Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 897370, 6 pages
http://dx.doi.org/10.1155/2014/897370

Research Article

Iterative Algorithms for the Split Problem and Its Convergence Analysis

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Received 12 May 2014; Accepted 18 May 2014; Published 28 May 2014

Academic Editor: Jen-Chih Yao

Now, it is known that the split common fixed point problem is a generalization of the split feasibility problem and of the convex feasibility problem. In this paper, the split common fixed point problem associated with the pseudocontractions is studied. An iterative algorithm has been presented for solving the split common fixed point problem. Strong convergence result is obtained.

1. Introduction

Now, we know that the convex feasibility problem can be formulated as finding a point $x^\dagger$ such that

$$x^\dagger \in \bigcap_{i=1}^{n} \mathbb{C}_i, \qquad n \in \mathbb{N}, \tag{1}$$

where $\mathbb{C}_i\ (\neq \emptyset)$ is a closed convex subset of a Hilbert space $\mathbb{H}$. The convex feasibility problem has extensive applications in many applied disciplines such as signal processing, biomedical engineering, and communications. For related works, please see [1–3].

If $n = 2$ in (1), then a special case of (1) is the following split feasibility problem.

Problem 1. The split feasibility problem: let $\mathbb{H}_1$ and $\mathbb{H}_2$ be two Hilbert spaces. Let $\mathbb{C}_1 \subset \mathbb{H}_1$ and $\mathbb{C}_2 \subset \mathbb{H}_2$ be two nonempty closed convex sets. Let $A : \mathbb{H}_1 \to \mathbb{H}_2$ be a bounded linear operator. The split feasibility problem is

$$\text{find a vector } x^\dagger \in \mathbb{C}_1 \text{ such that } Ax^\dagger \in \mathbb{C}_2. \tag{2}$$

Such problem arises in the intensity-modulated radiation therapy. In the finite-dimensional space, Censor and Elfving [4] firstly constructed the following iterative algorithm to solve (2):

$$x_{m+1} = A^{-1}\mathrm{proj}_{\mathbb{C}_2}\left(\mathrm{proj}_{A(\mathbb{C}_1)}\left(Ax_m\right)\right), \qquad m \in \mathbb{N}, \tag{3}$$

where $\mathbb{C}_1 \subset \mathbb{R}^n$ and $\mathbb{C}_2 \subset \mathbb{R}^n$ are closed convex sets and A is an $n \times n$ matrix.

However, we note that calculating inverse A^{-1} is very time-consuming, if the dimension n is large. For overcoming this problem, Byrne [5] introduced the following more popular algorithm:

$$x_{m+1} = \mathrm{proj}_{\mathbb{C}_1}\left(x_m - \varpi A^T\left(I - \mathrm{proj}_{\mathbb{C}_2}\right)Ax_m\right), \qquad m \in \mathbb{N}, \tag{4}$$

where A^T denotes the transposition of A. Consequently, (4) and its variant have been studied extensively. For related results, please refer to [6–13].

In the case where $\mathbb{C}_1$ and $\mathbb{C}_2$ in (2) are the fixed point sets of nonlinear operators, problem (2) is called by Censor and Segal [14] the split common fixed point problem.

Problem 2. The split common fixed point problem: this problem is to find a fixed point $x^\dagger$ of an operator S in the space $\mathbb{H}_1$ such that its image $Ax^\dagger$ under a linear transformation A

is a fixed point $y^\dagger$ of another operator T in the image space $\mathbb{H}_2$. Namely, find a vector $x^\dagger \in H$ such that

$$x^\dagger \in \mathrm{Fix}(S), \qquad y^\dagger = Ax^\dagger \in \mathrm{Fix}(T), \tag{5}$$

where Fix(S) and Fix(T) denote the fixed point sets of nonlinear operators $S : \mathbb{H}_1 \to \mathbb{H}_1$ and $T : \mathbb{H}_2 \to \mathbb{H}_2$, respectively.

A natural idea is to apply (4) to the split common fixed point problem (5). That is, taking $\mathbb{C}_1 = \mathrm{Fix}(S)$ and $\mathbb{C}_2 = \mathrm{Fix}(T)$ in (4), we get

$$x_{m+1} = \mathrm{proj}_{\mathrm{Fix}(S)}\left(x_m - \varpi A^T \times \left(I - \mathrm{proj}_{\mathrm{Fix}(T)}\right) Ax_m\right), \quad m \in \mathbb{N}. \tag{6}$$

However, $\mathrm{proj}_{\mathrm{Fix}(S)}$ and $\mathrm{proj}_{\mathrm{Fix}(T)}$ are generally not easy to calculate. Thus, (6) may fail. We have to find new algorithm to solve (5). In this respect, Censor and Segal [14] proposed the following iterative method: for any initial guess $x_1 \in \mathbb{H}_1$, define a sequence $\{x_m\}$ by

$$x_{m+1} = S\left(x_m - \lambda A^* (I - T) Ax_m\right), \quad m \in \mathbb{N}, \tag{7}$$

where S and T are directed operators. Moudafi [15] relaxed (7) to the following form:

$$\begin{gathered} y_m = x_m - \varpi A^* (I - T) Ax_m, \\ x_{m+1} = (1 - \alpha_m) y_m + \alpha_m S(y_m), \quad m \in \mathbb{N}, \end{gathered} \tag{8}$$

where U and T are demicontractive operators.

Note that (7) and (8) have weak convergence. Some strong convergence results have been given in the literature; see, for instance, [16, 17]. In the present paper, we consider an interesting respect: could we extend the classes of directed and demicontractive operators to the class of pseudocontractive mappings?

Our main purpose of this paper is to solve the above problem. We construct an iterative algorithm in which the involved operators are pseudocontractions and show its strong convergence.

2. Definitions and Lemmas

In this section, we collect some definitions and lemmas. Let $\mathbb{H}$ be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively. Let $(\mathbb{H} \supset)\mathbb{E} \neq \emptyset$ be a closed convex set.

Definition 3. An operator $T : \mathbb{E} \to \mathbb{E}$ is called Lipschitzian, if

$$\left\|Tx^\ddagger - Ty^\ddagger\right\| \le L\left\|x^\ddagger - y^\ddagger\right\|, \tag{9}$$

for some $L > 0$ and all $x^\ddagger, y^\ddagger \in \mathbb{E}$.

In this case, we call TL-Lipschitzian continuous. If $L = 1$ in (9), we call T nonexpansive.

Definition 4. An operator $T : \mathbb{E} \to \mathbb{E}$ is called a directed operator, if

$$\left\langle Tx^\dagger - x^\dagger, Tx^\dagger - x\right\rangle \le 0; \tag{10}$$

equivalently,

$$\left\|Tx^\dagger - x\right\|^2 \le \left\|x^\dagger - x\right\|^2 - \left\|Tx^\dagger - x^\dagger\right\|^2 \tag{11}$$

for all $x^\dagger \in \mathbb{E}$ and $x \in \mathrm{Fix}(T)$, the fixed points set of T.

Definition 5. An operator $T : \mathbb{E} \to \mathbb{E}$ is called a demicontractive operator, if

$$\begin{gathered} \left\|Tx^\dagger - x\right\|^2 \le \left\|x^\dagger - x\right\|^2 + \kappa\left\|Tx^\dagger - x^\dagger\right\|^2, \\ \forall x^\dagger \in \mathbb{E},\ x \in \mathrm{Fix}(T), \end{gathered} \tag{12}$$

where $\kappa \in (0,1)$

From the above definitions, we note that the class of demicontractive operators contains important operators such as the directed operators and the nonexpansive operators with fixed points. Such a class of operators is fundamental because they include many types of nonlinear operators arising in applied mathematics and optimization; see, for example, [18] and references therein.

Definition 6. An operator $T : \mathbb{E} \to \mathbb{E}$ is called pseudocontractive, if

$$\left\langle Tx^\dagger - Ty^\dagger, x^\dagger - y^\dagger\right\rangle \le \left\|x^\dagger - y^\dagger\right\|^2; \tag{13}$$

equivalently,

$$\left\|Tx^\dagger - Ty^\dagger\right\|^2 \le \left\|x^\dagger - y^\dagger\right\|^2 + \left\|(I - T)x^\dagger - (I - T)y^\dagger\right\|^2 \tag{14}$$

for all $x^\dagger, y^\dagger \in \mathbb{E}$.

It is obvious that the class of pseudocontractive mappings with fixed points includes the class of demicontractive mappings.

Lemma 7 (see [19]). *Let $\mathbb{H}$ be a real Hilbert space; let $\mathbb{C} \subset \mathbb{H}$ be a closed convex set. Let $W : \mathbb{C} \to \mathbb{C}$ be a continuous pseudocontractive mapping. Then*

(i) Fix(W) *is a closed convex subset of* $\mathbb{C}$;

(ii) $(I - W)$ *is demiclosed at zero.*

3. Main Results

Let $\mathbb{H}_1$ and $\mathbb{H}_2$ be two real Hilbert spaces and let $A : \mathbb{H}_1 \to \mathbb{H}_2$ be a bounded linear operator with its adjoint A^*. This section is devoted to study problem (5), where $S : \mathbb{H}_1 \to \mathbb{H}_1$ and $T : \mathbb{H}_2 \to \mathbb{H}_2$ are two L-Lipschitzian pseudocontractive mappings. We denote the solution set of problem (5) by

$$\Gamma = \left\{x^\dagger \in \mathrm{Fix}(S); Ax^\dagger \in \mathrm{Fix}(T)\right\}. \tag{15}$$

In the sequel, we assume $\Gamma \neq \emptyset$.

In order to solve problem (5), we present the following iterative algorithm.

Algorithm 8. Let $\varpi, \tau, \sigma, \varrho$, and ς be five constants. For $u \in \mathbb{H}_1$, arbitrarily, we define the following iterative manner:

$$\begin{gathered} \mathbb{C}_1 = \mathbb{H}_1, \qquad x_1 = \operatorname{proj}_{\mathbb{C}_1}(u), \\ z_n = x_n + \varpi A^* \left[(1-\tau) I + \tau T\left((1-\sigma) I + \sigma T\right) - I\right] A x_n, \\ y_n = (1-\varrho) z_n + \varrho S\left[(1-\varsigma) z_n + \varsigma S z_n\right], \\ \mathbb{C}_{n+1} = \left\{z \in \mathbb{C}_n : \|y_n - z\| \le \|x_n - z\|\right\}, \\ x_{n+1} = \operatorname{proj}_{\mathbb{C}_{n+1}}(u), \quad n \in \mathbb{N}. \end{gathered} \tag{16}$$

Theorem 9. *Assume that ϖ, τ, σ, ϱ, and ς satisfy the following assumptions: $0 < \varpi < 1/\|A\|^2$, $0 < \tau \le \sigma < 1/(\sqrt{1+L^2}+1)$, and $0 < \varrho < \varsigma < 1/(\sqrt{1+L^2}+1)$. Then, $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ defined by (16) converge strongly to $\operatorname{proj}_\Gamma(u)$.*

Proof. The outline of our proof details is as follows:

(i) $\Gamma \subset \mathbb{C}_n$, for all $n \in \mathbb{N}$;

(ii) $\mathbb{C}_n$ is closed and convex, for all $n \in \mathbb{N}$;

(iii) $x_n \to x^*$;

(iv) $x^* = \operatorname{proj}_\Gamma(u)$.

Proof of (i). We show this by induction. (1) $\Gamma \subset \mathbb{C}_1 = \mathbb{H}_1$ is obvious.

(2) Suppose that $\Gamma \subset \mathbb{C}_k$ for some $k \in \mathbb{N}$. For any $p \in \Gamma \subset \mathbb{C}_k$, we have, from (14), that

$$\begin{aligned} &\left\|S\left((1-\varsigma) z_k + \varsigma S z_k\right) - p\right\|^2 \\ &\quad = \left\|S\left((1-\varsigma) z_k + \varsigma S z_k\right) - Sp\right\|^2 \\ &\quad \le \left\|(1-\varsigma) z_k + \varsigma S z_k - p\right\|^2 \\ &\qquad + \left\|(1-\varsigma) z_k + \varsigma S z_k - S((1-\varsigma) z_k + \varsigma S z_k)\right\|^2, \\ &\quad \|S z_k - p\|^2 \le \|z_k - p\|^2 + \|z_k - S z_k\|^2. \end{aligned} \tag{17}$$

Observing that in any Hilbert space, we have

$$\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2, \quad \forall t \in [0,1]. \tag{18}$$

Set $v_k = (1-\varsigma) z_k + \varsigma S z_k$, for all $k \in \mathbb{N}$. By (17) and (18), we obtain

$$\begin{aligned} \|S v_k - p\|^2 &\le \left\|(1-\varsigma)(z_k - p) + \varsigma(S z_k - p)\right\|^2 \\ &\quad + \left\|(1-\varsigma) z_k + \varsigma S z_k - S v_k\right\|^2 \\ &= \left\|(1-\varsigma)(z_k - S v_k) + \varsigma(S z_k - S v_k)\right\|^2 \\ &\quad + \left\|(1-\varsigma)(z_k - p) + \varsigma(S z_k - p)\right\|^2 \\ &= (1-\varsigma)\|z_k - S v_k\|^2 + \varsigma\|S z_k - S v_k\|^2 \\ &\quad - \varsigma(1-\varsigma)\|z_k - S z_k\|^2 \\ &\quad + (1-\varsigma)\|z_k - p\|^2 + \varsigma\|S z_k - p\|^2 \\ &\quad - \varsigma(1-\varsigma)\|z_k - S z_k\|^2 \\ &\le (1-\varsigma)\|z_k - p\|^2 + \varsigma\left(\|z_k - p\|^2 + \|z_k - S z_k\|^2\right) \\ &\quad - 2\varsigma(1-\varsigma)\|z_k - S z_k\|^2 + (1-\varsigma)\|z_k - S v_k\|^2 \\ &\quad + \varsigma\|S z_k - S v_k\|^2. \end{aligned} \tag{19}$$

Since S is L-Lipschitzian and $z_k - v_k = \varsigma(z_k - S z_k)$, we have

$$\begin{aligned} \|S v_k - p\|^2 &\le (1-\varsigma)\|z_k - p\|^2 + \varsigma\left(\|z_k - p\|^2 + \|z_k - S z_k\|^2\right) \\ &\quad - 2\varsigma(1-\varsigma)\|z_k - S z_k\|^2 + (1-\varsigma)\|z_k - S v_k\|^2 \\ &\quad + \varsigma^3 L^2\|z_k - S z_k\|^2 \\ &= \|z_k - p\|^2 + (1-\varsigma)\|z_k - S v_k\|^2 \\ &\quad - \varsigma\left(1 - 2\varsigma - \varsigma^2 L^2\right)\|z_k - S z_k\|^2. \end{aligned} \tag{20}$$

Since $\varsigma < 1/(\sqrt{1+L^2}+1)$, we deduce

$$1 - 2\varsigma - \varsigma^2 L^2 > 0. \tag{21}$$

This together with (20) implies that

$$\|S v_k - p\|^2 \le \|z_k - p\|^2 + (1-\varsigma)\|z_k - S v_k\|^2. \tag{22}$$

Hence,

$$\begin{aligned} \|y_k - p\|^2 &= \left\|(1-\varrho) z_k + \varrho S v_k - p\right\|^2 \\ &= \left\|(1-\varrho)(z_k - p) + \varrho(S v_k - p)\right\|^2 \\ &= (1-\varrho)\|z_k - p\|^2 + \varrho\|S v_k - p\|^2 \\ &\quad - \varrho(1-\varrho)\|S v_k - z_k\|^2 \\ &\le (1-\varrho)\|z_k - p\|^2 \\ &\quad + \varrho\left[\|z_k - p\|^2 + (1-\varsigma)\|z_k - S v_k\|^2\right] \\ &\quad - \varrho(1-\varrho)\|S v_k - z_k\|^2 \\ &= \|z_k - p\|^2 + (1-\varrho)(\varrho - \varsigma)\|S v_k - z_k\|^2. \end{aligned} \tag{23}$$

Noting that $\varrho \le \varsigma$, we deduce

$$\begin{aligned} \|y_k - p\| &= \left\|(1-\varrho) z_k + \varrho S\left((1-\varsigma) z_k + \varsigma S z_k\right) - p\right\| \\ &\le \|z_k - p\|. \end{aligned} \tag{24}$$

Similarly, we also have

$$\left\|[(1-\tau)I+\tau T((1-\sigma)I+\sigma T)]Ax_k-Ap\right\|\le\left\|Ax_k-Ap\right\|. \tag{25}$$

In Hilbert spaces, there holds

$$2\langle z^{\dagger},z\rangle=\left\|z^{\dagger}+z\right\|^2-\left\|z^{\dagger}\right\|^2-\|z\|^2. \tag{26}$$

With the help of (25) and (26), we get

$$\begin{aligned}
&\left\|z_k-p\right\|^2\\
&=\|x_k-p+\varpi A^*[(1-\tau)I+\tau T((1-\sigma)I+\sigma T)-I]\\
&\qquad\times Ax_k\|^2\\
&=2\varpi\langle A^*[(1-\tau)I+\tau T((1-\sigma)I+\sigma T)-I]\\
&\qquad\times Ax_k,x_k-p\rangle\\
&\quad+\varpi^2\left\|A^*[(1-\tau)I+\tau T((1-\sigma)I+\sigma T)-I]Ax_k\right\|^2\\
&\quad+\left\|x_k-p\right\|^2\\
&=2\varpi\langle[(1-\tau)I+\tau T((1-\sigma)I+\sigma T)-I]Ax_k,\\
&\qquad Ax_k-Ap\rangle\\
&\quad+\varpi^2\|A\|^2\left\|[(1-\tau)I+\tau T((1-\sigma)I+\sigma T)-I]Ax_k\right\|^2\\
&\quad+\left\|x_k-p\right\|^2\\
&=\left\|x_k-p\right\|^2\\
&\quad+\varpi\Big[\left\|[(1-\tau)I+\tau T((1-\sigma)I+\sigma T)]Ax_k-Ap\right\|^2\\
&\qquad-\left\|[(1-\tau)I+\tau T((1-\sigma)I+\sigma T)-I]Ax_k\right\|^2\\
&\qquad-\left\|Ax_k-Ap\right\|^2\Big]\\
&\quad+\varpi^2\|A\|^2\left\|[(1-\tau)I+\tau T((1-\sigma)I+\sigma T)-I]Ax_k\right\|^2\\
&\le\varpi\left(\varpi\|A\|^2-1\right)\|[(1-\tau)I+\tau T((1-\sigma)I+\sigma T)-I]\\
&\qquad\times Ax_k\|^2\\
&\quad+\left\|x_k-p\right\|^2\le\left\|x_k-p\right\|^2.
\end{aligned} \tag{27}$$

By (24) and (27), we have

$$\left\|y_k-p\right\|\le\left\|z_k-p\right\|\le\left\|x_k-p\right\|. \tag{28}$$

This shows that $p\in\mathbb{C}_{k+1}$. Thus, we get $\Gamma\subset\mathbb{C}_n$, for all $n\in\mathbb{N}$.

Proof of (ii). It is easy to verify that $\mathbb{C}_n$ is closed, for all $n\in\mathbb{N}$. Next, we only need to verify that $\mathbb{C}_n$ is convex, for all $n\in\mathbb{N}$. In fact, let $z^{\dagger},z^{\ddagger}\in\mathbb{C}_{n+1}$; for each $\zeta\in(0,1)$, we have

$$\begin{aligned}
\left\|y_n-(\zeta z^{\dagger}+(1-\zeta)z^{\ddagger})\right\|^2&=\left\|\zeta(y_n-z^{\dagger})+(1-\zeta)(y_n-z^{\ddagger})\right\|^2\\
&=\zeta\left\|y_n-z^{\dagger}\right\|^2+(1-\zeta)\left\|y_n-z^{\ddagger}\right\|^2\\
&\quad-\zeta(1-\zeta)\left\|z^{\dagger}-z^{\ddagger}\right\|^2\\
&\le\zeta\left\|z_n-z^{\dagger}\right\|^2+(1-\zeta)\left\|z_n-z^{\ddagger}\right\|^2\\
&\quad-\zeta(1-\zeta)\left\|z^{\dagger}-z^{\ddagger}\right\|^2\\
&=\left\|z_n-\left(\zeta z^{\dagger}+(1-\zeta)z^{\ddagger}\right)\right\|^2;
\end{aligned} \tag{29}$$

namely,

$$\left\|y_n-\left(\zeta z^{\dagger}+(1-\zeta)z^{\ddagger}\right)\right\|\le\left\|z_n-\left(\zeta z^{\dagger}+(1-\zeta)z^{\ddagger}\right)\right\|; \tag{30}$$

this shows $\zeta z^{\dagger}+(1-\zeta)z^{\ddagger}\in\mathbb{C}_{n+1}$ and $\mathbb{C}_{n+1}$ is a convex set, for all $n\in\mathbb{N}$.

Proof of (iii). Since $\Gamma\subset\mathbb{C}_{n+1}\subset\mathbb{C}_n$ and $x_{n+1}=P_{\mathbb{C}_{n+1}}(u)\subset\mathbb{C}_n$, we obtain

$$\left\|x_{n+1}-u\right\|\le\left\|p-u\right\|\quad\forall n\in\mathbb{N},\ p\in\Gamma. \tag{31}$$

It follows that $\{x_n\}$ is bounded.

It is known that the metric projection $\mathrm{proj}_{\mathbb{C}}$ can be characterized by

$$\left\langle x^{\dagger}-\mathrm{proj}_{\mathbb{C}}\left(x^{\dagger}\right),\mathrm{proj}_{\mathbb{C}}\left(x^{\dagger}\right)-x\right\rangle\ge0,\quad\forall x\in\mathbb{C}; \tag{32}$$

equivalently,

$$\left\|x-\mathrm{proj}_{\mathbb{C}}(x^{\dagger})\right\|^2+\left\|x^{\dagger}-\mathrm{proj}_{\mathbb{C}}(x^{\dagger})\right\|^2\le\left\|x-x^{\dagger}\right\|^2. \tag{33}$$

With the help of (33), we have

$$\begin{aligned}
\left\|x_{n+1}-x_n\right\|^2+\left\|u-x_n\right\|^2&=\left\|x_{n+1}-P_{\mathbb{C}_n}(u)\right\|^2+\left\|u-P_{\mathbb{C}_n}(u)\right\|^2\\
&\le\left\|x_{n+1}-u\right\|^2,
\end{aligned} \tag{34}$$

which implies that

$$0\le\left\|x_{n+1}-x_n\right\|^2\le\left\|x_{n+1}-u\right\|^2-\left\|u-x_n\right\|^2. \tag{35}$$

It follows that

$$\left\|x_n-u\right\|\le\left\|x_{n+1}-u\right\|. \tag{36}$$

Since $\{\|x_n-u\|\}$ is bounded, we get

$$\lim_{n\to\infty}\left\|x_n-u\right\|\text{ exists.} \tag{37}$$

This together with (35) implies that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{38}$$

The fact that $x_{n+1}=P_{\mathbb{C}_{n+1}}(u)\in\mathbb{C}_{n+1}$ gives

$$\|y_n-x_{n+1}\|\le\|x_n-x_{n+1}\|\longrightarrow 0. \tag{39}$$

By (38) and (39), we derive

$$\lim_{n\to\infty}\|y_n-x_n\|=0. \tag{40}$$

Next we show that $\{x_n\}$ is a Cauchy sequence. As a matter of fact, for any $m,n\in\mathbb{N}$ with $m>n$, we have

$$\begin{aligned}\|x_m-x_n\|^2+\|x_n-u\|^2&=\|x_m-P_{\mathbb{C}_n}(u)\|^2+\|P_{\mathbb{C}_n}(u)-u\|^2\\&\le\|x_m-u\|^2.\end{aligned} \tag{41}$$

It follows that

$$\|x_m-x_n\|^2\le\|x_m-u\|^2-\|x_n-u\|^2. \tag{42}$$

Note that $\lim_{n\to\infty}(\|x_m-u\|^2-\|x_n-u\|^2)=0$. Therefore, $\lim_{n\to\infty}\|x_m-x_n\|=0$. So, $\{x_n\}$ is a Cauchy sequence and hence $x_n\to x^*$.

Proof of (iv). From (24), (27), and (40), we have

$$\begin{aligned}&-\varpi\left(\varpi\|A\|^2-1\right)\|[(1-\tau)I+\tau T((1-\sigma)I+\sigma T)-I]Ax_n\|^2\\&\quad\le\|x_n-p\|^2-\|y_n-p\|^2\\&\quad\le\|x_n-y_n\|(\|x_n-p\|+\|y_n-p\|)\\&\quad\longrightarrow 0.\end{aligned} \tag{43}$$

Hence,

$$\lim_{n\to\infty}\|[(1-\tau)I+\tau T((1-\sigma)I+\sigma T)-I]Ax_n\|=0. \tag{44}$$

It follows that

$$\lim_{n\to\infty}\|T((1-\sigma)I+\sigma T)Ax_n-Ax_n\|=0. \tag{45}$$

Next, we firstly show that

$$\mathrm{Fix}(T)=\mathrm{Fix}(T((1-\sigma)I+\sigma T)). \tag{46}$$

As a matter of fact, $\mathrm{Fix}(T)\subset\mathrm{Fix}(T((1-\sigma)I+\sigma T))$ is obvious. We only need to show that $\mathrm{Fix}(T((1-\sigma)I+\sigma T))\subset\mathrm{Fix}(T)$.

Take any $y^\dagger\in\mathrm{Fix}(T((1-\sigma)I+\sigma T))$. We have $T((1-\sigma)I+\sigma T)y^\dagger=y^\dagger$. Set $R=(1-\sigma)I+\sigma T$. We have $TRy^\dagger=y^\dagger$. Write $Ry^\dagger=y^*$. Then, $Ty^*=y^\dagger$. Next, we show $y^\dagger=y^*$. In fact,

$$\begin{aligned}\|y^\dagger-y^*\|&=\|Ty^*-Ry^\dagger\|\\&=\|Ty^*-(1-\sigma)y^\dagger-\sigma Ty^\dagger\|\\&=\sigma\|Ty^*-Ty^\dagger\|\\&\le\sigma L\|y^*-y^\dagger\|.\end{aligned} \tag{47}$$

Since $\sigma<1/(\sqrt{1+L^2}+1)<1/L$, we deduce $y^*=y^\dagger\in\mathrm{Fix}(R)=\mathrm{Fix}(T)$. Thus, $y^\dagger\in\mathrm{Fix}(T)$. Hence, $\mathrm{Fix}(T((1-\sigma)I+\sigma T))\subset\mathrm{Fix}(T)$. Therefore, $\mathrm{Fix}(T((1-\sigma)I+\sigma T))=\mathrm{Fix}(T)$.

With (46) in hand, we prove that $T((1-\sigma)I+\sigma T)-I$ is demiclosed at 0. Let the sequence $\{u_n\}\subset\mathbb{H}_2$ satisfying $u_n\rightharpoonup y^\dagger$ and $u_n-T((1-\sigma)I+\sigma T)u_n\to 0$. Now, we show that $y^\dagger\in\mathrm{Fix}(T((1-\sigma)I+\sigma T))$.

Since T is L-Lipschitzian, we have

$$\begin{aligned}\|u_n-Tu_n\|&\le\|u_n-T((1-\sigma)I+\sigma T)u_n\|\\&\quad+\|T((1-\sigma)I+\sigma T)u_n-Tu_n\|\\&\le\|u_n-T((1-\sigma)I+\sigma T)u_n\|+\sigma L\|u_n-Tu_n\|.\end{aligned} \tag{48}$$

It follows that

$$\|u_n-Tu_n\|\le\frac{1}{1-\sigma L}\|u_n-T((1-\sigma)I+\sigma T)u_n\|. \tag{49}$$

Hence,

$$\lim_{n\to\infty}\|u_n-Tu_n\|=0. \tag{50}$$

Since $T-I$ is demiclosed at 0 by Lemma 7, we immediately deduce $y^\dagger\in\mathrm{Fix}(T)=\mathrm{Fix}(T((1-\sigma)I+\sigma T))$. Therefore, $T((1-\sigma)I+\sigma T)-I$ is demiclosed at 0. Since A is a bounded linear operator, we get $\|Ax_n-Ax^*\|\to 0$. From (45), we deduce $Ax^*\in\mathrm{Fix}(T((1-\sigma)I+\sigma T))=\mathrm{Fix}(T)$.

By (16), (40), and (44), we deduce

$$\lim_{n\to\infty}\|z_n-y_n\|=0. \tag{51}$$

So,

$$\lim_{n\to\infty}\|S[(1-\varsigma)z_n+\varsigma Sz_n]-z_n\|=\lim_{n\to\infty}\frac{1}{\varrho}\|y_n-z_n\|=0. \tag{52}$$

Similarly, we can show that $x^*\in\mathrm{Fix}(S)$. To this end, we have proven that $x^*\in\mathrm{Fix}(S)$ and $Ax^*\in\mathrm{Fix}(T)$. Therefore, $x^*\in\Gamma$. This completes the proof. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The third author was supported by the fund of the Research Promotion Program, Gyeongsang National University, 2013 (RPP-2013-023). The fourth author was supported in part by the NNSF of China (61362033) and NZ13087.

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