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Research Article

Positive Solutions for a Nonlinear Higher Order Differential System with Coupled Integral Boundary Conditions

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We investigate the existence of positive solutions for a nonlinear higher order differential system, where the differential system is coupled not only in the differential system but also through the boundary conditions. By constructing a special cone and using the fixed point theorem of cone expansion and compression of norm type, the existence of single and multiple positive solutions is established. As an application, we give some examples to demonstrate our results.

1. Introduction

In this paper, we consider the following nonlinear higher order differential system with coupled integral boundary conditions:

$$\begin{gathered} u^{(n)}(t) + a_1(t) f_1(t, u(t), v(t)) = 0, \quad t \in (0,1), \\ v^{(n)}(t) + a_2(t) f_2(t, u(t), v(t)) = 0, \quad t \in (0,1), \\ u^{(k)}(0) = v^{(k)}(0) = 0, \quad k = 0, 1, \ldots, n-2, \\ u(1) = \alpha[v], \qquad v(1) = \beta[u], \end{gathered} \tag{1}$$

where $f_i \in C([0,1] \times [0,+\infty) \times [0,+\infty), [0,+\infty))$, $a_i \in C([0,1], [0,+\infty))$, $i = 1, 2$, $n \geq 3$, $\alpha[v]$, $\beta[u]$ are bounded linear functions on $C[0,1]$ given by

$$\alpha[v] = \int_0^1 v(t)\, dA(t), \qquad \beta[u] = \int_0^1 u(t)\, dB(t), \tag{2}$$

involving Stieltjes integrals. In particular, A, B are functionals of bounded variation with positive measures.

In recent years, there were many works to be done for a variety of nonlinear higher order ordinary differential system. However, most papers only focus on paying attention to the differential system with uncoupled boundary conditions (see [1–5] and the reference therein). Coupled boundary conditions arise in the study of reaction-diffusion equations and Sturm-Liouville problems (see [6]) and have wide applications in various fields of sciences and engineering, for example, the heat equation [7, 8].

In a recent article [9], by applying a nonlinear alternative of Leray-Schauder type and Guo-Krasnoselskii's fixed point theorem on cone, the authors established the existence of multiple positive solutions of the following system with four-point coupled boundary conditions:

$$\begin{gathered} D_{0+}^{\alpha} u(t) + \lambda f(t, u(t), v(t)) = 0, \quad t \in (0,1), \ \lambda > 0, \\ D_{0+}^{\alpha} u(t) + \lambda h(t, u(t), v(t)) = 0, \quad t \in (0,1), \\ u^{(i)}(0) = v^{(i)}(0) = 0, \quad 0 \leq i \leq n-2, \\ u(1) = a v(\xi), \qquad v(1) = b u(\eta), \end{gathered} \tag{3}$$

where $D_{0+}^{\alpha} u$ is the Riemann-Liouville's fractional derivative.

In [10], by using fixed point index theory, Yang studied the following system with uncoupled boundary conditions:

$$\begin{gathered} u''(t) + f_1(t, u(t), v(t)) = 0, \quad t \in (0,1), \\ v''(t) + f_2(t, u(t), v(t)) = 0, \quad t \in (0,1), \\ u(0) = 0, \qquad u(1) = \beta[u], \\ v(0) = 0, \qquad v(1) = \beta[v], \end{gathered} \tag{4}$$

where $\beta[\cdot]$ are linear functionals defined by Stieltjes integrals.

The work of above-mentioned papers and wide applications of coupled boundary value conditions motivate us to study the system (1). Further, the system is coupled not only in the differential system but also through the boundary conditions. By constructing a special cone and using the fixed point theorem on cone expansion and compression, the existence of single and multiple positive solutions is established.

2. Preliminaries

Let $E = C[0,1]]$; we write $\|u\| = \max\{|u(t)| : t \in [0,1]\}$. Clearly, $(E, \|\cdot\|)$ is a Banach space. For each $(u,v) \in E \times E$, we write $\|(u,v)\| = \|u\| + \|v\|$. Define

$$P = \{u \in E : u(t) \geq 0, t \in [0,1]\},$$
$$Q = \Big\{(u,v) \in P \times P : \min_{a \leq t \leq b}(u,v) \overset{\text{def}}{=} \min_{a \leq t \leq b}(u(t) + v(t)) \geq \gamma \|(u,v)\|\Big\}, \tag{5}$$

where $[a,b]$ is some subset of $(0,1)$, $0 < \gamma \leq (a^{n-1}\nu/\rho)$; consider

$$\rho = \max\left\{\frac{\alpha[t^{n-1}]}{k}\beta[1] + 1, \frac{\beta[t^{n-1}]}{k}\alpha[1] + 1, \frac{\beta[1]}{k}, \frac{\alpha[1]}{k}\right\},$$
$$\nu = \min\left\{\frac{\alpha[t^{n-1}]}{k}\beta[\gamma_n(t)], \frac{\beta[t^{n-1}]}{k}\alpha[\gamma_n(t)], \frac{\beta[\gamma_n(t)]}{k}, \frac{\alpha[\gamma_n(t)]}{k}\right\}, \tag{6}$$
$$\alpha[t^{n-1}] = \int_0^1 t^{n-1} dA(t) > 0,$$
$$\beta[t^{n-1}] = \int_0^1 t^{n-1} dB(t) > 0,$$
$$k = 1 - \alpha[t^{n-1}]\beta[t^{n-1}] > 0,$$

where $\gamma_n(t)$ is defined by the following Lemma 2. Clearly, $(E \times E, \|\cdot\|)$ is a Banach space and P is a cone of E.

Lemma 1. *Let $u, v \in E$. Then differential system*

$$u^{(n)}(t) + x(t) = 0, \quad t \in (0,1),$$
$$v^{(n)}(t) + y(t) = 0, \quad t \in (0,1),$$
$$u^{(k)}(0) = v^{(k)}(0) = 0, \quad k = 0, 1, \ldots, n-2, \tag{7}$$
$$u(1) = \alpha[v], \qquad v(1) = \beta[u],$$

has the following integral representation

$$u(t) = \int_0^1 F_1(t,s)\,x(s)\,ds + \int_0^1 G_1(t,s)\,y(s)\,ds,$$
$$v(t) = \int_0^1 F_2(t,s)\,y(s)\,ds + \int_0^1 G_2(t,s)\,x(s)\,ds, \tag{8}$$

where

$$F_1(t,s) = \frac{\alpha[t^{n-1}]t^{n-1}}{k}\int_0^1 K_n(\xi,s)\,dB(\xi) + K_n(t,s),$$
$$G_1(t,s) = \frac{t^{n-1}}{k}\int_0^1 K_n(\xi,s)\,dA(\xi),$$
$$F_2(t,s) = \frac{\beta[t^{n-1}]t^{n-1}}{k}\int_0^1 K_n(\xi,s)\,dA(\xi) + K_n(t,s), \tag{9}$$
$$G_2(t,s) = \frac{t^{n-1}}{k}\int_0^1 K_n(\xi,s)\,dB(\xi),$$

$$K_n(t,s) = \frac{1}{(n-1)!}\begin{cases} t^{n-1}(1-s)^{n-1} - (t-s)^{n-1}, & 0 \leq s \leq t \leq 1, \\ t^{n-1}(1-s)^{n-1}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{10}$$

Proof. By Taylor's formula, we have

$$u(t) = u(0) + tu'(0) + \cdots + \frac{t^{n-1}}{(n-1)!}u^{(n-1)}(0) + \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}u^{(n)}(s)\,ds,$$
$$v(t) = v(0) + tv'(0) + \cdots + \frac{t^{n-1}}{(n-1)!}v^{(n-1)}(0) + \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}v^{(n)}(s)\,ds. \tag{11}$$

So, we reduce the equation of problems (7) to the following equivalent integral equation:

$$u(t) = -\frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}x(s)\,ds + \frac{t^{n-1}}{(n-1)!}u^{(n-1)}(0),$$
$$v(t) = -\frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}y(s)\,ds + \frac{t^{n-1}}{(n-1)!}v^{(n-1)}(0). \tag{12}$$

Let $t = 1$; we have

$$u^{(n-1)}(0) = \int_0^1 (1-s)^{n-1}x(s)\,ds + (n-1)!u(1),$$
$$v^{(n-1)}(0) = \int_0^1 (1-s)^{n-1}y(s)\,ds + (n-1)!v(1). \tag{13}$$

By substituting $u^{(n-1)}(0)$ and $v^{(n-1)}(0)$ into (12), we have

$$\begin{aligned} u(t) &= -\frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}x(s)\,ds \\ &\quad + \frac{1}{(n-1)!}\int_0^1 t^{n-1}(1-s)^{n-1}x(s)\,ds + t^{n-1}u(1) \\ &= \frac{1}{(n-1)!}\int_0^t \left[t^{n-1}(1-s)^{n-1} - (t-s)^{n-1}\right]x(s)\,ds \\ &\quad + \frac{1}{(n-1)!}\int_t^1 t^{n-1}(1-s)^{n-1}x(s)\,ds + t^{n-1}u(1) \\ &= \int_0^1 K_n(t,s)\,x(s)\,ds + t^{n-1}u(1), \\ v(t) &= \int_0^1 K_n(t,s)\,y(s)\,ds + t^{n-1}v(1); \end{aligned} \tag{14}$$

that is,

$$\begin{aligned} u(t) &= \int_0^1 K_n(t,s)\,x(s)\,ds + t^{n-1}u(1), \\ v(t) &= \int_0^1 K_n(t,s)\,y(s)\,ds + t^{n-1}v(1). \end{aligned} \tag{15}$$

By applying β and α to (15), combined with the conditions $u(1) = \alpha[v]$, $v(1) = \beta[u]$, respectively, we obtain

$$\begin{aligned} u(1) &= \int_0^1 v(t)\,dA(t) = \iint_0^1 K_n(t,s)\,y(s)\,ds\,dA(t) \\ &\quad + u(1)\int_0^1 t^{n-1}dA(t), \\ v(1) &= \int_0^1 u(t)\,dB(t) = \iint_0^1 K_n(t,s)\,x(s)\,ds\,dB(t) \\ &\quad + u(1)\int_0^1 t^{n-1}dB(t). \end{aligned} \tag{16}$$

Therefore

$$\begin{pmatrix} -\beta\left[t^{n-1}\right] & 1 \\ 1 & -\alpha\left[t^{n-1}\right] \end{pmatrix}\begin{pmatrix} u(1) \\ v(1) \end{pmatrix} = \begin{pmatrix} \iint_0^1 K_n(t,s)\,x(s)\,ds\,dB(t) \\ \iint_0^1 K_n(t,s)\,y(s)\,ds\,dA(t) \end{pmatrix}, \tag{17}$$

and so

$$\begin{pmatrix} u(1) \\ v(1) \end{pmatrix} = \frac{1}{k}\begin{pmatrix} \alpha\left[t^{n-1}\right] & 1 \\ 1 & \beta\left[t^{n-1}\right] \end{pmatrix} \times \begin{pmatrix} \iint_0^1 K_n(t,s)\,x(s)\,ds\,dB(t) \\ \iint_0^1 K_n(t,s)\,y(s)\,ds\,dA(t) \end{pmatrix}. \tag{18}$$

By substituting (18) into (15), we obtain

$$\begin{aligned} u(t) &= \frac{\alpha\left[t^{n-1}\right]t^{n-1}}{k}\iint_0^1 K_n(t,s)\,x(s)\,ds\,dB(t) \\ &\quad + \frac{t^{n-1}}{k}\iint_0^1 K_n(t,s)\,y(s)\,ds\,dA(t) \\ &\quad + \int_0^1 K_n(t,s)\,x(s)\,ds, \\ v(t) &= \frac{t^{n-1}}{k}\iint_0^1 K_n(t,s)\,x(s)\,ds\,dB(t) \\ &\quad + \frac{\beta\left[t^{n-1}\right]t^{n-1}}{k}\iint_0^1 K_n(t,s)\,y(s)\,ds\,dA(t) \\ &\quad + \int_0^1 K_n(t,s)\,y(s)\,ds. \end{aligned} \tag{19}$$

Therefore

$$\begin{aligned} u(t) &= \int_0^1 \left[\frac{\alpha\left[t^{n-1}\right]t^{n-1}}{k}\int_0^1 K_n(\xi,s)\,dB(\xi)\right]x(s)\,ds \\ &\quad + \int_0^1 \left[\frac{t^{n-1}}{k}\int_0^1 K_n(\xi,s)\,dA(\xi)\right]y(s)\,ds \\ &\quad + \int_0^1 K_n(t,s)\,x(s)\,ds, \\ v(t) &= \int_0^1 \left[\frac{t^{n-1}}{k}\int_0^1 K_n(\xi,s)\,dB(\xi)\right]x(s)\,ds \\ &\quad + \int_0^1 \left[\frac{\beta\left[t^{n-1}\right]t^{n-1}}{k}\int_0^1 K_n(\xi,s)\,dA(\xi)\right]y(s)\,ds \\ &\quad + \int_0^1 K_n(t,s)\,y(s)\,ds. \end{aligned} \tag{20}$$

which is equivalent to system (8). □

Lemma 2 (see [11]). *The continuous function $K_n(t,s)$ has the following properties:*

(i) $0 \le K_n(t,s) \le K_n(s)$, *for all* $t, s \in [0,1]$, *where* $K_n(s) = s(1-s)^{n-1}/(n-2)!$;

(ii) $K_n(t,s) \ge \gamma_n(t)K_n(s)$, *for all* $t, s \in [0,1]$, *where* $\gamma_n(t) = (1/(n-1))\min\{t^{n-1}, (1-t)t^{n-2}\}$.

Remark 3. By combining (i) and (ii), we can easily see

$$K_n(s) \ge K_n(t,s) \ge \gamma_n(t)\,K_n(s), \quad \forall t, s \in [0,1]. \tag{21}$$

Remark 4. From Remark 3, and (9), for $t, s \in [0,1]$, we have

$$\begin{aligned} F_i(t,s) &\le \rho K_n(s), \quad G_i(t,s) \le \rho K_n(s), \quad i = 1,2, \\ F_i(t,s) &\ge \nu t^{n-1}K_n(s), \quad G_i(t,s) \ge \nu t^{n-1}K_n(s), \quad i = 1,2. \end{aligned} \tag{22}$$

Define the operator $T : Q \to P \times P$ by

$$T(u,v) = (T_1(u,v), T_2(u,v)), \tag{23}$$

where operators $T_1, T_2 : Q \to P$ are defined by

$$\begin{aligned} T_1(u,v)(t) &= \int_0^1 F_1(t,s)\, a_1(s)\, f_1(s,u(s),v(s))\, ds \\ &\quad + \int_0^1 G_1(t,s)\, a_2(s)\, f_2(s,u(s),v(s))\, ds, \\ &\qquad t \in [0,1], \end{aligned} \tag{24}$$

$$\begin{aligned} T_2(u,v)(t) &= \int_0^1 F_2(t,s)\, a_2(s)\, f_2(s,u(s),v(s))\, ds \\ &\quad + \int_0^1 G_2(t,s)\, a_1(s)\, f_1(s,u(s),v(s))\, ds, \\ &\qquad t \in [0,1]. \end{aligned} \tag{25}$$

Moreover, by Lemma 1, if $(u,v) \in Q$ is a fixed point of the operator T, then (u,v) is a solution of the system (1).

Lemma 5. *The operator $T : Q \to Q$ is completely continuous.*

Proof. By Remark 4, for $s \in [0,1]$, we obtain

$$\begin{aligned} &\min_{t\in[a,b]} F_i(t,s) \ge \nu a^{n-1} K_n(s), \\ &\min_{t\in[a,b]} G_i(t,s) \ge \nu a^{n-1} K_n(s), \quad i = 1,2. \end{aligned} \tag{26}$$

Therefore, By Lemma 2 and Remark 3, for $(u,v) \in P$, we have

$$\begin{aligned} \|T_1(u,v)\| &\le \rho \int_0^1 K_n(s)\, a_1(s)\, f_1(s,u(s),v(s))\, ds \\ &\quad + \rho \int_0^1 K_n(s)\, a_2(s)\, f_2(s,u(s),v(s))\, ds. \end{aligned} \tag{27}$$

Moreover, we have

$$\begin{aligned} &\min_{t\in[a,b]} T_1(u,v)(t) \\ &= \min_{t\in[a,b]} \left[\int_0^1 F_1(t,s)\, a_1(s)\, f_1(s,u(s),v(s))\, ds \right. \\ &\qquad \left. + \int_0^1 G_1(t,s)\, a_2(s)\, f_2(s,u(s),v(s))\, ds \right] \\ &\ge \nu a^{n-1} \left[\int_0^1 K_n(s)\, a_1(s)\, f_1(s,u(s),v(s))\, ds \right. \\ &\qquad \left. + \int_0^1 K_n(s)\, a_2(s)\, f_2(s,u(s),v(s))\, ds \right] \\ &\ge \gamma \|T_1(u,v)\|. \end{aligned} \tag{28}$$

In the same way, we can prove that

$$\min_{t\in[a,b]} T_2(u,v)(t) \ge \gamma \|T_2(u,v)\|. \tag{29}$$

Thus

$$\begin{aligned} \min_{t\in[a,b]} T(u,v) &= \min_{t\in[a,b]} \left(T_1(u,v)(t) + T_2(u,v)(t)\right) \\ &\ge \gamma \|T_1(u,v)\| + \gamma \|T_2(u,v)\| = \gamma \|T(u,v)\|. \end{aligned} \tag{30}$$

Then operator $T : Q \to Q$ is continuous since $K_n(t,s)$, $f_1(t,u,v)$, $f_2(t,u,v)$, $a_1(t)$, $a_2(t)$ are continuous. Standard applications of Arzelà-Ascoli theorem; it is easy to prove that operator $T : Q \to Q$ is completely continuous. □

Lemma 6 (see [12]). *Suppose E is a real Banach space and P is cone in E, and let Ω_1, Ω_2 be bounded open sets in E such that $\theta \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$. Let operator $T : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ be completely continuous. Suppose that one of the following two conditions holds:*

(i) $\|Tu\| \le \|u\|$, *for all* $u \in P \cap \partial\Omega_1$; $\|Tu\| \ge \|u\|$, *for all* $u \in P \cap \partial\Omega_2$,

(ii) $\|Tu\| \ge \|u\|$, *for all* $u \in P \cap \partial\Omega_1$; $\|Tu\| \le \|u\|$, *for all* $u \in P \cap \partial\Omega_2$,

then operator T has at least one fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. Main Results

In this section, we show the existence of positive solutions to the system (1). For convenience, we first introduce the following notations:

$$\begin{aligned} f_{1\lambda} &= \liminf_{u+v\to\lambda} \min_{t\in[0,1]} \frac{f_1(t,u,v)}{u+v}, \\ f^{1\lambda} &= \limsup_{u+v\to\lambda} \max_{t\in[0,1]} \frac{f_1(t,u,v)}{u+v}, \\ f_{2\lambda} &= \liminf_{u+v\to\lambda} \min_{t\in[0,1]} \frac{f_2(t,u,v)}{u+v}, \\ f^{2\lambda} &= \limsup_{u+v\to\lambda} \max_{t\in[0,1]} \frac{f_2(t,u,v)}{u+v}, \end{aligned} \tag{31}$$

where $\lambda = 0$ or ∞. Let $r = \min\{r_1, r_2\}$, $R = \max\{R_1, R_2\}$, where

$$\begin{aligned} r_i &= \left(4\rho \int_0^1 a_i(s) K_n(s) ds\right)^{-1}, \\ R_i &= \left(4\gamma\nu a^{n-1} \int_a^b a_i(s)\, K_n(s)\, ds\right)^{-1}, \\ &\qquad i = 1,2. \end{aligned} \tag{32}$$

Theorem 7. *If $f^{10}, f^{20} \in [0,r)$ and $f_{1\infty}, f_{2\infty} \in (R,+\infty]$, then system (1) has at least one positive solution.*

Proof. At first, it follows from the assumption $f^{10}, f^{20} \in [0, r)$ that there exists $\mu_1 > 0$ and a sufficiently small $\varepsilon_1 > 0$ such that

$$f_1(t,u,v) \le \left(f^{10}+\varepsilon_1\right)(u+v), \quad \forall t \in [0,1],\ u+v \le \mu_1,$$
$$f_2(t,u,v) \le \left(f^{20}+\varepsilon_1\right)(u+v), \quad \forall t \in [0,1],\ u+v \le \mu_1, \tag{33}$$

where ε_1 satisfies $f^{10}+\varepsilon_1 \le r$ and $f^{20}+\varepsilon_1 \le r$.

Set $\Omega_1 = \{(u,v) \in P \times P : \|(u,v)\| < \mu_1\}$. For any $(u,v) \in \partial\Omega_1 \cap Q$, by (24), (25), and (33), we have

$$\begin{aligned}
\|T_1(u,v)\| &\le \rho \int_0^1 K_n(s)\,a_1(s)\,f_1(s,u(s),v(s))\,ds + \rho \int_0^1 K_n(s)\,a_2(s)\,f_2(s,u(s),v(s))\,ds \\
&\le \rho\left[\left(f^{10}+\varepsilon_1\right)\int_0^1 K_n(s)\,a_1(s)\,ds + \left(f^{20}+\varepsilon_1\right)\int_0^1 K_n(s)\,a_2(s)\,ds\right]\cdot\|(u,v)\| \\
&\le \rho\left[\frac{1}{4\rho}+\frac{1}{4\rho}\right]\|(u,v)\| \le \frac{1}{2}\|(u,v)\|, \\
\|T_2(u,v)\| &\le \rho \int_0^1 K_n(s)\,a_2(s)\,f_2(s,u(s),v(s))\,ds + \rho \int_0^1 K_n(s)\,a_1(s)\,f_1(s,u(s),v(s))\,ds \\
&\le \rho\left[\left(f^{20}+\varepsilon_1\right)\int_0^1 K_n(s)\,a_2(s)\,ds + \left(f^{10}+\varepsilon_1\right)\int_0^1 K_n(s)\,a_1(s)\,ds\right]\cdot\|(u,v)\| \\
&\le \rho\left[\frac{1}{4\rho}+\frac{1}{4\rho}\right]\|(u,v)\| \le \frac{1}{2}\|(u,v)\|.
\end{aligned} \tag{34}$$

Therefore

$$\|T(u,v)\| = \|T_1(u,v)\| + \|T_2(u,v)\| \le \|(u,v)\|, \quad \text{for } (u,v) \in \partial\Omega_1 \cap Q. \tag{35}$$

Further, it follows from the the assumption $f_{1\infty}, f_{2\infty} \in (R, +\infty]$ that there exists $l > \mu_1 > 0$ and a sufficiently small $\varepsilon_2 > 0$ such that

$$f_1(t,u,v) \ge \left(f_{1\infty}-\varepsilon_2\right)(u+v), \quad \forall t \in [0,1],\ u+v \ge l,$$
$$f_2(t,u,v) \ge \left(f_{2\infty}-\varepsilon_2\right)(u+v), \quad \forall t \in [0,1],\ u+v \ge l, \tag{36}$$

where ε_2 satisfies $f_{1\infty}-\varepsilon_2 \ge R$ and $f_{2\infty}-\varepsilon_2 \ge R$. Let $\mu_2 = \max\{2\mu_1, l/\gamma\}$; set $\Omega_2 = \{(u,v) \in P \times P : \|(u,v)\| < \mu_2\}$. Then $(u,v) \in \partial\Omega_2 \cap Q$ implies that $\min_{t\in[a,b]}(u,v) \ge \gamma\|(u,v)\| = \gamma\mu_2 \ge l$. So, by (24), (25), and (36), we have

$$\begin{aligned}
\min_{t\in[a,b]} T_1(u,v)(t) &\ge \nu a^{n-1}\left[\int_a^b K_n(s)\,a_1(s)\,f_1(s,u(s),v(s))\,ds + \int_a^b K_n(s)\,a_2(s)\,f_2(s,u(s),v(s))\,ds\right] \\
&\ge \gamma\nu a^{n-1}\left[\left(f_{1\infty}-\varepsilon_2\right)\int_a^b K_n(s)\,a_1(s)\,ds + \left(f_{2\infty}-\varepsilon_2\right)\int_a^b K_n(s)\,a_2(s)\,ds\right]\cdot\|(u,v)\| \\
&\ge \gamma\nu a^{n-1}\left[\frac{1}{4\gamma\nu a^{n-1}}+\frac{1}{4\gamma\nu a^{n-1}}\right]\|(u,v)\| \\
&\ge \frac{1}{2}\|(u,v)\|, \\
\min_{t\in[a,b]} T_2(u,v)(t) &\ge \nu a^{n-1}\left[\int_a^b K_n(s)\,a_2(s)\,f_2(s,u(s),v(s))\,ds + \int_a^b K_n(s)\,a_1(s)\,f_1(s,u(s),v(s))\,ds\right] \\
&\ge \gamma\nu a^{n-1}\left[\left(f_{2\infty}-\varepsilon_2\right)\int_a^b K_n(s)\,a_2(s)\,ds + \left(f_{1\infty}-\varepsilon_1\right)\int_a^b K_n(s)\,a_1(s)\,ds\right]\cdot\|(u,v)\| \\
&\ge \frac{1}{2}\|(u,v)\|.
\end{aligned} \tag{37}$$

Therefore,

$$\begin{aligned}
\|T(u,v)\| &= \|T_1(u,v)\| + \|T_2(u,v)\| \\
&\ge \min_{t\in[a,b]} T_1(u,v)(t) + \min_{t\in[a,b]} T_2(u,v)(t) \\
&\ge \|(u,v)\|, \quad (u,v) \in \partial\Omega_2 \cap Q.
\end{aligned} \tag{38}$$

By applying Lemmas 5 and 6 to (35) and (38), it follows that operator T has at least one fixed point (u,v) in $Q \cap (\overline{\Omega}_2 \setminus \Omega_1)$. This means that system (1) has at least one positive solution (u,v). □

Using similar arguments as those used in the proof of Theorem 7, we can also obtain the following result.

Theorem 8. *If $f^{1\infty}, f^{2\infty} \in [0,r)$ and $f_{10}, f_{20} \in (R,+\infty]$, then system* (1) *has at least one positive solution.*

Next we discuss the multiplicity of positive solutions for system (1).

Theorem 9. *If $f_{10}, f_{2\infty} \in (4R, +\infty]$, and there exist an $m > 0$ such that $f_1(t,u,v), f_2(t,u,v) \in (0, mr)$, for for all $t \in [0,1], (u,v) \in \partial\Omega_3 \cap Q$, where $\Omega_3 = \{(u,v) \in P \times P, \|(u,v)\| < m\}$, then system* (1) *has at least two positive solutions.*

Proof. At first, it follows from the assumption $f_{10} \in (4R, +\infty], +\infty]$ that there exists an $0 < m_1 < m$ and a sufficiently small $\varepsilon_4 > 0$ such that

$$f_1(t,u,v) \geq (f_{10} - \varepsilon_4)(u+v), \quad \forall t \in [0,1], \ u+v \leq m_1, \tag{39}$$

where ε_4 satisfies $f_{10} - \varepsilon_4 \geq 4R$.

Set $\Omega_4 = \{(u,v) \in P \times P : \|(u,v)\| < m_1\}$ and $(u,v) \in \partial\Omega_4 \cap Q$. By (24), we have

$$\begin{aligned} \|T(u,v)\| &\geq \min_{t\in[a,b]} T(u,v)(t) \geq \min_{t\in[a,b]} T_1(u,v)(t) \\ &\geq \nu a^{n-1} \int_a^b K_n(s) a_1(s) f_1(s,u(s),v(s))\, ds \\ &\geq \gamma \nu a^{n-1} (f_{10} - \varepsilon_4) \int_a^b K_n(s) a_1(s)\, ds \cdot \|(u,v)\| \\ &\geq \|(u,v)\|. \end{aligned} \tag{40}$$

Further, by using $f_{2\infty} \in (4R, +\infty]$, there exists $m_2 > m > 0$ and a sufficiently small $\varepsilon_5 > 0$ such that

$$f_2(t,u,v) \geq (f_{2\infty} - \varepsilon_5)(u+v), \quad \forall t \in [0,1], \ u+v \geq m_2, \tag{41}$$

where ε_5 satisfies $f_{2\infty} - \varepsilon_5 \geq 4R$. Set $\Omega_5 = \{(u,v) \in P \times P : \|(u,v)\| < m_3\}$, where $m_3 > m_2$. For all $(u,v) \in \partial\Omega_5 \cap Q$, by (24), we have

$$\begin{aligned} \|T(u,v)\| &\geq \min_{t\in[a,b]} T(u,v)(t) \geq \min_{t\in[a,b]} T_1(u,v)(t) \\ &\geq \nu a^{n-1} \int_a^b K_n(s) a_2(s) f_2(s,u(s),v(s))\, ds \\ &\geq \gamma \nu a^{n-1} (f_{2\infty} - \varepsilon_5) \int_a^b K_n(s) a_2(s)\, ds \cdot \|(u,v)\| \\ &\geq \|(u,v)\|. \end{aligned} \tag{42}$$

By assumption, for for all $(u,v) \in \partial\Omega_3 \cap Q$, we have

$$\begin{aligned} \|T(u,v)\| &= \|T_1(u,v)\| + \|T_2(u,v)\| \\ &\leq 2rm\rho \left[\int_0^1 K_n(s) a_1(s)\, ds + \int_0^1 K_n(s) a_2(s)\, ds \right] \\ &\leq m = \|(u,v)\|. \end{aligned} \tag{43}$$

From (40)–(43), it is easy to know that two conditions of Lemma 6 are both satisfied. By applying Lemmas 5 and 6 to (40)–(43), it follows that operator T has at least a fixed point $(u_1,v_1) \in Q \cap (\overline{\Omega}_3 \setminus \Omega_4)$ and a fixed point $(u_2,v_2) \in Q \cap (\overline{\Omega}_5 \setminus \Omega_3)$. Both are positive solutions of system (1) and satisfy $m_1 \leq \|(u_1,v_1)\| < m < \|(u_2,v_2)\| \leq m_3$. This means system (1) has at least two positive solutions. □

Similarly, we have the following results.

Theorem 10. *If $f_{10}, f_{1\infty} \in (4R, +\infty]$, and there exist an $m > 0$ such that $f_1(t,u,v), f_2(t,u,v) \in (0, mr)$, for for all $t \in [0,1], (u,v) \in \partial\Omega_6 \cap Q$, where $\Omega_6 = \{(u,v) \in P \times P, \|(u,v)\| < m\}$, then system* (1) *has at least two positive solutions.*

Theorem 11. *If $f_{20}, f_{1\infty} \in (4R, +\infty]$, and there exist an $m > 0$ such that $f_1(t,u,v), f_2(t,u,v) \in (0, mr)$, for all $t \in [0,1], (u,v) \in \partial\Omega_7 \cap Q$, where $\Omega_7 = \{(u,v) \in P \times P, \|(u,v)\| < m\}$. Then system* (1) *has at least two positive solutions.*

Theorem 12. *If $f_{20}, f_{2\infty} \in (4R, +\infty]$, and there exist an $m > 0$ such that $f_1(t,u,v), f_2(t,u,v) \in (0, mr)$, for all $t \in [0,1], (u,v) \in \partial\Omega_8 \cap Q$, where $\Omega_8 = \{(u,v) \in P \times P, \|(u,v)\| < m\}$. Then system* (1) *has at least two positive solutions.*

4. Some Examples

In order to illustrate our result, we consider some examples.

Example 1. Consider the following system

$$\begin{aligned} &-u^{(3)}(t) = \frac{1+t}{8}\left[\left(u^2+v^2\right) + \lambda \sin(u+v)\right], \quad t \in (0,1), \\ &-v^{(3)}(t) = \frac{1}{4}\left[\left(u^2+v^2\right)^2 + e^{-(u^2+v^2)}\right], \quad t \in (0,1), \\ &u(0) = u'(0) = 0, \qquad u(1) = 2\int_0^1 v(t)\, dt, \\ &v(0) = v'(0) = 0, \qquad v(1) = \int_0^1 u(t)\, dt, \end{aligned} \tag{44}$$

where $n = 3$, $a_1(t) = (1+t)/8$, $a_2(t) = 1/4$, $f_1(t,u,v) = (u^2+v^2) + \lambda\sin(u+v)$, $f_2(t,u,v) = (u^2+v^2)^2 + e^{-(u^2+v^2)}$, $A(t) = 2t, B(t) = t, K_n(s) = s(1-s)^2, \gamma_n(t) = (1/2)\min\{t^2, (1-t)t\}$. By direct calculation, we can obtain that $\rho = 18/7$, $\nu = 3/56$, $r = 14/15$. Choose $\lambda \in [0, 14/15))$; then conditions of Theorem 7 are satisfied. This means that system (44) has at least one position solution.

Example 2. Consider the following system

$$\begin{aligned} &-u^{(3)}(t) = \frac{1+t}{16}(u^2+v^2)^{(1/3)}, \quad t \in (0,1), \\ &-v^{(3)}(t) = \frac{1}{32}\left(u^2+v^2\right), \quad t \in (0,1), \end{aligned}$$

$$u(0) = u'(0) = 0, \qquad u(1) = 2\int_0^1 v(t)\,dt,$$
$$v(0) = v'(0) = 0, \qquad v(1) = \int_0^1 u(t)\,dt, \tag{45}$$

where $n = 3$, $a_1(t) = (1+t)/16$, $a_2(t) = 1/32$, $f_1(t,u,v) = (u^2+v^2)^{1/3}$, $f_2(t,u,v) = (u^2+v^2)$, $A(t) = 2t$, $B(t) = t$, $K_n(s) = s(1-s)^2$, $\gamma_n(t) = (1/2)\min\{t^2,(1-t)t\}$. By direct calculation, we can obtain that $\rho = (18/7)$, $\upsilon = (3/56)$, \, $r = (112/15)$. Choose $m = 2$; then conditions of Theorem 9 are satisfied. This means that system (45) has at least two position solutions.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' Contribution

The work presented here was carried out in collaboration between all authors. All authors read and approved the final paper.

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