and Yu to general Banach space, respectively. Wang [8] also discussed continuity of the metric projection on the hyperplane $H_{x_0^*,c}$ in Banach space.

In this paper, let X be a Banach space and let X^* be the dual of X. Let $S(X)$ and $B(X)$ be the unit sphere and unit ball of X, respectively. Let $x_0^* \in X^* \setminus \{\theta\}$, let $c \in \mathbb{R}$, let $K_{x_0^*,c} = \{x \in X : x_0^*(x) \leq c\}$, let $D^{-1}(x_0^*) = \{x \in X : x_0^*(x) = \|x_0^*\|\|x\| = \|x_0^*\|^2 = \|x\|^2\}$, let $x_0 \in X \setminus \{\theta\}$, let $K_{x_0,c} = \{x^* \in X^* : x^*(x_0) \leq c\}$, and let $D(x_0) = \{x^* \in X^* : x^*(x_0) = \|x^*\|\|x_0\| = \|x^*\|^2 = \|x_0\|^2\}$. It is easily proved that $D(\alpha x) = \alpha D(x)$, $D^{-1}(\alpha x^*) = \alpha D^{-1}(x^*)$, and for all $\alpha \in \mathbb{R}$. For $M \subset X$, the metric projection $P_M : X \to 2^M$ is defined by $P_M(x) = \{y \in M : \|x - y\| = d(x, M)\}$, where $d(x, M) = \inf\{\|x - y\| : y \in M\}$. Obviously, P_M is a set-valued mapping. If $P_M(x) \neq \emptyset$ for each $x \in X$, then M is said to be proximinal. It is well known that P_M is single-valued when X is strictly convex and M is proximal.

Cabrera and Sadarangani [10] introduced the geometrical properties of Banach spaces as follows.

A Banach space X is called nearly strictly convex (resp., weakly nearly strictly convex) whenever, for any $x^* \in S(X^*)$, the set $D^{-1}(x^*)$ is compact (resp., weakly compact). A Banach space X is called nearly smooth (resp., weakly nearly smooth) whenever, for any $x \in S(X)$, the set $D(x)$ is compact (resp., weakly compact).

The metric projection P_M is said to be norm-norm (resp., norm-weakly) upper semicontinuous if, for all x in X and for all norm (resp., weakly) open set $W \supset P_M(x)$, there exists a norm neighborhood U of x such that $P_M(U) \subset W$.

In this paper, firstly, we established a necessary and sufficient condition for the existence of metric projection on a class of half-space $K_{x_0^*,c}$ in Banach space. Secondly, we give two representations of the metric projections $P_{K_{x_0^*,c}}$ and

$P_{K_{x_0,c}}$ by using a different method from the literatures [5–9]. Thirdly, by these representations, we prove that if X is weakly nearly strictly convex (resp., weakly nearly smooth), then metric projection $P_{K_{x_0^*,c}}$ (resp., $P_{K_{x_0,c}}$) is norm-weakly upper semicontinuous. Finally, the characterization of the metric projection P_M from X to a subspace M to be a linear bounded operator is given. We extend the corresponding results in [5–9].

2. The Representations of the Metric Projection on Two Classes of Half-Spaces in Banach Spaces

Lemma 1. *Let X be a Banach space and let $x_0^* \in X^* \setminus \{\theta\}$; then $d(x, K_{x_0^*,c}) = |x_0^*(x) - c|/\|x_0^*\|$ for all $x \in X \setminus K_{x_0^*,c}$.*

Proof. Firstly, suppose that $\|x_0^*\| = 1$. Let $x \in X/K_{x_0^*,c}$. For any $y \in K_{x_0^*,c}$, since

$$x_0^*(x-y) \geq x_0^*(x) - c > 0, \tag{1}$$

we deduce that

$$d\left(x, K_{x_0^*,c}\right) \geq \left|x_0^*(x) - c\right|. \tag{2}$$

On the other hand, for any $\varepsilon > 0\,(\varepsilon < 1/4)$, there exists u_ε in $S(X)$ such that $1 - \varepsilon < x_0^*(u_\varepsilon) \leq 1$. Set $y_\varepsilon = x - (1 + 2\varepsilon)(x_0^*(x) - c)u_\varepsilon$. Then

$$\begin{aligned} x_0^*(y_\varepsilon) &= x_0^*(x) - (1+2\varepsilon)\left(x_0^*(x) - c\right)x_0^*(u_\varepsilon) \\ &< x_0^*(x) - (1+2\varepsilon)\left(x_0^*(x) - c\right)(1-\varepsilon) \\ &= x_0^*(x) - \left(1 + \varepsilon - 2\varepsilon^2\right)\left(x_0^*(x) - c\right) \\ &\leq x_0^*(x) - \left(x_0^*(x) - c\right) = c. \end{aligned} \tag{3}$$

Consequently, $y_\varepsilon \in K_{x_0^*,c}$ and

$$\left\|x - y_\varepsilon\right\| = (1+2\varepsilon)\left|x_0^*(x) - c\right|. \tag{4}$$

It follows that

$$d\left(x, K_{x_0^*,c}\right) \leq (1+2\varepsilon)\left|x_0^*(x) - c\right|. \tag{5}$$

By arbitrariness of ε, we deduce that

$$d\left(x, K_{x_0^*,c}\right) \leq \left|x_0^*(x) - c\right|. \tag{6}$$

This means that

$$d\left(x, K_{x_0^*,c}\right) = \left|x_0^*(x) - c\right|. \tag{7}$$

Secondly, for $x^* \in X^* \setminus \theta$ and $\|x^*\| \neq 1$, since

$$K_{x_0^*,c} = \{x \in X : x_0^*(x) \leq c\} = \left\{x \in X : \frac{x_0^*}{\|x_0^*\|}(x) \leq \frac{c}{\|x_0^*\|}\right\}, \tag{8}$$

from (7), we may obtain that

$$d\left(x, K_{x_0^*,c}\right) = \frac{\left|x_0^*(x) - c\right|}{\|x_0^*\|}, \quad \forall x \in X \setminus K_{x_0^*,c}. \tag{9}$$

□

Remark 2. For given $x_0^* \in X^* \setminus \{\theta\}$ and $c \in \mathbb{R}$, by Lemma 1, we have that

$$d\left(x, K_{x_0^*,c}\right) = d\left(x, H_{x_0^*,c}\right), \tag{10}$$

for any $x \in X \setminus K_{x_0^*,c}$.

Theorem 3. *Let X be a Banach space, let $x_0^* \in X^* \setminus \theta$, and let $c \in \mathbb{R}$; then $P_{K_{x_0^*,c}}(x) \neq \emptyset$ for any $x \in X$ if and only if $D^{-1}(x_0^*) \neq \emptyset$.*

Proof. On necessity: take $x \in X \setminus K_{x_0^*,c}$; then there exists a $y \in P_{K_{x_0^*,c}}(x)$. Set $u = (\|x_0^*\|^2/(x_0^*(x) - c))(x - y)$; by Lemma 1, we have that

$$\begin{aligned} \|u\| &= \frac{\|x_0^*\|^2}{\left|x_0^*(x) - c\right|}\|x - y\| \\ &= \frac{\|x_0^*\|^2}{\left|x_0^*(x) - c\right|}\frac{\left|x_0^*(x) - c\right|}{\|x_0^*\|} = \|x_0^*\|. \end{aligned} \tag{11}$$

Hence, $x_0^*(u) \leqslant \|x_0^*\|\|u\| = \|x_0^*\|^2$.

On the other hand,

$$\begin{aligned} x_0^*(u) &= \frac{\|x_0^*\|^2}{x_0^*(x) - c}\left(x_0^*(x) - x_0^*(y)\right) \\ &\geqslant \frac{\|x_0^*\|^2}{x_0^*(x) - c}\left(x_0^*(x) - c\right) = \|x_0^*\|^2. \end{aligned} \tag{12}$$

This shows that $x_0^*(u) = \|x_0^*\|^2 = \|u\|^2$, that is, $u \in D^{-1}(x_0^*)$ and $D^{-1}(x_0^*) \neq \emptyset$.

On sufficiency: take $x \in S(X)$ such that $x_0^*(x) = \|x_0^*\|\|x\| = \|x_0^*\|^2 = \|x\|^2$. We discuss that in two cases.

Case 1. If $x \in K_{x_0^*,c}$, then $x \in P_{K_{x_0^*,c}}(x)$.

Case 2. If $x \notin K_{x_0^*,c}$, since

$$x_0^*\left(x - \frac{x_0^*(x) - c}{\|x_0^*\|^2}x_0\right) = x_0^*(x) - \left(x_0^*(x) - c\right) = c; \tag{13}$$

then we have that $x - ((x_0^*(x) - c)/\|x_0^*\|^2)x_0 \in K_{x_0^*,c}$. By Lemma 1,

$$\left\|x - \left(x - \frac{x_0^*(x) - c}{\|x_0^*\|^2}x_0\right)\right\| = \frac{x_0^*(x) - c}{\|x_0^*\|} = d\left(x, K_{x_0^*}\right). \tag{14}$$

It follows that $x - ((x_0^*(x) - c)/\|x_0^*\|^2)x_0 \in P_{K_{x_0^*,c}}(x)$. □

Theorem 4. *Let X be a Banach space, let $x_0^* \in X^* \setminus \{\theta\}$, let x_0^* attain its norm on $S(X)$, and let $c \in \mathbb{R}$. Then*

$$P_{K_{x_0^*,c}}(x) = x - \max\left\{0, \frac{x_0^*(x) - c}{\|x_0^*\|^2}\right\}D^{-1}\left(x_0^*\right). \tag{15}$$

Proof. Take $x \in X$. We discuss that in two cases.

Case 1. If $x \in K_{x_0^*,c}$, then $P_{K_{x_0^*,c}}(x) = \{x\}$.

Case 2. If $x \notin K_{x_0^*,c}$, we arbitrarily take $x_0 \in D^{-1}(x_0^*)$. Let $y = x - ((x_0^*(x) - c)/\|x_0^*\|^2)x_0$. Similar to the proof of Theorem 3, we may obtain that $y \in P_{K_{x_0^*,c}}(x)$. Therefore,

$$x - \frac{x_0^*(x) - c}{\|x_0^*\|^2} D^{-1}(x_0^*) \subset P_{K_{x_0^*,c}}(x). \tag{16}$$

On the other hand, we arbitrarily take $y \in P_{K_{x_0^*,c}}(x)$. Let $u = (\|x_0^*\|^2/(x_0^*(x) - c))(x - y)$; similar to the proof of Theorem 3, we may obtain that $u \in D^{-1}(x_0^*)$. Therefore,

$$y = x - \frac{x_0^*(x) - c}{\|x_0^*\|^2} u \in x - \frac{x_0^*(x) - c}{\|x_0^*\|^2} D^{-1}(x_0^*), \tag{17}$$

that is,

$$P_{K_{x_0^*,c}}(x) \subset x - \frac{x_0^*(x) - c}{\|x_0^*\|^2} D^{-1}(x_0^*). \tag{18}$$

By Case 1 and Case 2, we have

$$P_{K_{x_0^*,c}}(x) = x - \max\left\{0, \frac{x_0^*(x) - c}{\|x_0^*\|^2}\right\} D^{-1}(x_0^*), \tag{19}$$

for any $x \in X$. □

By the similar proof to that in Lemma 1, we can obtain the following result.

Lemma 5. *Let X be a Banach space, let $x_0 \in X \setminus \{\theta\}$, and let $c \in \mathbb{R}$. Then*

$$d(x^*, K_{x_0,c}) = \frac{|x^*(x_0) - c|}{\|x_0\|}, \tag{20}$$

for any $x^* \in X^* \setminus K_{x_0,c}$.

By a similar proof to that in Theorem 4, we can also prove the following result according to Lemma 5.

Theorem 6. *Let X be a Banach space, let $x_0 \in X \setminus \{\theta\}$, and let $c \in \mathbb{R}$. Then*

$$P_{K_{x_0,c}}(x^*) = x^* - \max\left\{0, \frac{x^*(x_0) - c}{\|x_0\|^2}\right\} D(x_0), \tag{21}$$

for any $x^* \in X^*$.

3. Continuity of the Metric Projection on the Two Classes of Half-Spaces in Banach Spaces

Theorem 7. *Let $x_0^* \in X \setminus \{\theta\}$, let x_0^* attain its norm on $S(X)$, and let $c \in \mathbb{R}$. If X is weakly nearly strictly convex, then the metric projection $P_{K_{x_0^*,c}}$ is norm-weakly upper semicontinuous.*

Proof. Let $x, x_n \in X$, and let $x_n \to x$ as $n \to \infty$. Our proof will be divided into two cases.

Case 1. Suppose that $\{x_n\} \subset K_{x_0^*,c}$. Since $K_{x_0^*,c}$ is a closed set, $x \in K_{x_0^*,c}$. Clearly, $P_{K_{x_0^*,c}}(x_n) = x_n \to x = P_{K_{x_0^*,c}}(x)$.

Case 2. Suppose that $\{x_n\} \not\subset K_{x_0^*,c}$.

If there are an infinite number of n for which $x_n \in K_{x_0^*,c}$, then we can choose a subsequence $\{x_{n_k}\} \subset \{x_n\}$ with $\{x_{n_k}\} \subset K_{x_0^*,c}$. Therefore, $P_{K_{x_0^*,c}}(x_{n_k}) = x_{n_k} \to x = P_{K_{x_0^*,c}}(x)$ as $k \to \infty$.

If there are an infinite number of n for which $x_n \notin K_{x_0^*,c}$, without loss of generality, we may assume that $\{x_n\} \subset X \setminus K_{x_0^*,c}$. Taking $y_n \in P_{K_{x_0^*,c}}(x_n)$, by Theorem 4, we have

$$P_{K_{x_0^*,c}}(x_n) = x_n - \frac{x_0^*(x_n) - c}{\|x_0\|^2} D^{-1}(x_0^*). \tag{22}$$

We assume that $y_n = x_n - ((x_0^*(x_n) - c)/\|x_0^*\|^2)z_n$, where $z_n \in D^{-1}(x_0^*)$. Since X is weakly nearly strictly convex, we know that $\{z_n\}$ has a weakly convergent subsequence $\{z_{n_k}\}$ with $z_{n_k} \xrightarrow{w} z$ as $k \to \infty$. Consequently,

$$y_{n_k} = x_{n_k} - \frac{x_0^*(x_{n_k}) - c}{\|x_0^*\|} z_{n_k} \xrightarrow{w} x - \frac{x_0^*(x) - c}{\|x_0^*\|} z. \tag{23}$$

Noting $x_0^*(z) = \lim_k x_0^*(z_{n_k}) = \lim_k \|x_0^*\|^2 = \|z_{n_k}\|^2$ and $\|z\| \leqslant \underline{\lim}_k \|z_{n_k}\|$, we know that $x_0^*(z) \geqslant \|x_0^*\| \cdot \|z\|$. Therefore,

$$x_0^*(z) = \|x_0^*\| \cdot \|z\| = \|x_0^*\|^2 = \|z\|^2, \tag{24}$$

where $z \in D^{-1}(x_0^*)$. This shows that $y_{n_k} \xrightarrow{w} x - ((x_0^*(x) - c)/\|x_0^*\|)z \in P_{K_{x_0^*,c}}(x)$.

Now, we will show that $P_{K_{x_0^*,c}}$ is norm-weakly upper semicontinuous at x. Otherwise, there exist a weakly open set $W_0 \supset P_{K_{x_0^*,c}}(x)$ and a sequence $\{x_m\}$ with $x_m \to x$ as $m \to \infty$, but $P_{K_{x_0^*,c}}(x_m) \not\subset W_0$ for all m. Taking $y_m \in P_{K_{x_0^*,c}}(x_m) \setminus W_0$, $m = 1, 2, \ldots$, similar to previous arguments, we can observe the fact that there exists a subsequence $\{y_{m_k}\}$ of $\{y_m\}$ such that $y_{m_k} \xrightarrow{w} y$ as $k \to \infty$ and $y \in P_{K_{x_0^*,c}}(x)$. This means that there exists $y_{m_k} \in W_0$ for some k large enough, which is a contradiction. □

Similar to the proof of Theorem 8, we may prove the following theorem.

Theorem 8. *Let X be a Banach space.*

(1) *Let $x_0^* \in X^* \setminus \{\theta\}$, let x_0^* attain its norm on $S(X)$, and let $c \in \mathbb{R}$. If X is nearly strictly convex, then the metric projection $P_{K_{x_0^*,c}}$ is norm-norm upper semicontinuous.*

(2) *Let $x_0 \in X \setminus \{\theta\}$ and let $c \in \mathbb{R}$. If X is weakly nearly smooth, then the metric projection $P_{K_{x_0,c}}$ is norm-weakly upper semicontinuous.*

(3) *Let $x_0 \in X \setminus \{\theta\}$ and let $c \in \mathbb{R}$. If X is nearly smooth, then the metric projection $P_{K_{x_0,c}}$ is norm-norm upper semicontinuous.*

Lemma 9 (see [11]). *Let M be a proximal subspace. Then for any $x \in X$, one has the decomposition*

$$x = x_1 + x_2, \quad x_1 \in P_M(x), \quad x_2 \in D^{-1}(M^\perp), \tag{25}$$

where $M^\perp = \{x^* \in X^* : x^*(x) = 0, \forall x \in M\}$ and

$$D^{-1}(M^\perp) = \{x \in X : D(x) \cap M^\perp \neq \emptyset\}. \tag{26}$$

If M is a Chebyshev subspace, the decomposition is unique, and

$$x = P_M(x) + x_2, \quad x_2 \in D^{-1}(M^\perp). \tag{27}$$

Lemma 10. *Let X be a strictly convex Banach space and let M be a proximal subspace. Then, for any $x \in X$, one has*

$$P_M(x+y) = P_M(x) + y, \quad y \in M. \tag{28}$$

Proof. Let $y \in M$, for any $z \in M$, we have that $w = z - y \in M$. Consider

$$\begin{aligned} &\|P_M(x) + y - (x+y)\| \\ &= \|P_M(x) - x\| \le \|w - x\| \\ &= \|(w+y) - (x+y)\| = \|z - (x+y)\|. \end{aligned} \tag{29}$$

By the definition of P_M, we obtain $P_M(x) + y \in P_M(x+y)$. Since X is strictly convex, we know that P_M is single-valued, and hence we have $P_M(x+y) = P_M(x) + y$. □

Similar to the proof Theorem 2.1(1) in [6], we can prove the following result by Lemmas 9 and 10.

Lemma 11. *Let X be a strictly convex Banach space and let M be a proximal subspace. P is single-valued operator from X into M, and P_M is a metric projection from X into M. Then $P = P_M$ if and only if the following conditions are satisfied:*

(1) *$P^{-1}(\theta) = D^{-1}(M^\perp)$;*

(2) *$P(x+y) = P(x) + y$, for all $y \in M$.*

Theorem 12. *Let X be a strictly convex Banach space and let M be a proximal subspace. Then the metric projection P_M is a linear bounded operator if and only if $D^{-1}(M^\perp)$ is a linear subspace.*

Proof. On necessity: let P_M be a linear operator. Since X is strictly convex and M is proximal, then P_M is single valued. By Lemma 11(1), for any $x, y \in D^{-1}(M^\perp) = P_M^{-1}(\theta)$, $\alpha, \beta \in \mathbb{R}$, then

$$P_M(\alpha x + \beta y) = \alpha P_M(x) + \beta P_M(y) = 0, \tag{30}$$

and hence $\alpha x + \beta y \in P_M^{-1}(\theta) = D^{-1}(M^\perp)$. This shows that $D^{-1}(M^\perp)$ is a linear subspace.

On sufficiency: let $D^{-1}(M^\perp)$ be a linear subspace and let P_M be a metric projection; since X is strictly convex, by Lemma 11(1), $P_M^{-1}(\theta)$ is also a linear subspace. For any $x, y \in X$, $x - P_M(x), y - P_M(y) \in \{x - P_M(x) : x \in X\}$, we have that

$$\begin{aligned} &(x+y) - (P_M(x) + P_M(y)) \\ &= (x - P_M(x)) + (y - P_M(y)) \\ &\in \{z - P_M(z) : z \in X\} = P_M^{-1}(\theta). \end{aligned} \tag{31}$$

By Lemma 11(2), we have that

$$\begin{aligned} 0 &= P_M((x+y) - (P_M(x) + P_M(y))) \\ &= P_M(x+y) - (P_M(x) + P_M(y)). \end{aligned} \tag{32}$$

It follows that $P_M(x+y) = P_M(x) + P_M(y)$. Note that P_M is homogeneous; we obtain that P_M is a linear operator. In addition, for any $x \in X$, since $\theta \in M$, we have that

$$\begin{aligned} \|P_M(x)\| &= \|P_M(x) - x + x\| \\ &\le \|P_M(x) - x\| + \|x\| \\ &\le \|\theta - x\| + \|x\| = 2\|x\|. \end{aligned} \tag{33}$$

This shows that P_M is a bounded operator. □

Conflict of Interests

The authors declare that they have no conflict of interests.

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