Research Article
An Efficient Series Solution for Fractional Differential Equations

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We introduce a simple and efficient series solution for a class of nonlinear fractional differential equations of Caputo's type. The new approach is a modified form of the well-known Taylor series expansion where we overcome the difficulty of computing iterated fractional derivatives, which do not compute in general. The terms of the series are determined sequentially with explicit formula, where only integer derivatives have to be computed. The efficiency of the new algorithm is illustrated through several examples. Comparison with other series methods such as the Adomian decomposition method and the homotopy perturbation method is made to indicate the efficiency of the new approach. The algorithm can be implemented for a wide class of fractional differential equations with different types of fractional derivatives.

1. Introduction

During the last three decades, fractional calculus caught the attention of many researchers in differential fields of science and engineering. This is, mainly, due to the importance of noninteger order derivatives in modeling certain physical phenomena [1–4]. It turns out that, in some cases, modeling using fractional calculus is more realistic than integer calculus. This is because of the fact that the behavior of many physical phenomena depends not only upon the instantaneous state but also on the previous time history. Fractional derivative, comprising in its definition previous time history about the function, makes it more suitable for modeling systems whose evolution depends upon their current and previous states.

Recently, many researchers got interested in looking at fractional differential equations (FDEs) as new model equations for many physical problems. However, many of these such FDEs do not possess exact analytic solutions. This difficulty prompted many researchers to develop numerical schemes to find approximate solutions. Many numerical methods used to solve integer order differential equations have been adapted to treat FDEs such as the variational iteration (VIM) [5–8], the homotopy analysis method (HAM) [9–14], and the Adomian decomposition method (ADM)

[15–20], just to name a few. For a survey of recent development of methods in fractional calculus, the reader is referred to [21]. All these methods can be classified as iterative methods which produce a solution in the form of a series expansion whose terms are generated iteratively. However, for many cases, the iterative process of these methods is not easily implemented. For example, the ADM requires integration at each step to find the next iterate and the ADM requires solving a differential equation. Another approach is the upper-lower iterative method [22]. Quadrature techniques have been implemented to construct different formulations of fractional backward difference methods [23–25]. Also, fractional linear multistep methods presented for special types of the Volterra integral equation [26, 27] have been implemented for several types of fractional differential equations. As a result, a class of higher order backward difference methods have been obtained [28]. For more details one can refer to [29] and the references therein. Convenient and easy presentations to discretize fractional derivative of arbitrary order have been obtained in a form of triangular strip matrices; see [30, 31]. The suggested approach leads to a significant simplification of the solutions of differential equations of fractional order.

In our present work we present a series solution method in the spirit of the Taylor series expansion for a class of

nonlinear differential equation of fractional order. The coefficients of the series expansion are also iteratively computed but the iteration process involves only differentiation. Naturally, if the problem is of fractional order, the differentiation is also of fractional order. However, to overcome the use of fractional differentiation, we employ a transformation that allows us to use ordinary differentiation rather than fractional differentiation to recursively compute the coefficient of the series expansion. We see this as an advantage to the above-mentioned methods.

In this paper, we consider the initial value problem of fractional order:

$$D_t^\alpha u(x,t) = u_{xx}(x,t) + h(x,t,u), \quad t > 0, \ x \in \mathbb{R}, \quad (1)$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}, \quad (2)$$

where $0 < \alpha < 1, h \in C^\infty(\mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}, \mathbb{R}), u_0(x) \in C^\infty(\mathbb{R}, \mathbb{R})$, and D_t^α is the Caputo partial fractional derivative of order α. For $\alpha \in \mathbb{R}, \ n-1 < \alpha \le n, n \in \mathbb{N}^+$, the left Caputo fractional derivative is defined by [3]

$$D_t^\alpha u(x,t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{\partial u(x,s)}{\partial s} \frac{1}{(t-s)^{\alpha+1-n}} ds \quad (3)$$

and satisfies the following properties:

(1) $D_t^\alpha f(x) = 0$;

(2) $D_t^\alpha t^r = \dfrac{\Gamma(r+1)}{\Gamma(r-\alpha+1)} t^{r-\alpha}, \ r > n-1, \ r \in \mathbb{R}$;

(3) $D_t^\alpha(\sum_{i=0}^m c_i f_i(x,t)) = \sum_{i=0}^m c_i D_t^\alpha f_i(x,t)$, where $c_0, c_1, \ldots, c_m$ are constants.

The Caputo partial fractional derivative in (3) is related to the Riemann-Liouville partial fractional integral, I_t^α, of order α, by

$$D_t^\alpha f(x,t) = I_t^{n-\alpha} \frac{\partial^n f(x,t)}{\partial t^n}, \quad (4)$$

where, for $n-1 \le \alpha < n$,

$$I_t^\alpha f(x,t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(x,s) \, ds. \quad (5)$$

I_t^α can be considered as the inverse operator of D_t^α in the sense

$$I_t^\alpha D_t^\alpha f(x,t) = f(x,t) - \sum_{k=0}^{n-1} \frac{\partial^k f(x,0^+)}{\partial t^k} \frac{t^k}{k!}, \quad (6)$$

$$D_t^\alpha I_t^\alpha f(x,t) = f(x,t).$$

In this paper, we consider $\alpha = p/q$ rational with $gcd(p,q) = 1$. The paper is organized as follows. In Section 2, we present the series solution method to problem (1) and (2). In Section 3, we present numerical results to illustrate the efficiency of the presented technique. Comparison with other methods such as the Adomian decomposition method (ADM) and the homotopy perturbation method (HPM) will be also presented in Section 3. Finally, we conclude with some remarks in Section 4.

2. Series Method

In this section, we present the series solution method to solve problem (1) and (2) and we give the final result for the ODE version of (1) and (2). Given the order $\alpha = p/q$, we assume that the solution $u(x,t)$ takes the form

$$u(x,t) = \sum_{k=0}^\infty a_k(x) t^{k/q}, \quad (7)$$

where $u(x,0) = u_0(x)$ and $a_k(x)$ are functional coefficients to be determined. Clearly, $a_0(x) = u_0(x)$. Formal substitution of (7) into (1) gives

$$D_t^\alpha \left(\sum_{k=0}^\infty a_k(x) t^{k/q} \right) = \sum_{k=0}^\infty a_k''(x) t^{k/q} + h\left(x, t, \sum_{k=0}^\infty u_k(x) t^{k/q} \right). \quad (8)$$

Assuming we can interchange the summation and the fractional derivative operator and using property 2 above, we obtain

$$\sum_{k=1}^\infty a_k(x) s_k t^{(k-p)/q} = \sum_{k=0}^\infty a_k''(x) t^{k/q} + h\left(x, t, \sum_{k=0}^\infty a_k(x) t^{k/q} \right), \quad (9)$$

where $s_k = \Gamma(k/q + 1)/\Gamma(k/q - \alpha + 1)$. Note that if $p > 1$, we will have negative powers of t (for $k = 1, 2, \ldots, p-1$) in the sum on the left hand side of (9). To avoid this, we multiply (9) by $t^{(p-1)/q}$ to get

$$\sum_{k=0}^\infty a_{k+1}(x) s_{k+1} t^{k/q} = \sum_{k=0}^\infty a_k''(x) t^{(k+p-1)/q} + t^{(p-1)/q} h \times \left(x, t, \sum_{k=0}^\infty a_k(x) t^{k/q} \right). \quad (10)$$

One way of finding the coefficients $a_k(x)$, in line with finding the coefficients of a Taylor series, is to recursively apply the operator $D_t^{1/q}$ to (10) and substitute $t = 0$. However, this is not convenient for implementation. To avoid the use of the fractional differentiation, we introduce the change of variable $w = t^{1/q}$ which transforms (10) into

$$\sum_{k=0}^\infty a_{k+1}(x) s_{k+1} w^k = \sum_{k=0}^\infty a_k''(x) w^{k+p-1} + w^{p-1} h \times \left(x, w^q, \sum_{k=0}^\infty a_k(x) w^k \right). \quad (11)$$

Now, differentiating ordinarily k times with respect to w and substituting $w = 0$, we find the following recursion relation for $k \geq 0$:

$$a_{k+1}(x) = \frac{1}{s_{k+1}} a''_{k-p+1}(x) + \frac{1}{s_{k+1}k!}$$
$$\times \left[\frac{\partial^k}{\partial w^k} \left[w^{p-1} h \left(x, w^q, \sum_{m=0}^{\infty} a_m(x) w^m \right) \right] \right]_{w=0}, \tag{12}$$

where we assume that $a_l(x) \equiv 0$ for $l < 0$.

We note that if problem (1) and (2) is an ordinary differential equation of fractional order, that is, $u \equiv u(t)$ and $h \equiv h(t, u)$, the coefficients a_k are real numbers and the recursion relation (12) reduces to

$$a_{k+1} = \frac{1}{s_{k+1}k!} \left[\frac{\partial^k}{\partial w^k} \left[w^{p-1} h \left(w^q, \sum_{m=0}^{\infty} a_m w^m \right) \right] \right]_{w=0}. \tag{13}$$

Remark 1. We remark that the present method is different in many ways from the ADM. A main difference between the two methods is that the ADM, in its generation of successive terms, uses fractional integration while the present method uses ordinary differentiation. However, when $\alpha = 1$ ($p = q = 1$), formula (13) will reduce to

$$a_{k+1} = \frac{1}{(k+1)!} \left[\frac{\partial^k}{\partial w^k} \left[h \left(w, \sum_{m=0}^{\infty} a_m w^m \right) \right] \right]_{w=0}, \tag{14}$$

which is the well-known Adomian polynomial formula [32].

In Section 3, we present several examples to show the practicality of this approach and make a comparison with other techniques such as the Adomian decomposition method and homotopy perturbation method.

3. Numerical Results

Example 1. Consider the fractional initial value problem

$$D_t^{1/2} y = \Gamma \left(\frac{3}{2} \right) \left(y^2 - t + 1 \right), \quad y(0) = 0, \tag{15}$$

with $y(t) = t^{1/2}$ being the exact solution.

Applying the proposed algorithm, the solution takes the form $y = \sum_{k=0}^{\infty} a_k t^{k/2}$. The zeroth coefficient is readily given by $a_0 = y(0) = 0$. For $k \geq 0$, we have from (13)

$$a_{k+1} = \frac{\Gamma(3/2)}{s_{k+1}k!} \left[\frac{\partial^k}{\partial w^k} \left[\left(\sum_{m=0}^{\infty} a_m w^m \right)^2 - w^2 + 1 \right] \right]_{w=0} \tag{16}$$

$$= \frac{\Gamma(3/2)}{s_{k+1}k!} \left[k! \sum_{m=0}^{k} a_m a_{k-m} - 2\delta_{k-2} + \delta_k \right], \tag{17}$$

where $\delta_j = 1$ if $j = 0$ and 0 otherwise. With $k = 0$, (17) gives $a_1 = (\Gamma(3/2)/s_1)(a_0^2 + 1) = 1$, since $s_1 = \Gamma(3/2)$. It can be easily

verified that (17) gives $a_k = 0$ for $k \geq 2$. Hence the solution is

$$y(t) = \sum_{k=0}^{\infty} a_k t^{k/2} = a_1 t^{1/2} = t^{1/2}, \tag{18}$$

which is the exact solution.

First, we compare our results with the Adomian decomposition method (ADM). To apply the ADM, assume that the solution $y(t)$ of (15) and the nonlinear function $f(y) = y^2$ can be written in the series form as

$$y(t) = \sum_{n=0}^{\infty} y_n(t), \qquad f(y) = \sum_{n=0}^{\infty} A_n, \tag{19}$$

where A_n, $n = 0, 1, 2, \ldots$, are called the Adomian polynomials. These polynomials can be derived by expanding the function $f(y)$ about y_0 as follows:

$$f(y) = f(y_0) + f'(y_0) \frac{y - y_0}{1!} + f''(y_0) \frac{(y - y_0)^2}{2!} + \cdots \tag{20}$$

or

$$f(y) = f(y_0) + f'(y_0) \frac{\sum_{n=1}^{\infty} y_n}{1!} + f''(y_0) \frac{\left(\sum_{n=1}^{\infty} y_n \right)^2}{2!} + \cdots . \tag{21}$$

Thus, A_k can be derived as

$$A_k = \frac{1}{k!} \frac{d^k}{d\beta^k} \left[f \left(\sum_{j=0}^{\infty} \beta^j y_j \right) \right]_{\beta=0}, \quad j \geq 0. \tag{22}$$

Next, define the fractional differential operator L as $L = D_t^{1/2}$; then (15) can be written in the form

$$L(y) = \Gamma \left(\frac{3}{2} \right) \left(f(y) - t + 1 \right). \tag{23}$$

And defining the inverse operator as $L^{-1}(\cdot) = I_t^{1/2}$, then the solution $y(t)$ of (15) can be written in the form

$$y(t) = \Gamma \left(\frac{3}{2} \right) I_t^{1/2} (f(y)) + \Gamma \left(\frac{3}{2} \right) I_t^{1/2} ((-t + 1))$$
$$= \Gamma \left(\frac{3}{2} \right) I_t^{1/2} \left(\sum_{n=0}^{\infty} A_n \right) + \Gamma \left(\frac{3}{2} \right) I_t^{1/2} ((-t + 1)) \tag{24}$$
$$= \sum_{n=0}^{\infty} y_n(t).$$

Now balancing the last equality in (24) yields

$$y_0(t) = \Gamma \left(\frac{3}{2} \right) I_t^{1/2} (-t + 1),$$
$$y_{k+1}(t) = \Gamma \left(\frac{3}{2} \right) I_t^{1/2} (A_k), \quad k = 0, 1, \ldots . \tag{25}$$

The first few terms generated by ADM are given below:

$$y_0(t) = \frac{1}{3}(3 - 2t)t^{1/2},$$

$$y_1(t) = \frac{2}{3}t^{3/2} - \frac{32}{45}t^{5/2} + \frac{64}{315}t^{7/2},$$

$$y_2(t) = \frac{32}{45}t^{5/2} - \frac{1664}{1575}t^{7/2} + \frac{32768}{59535}t^{9/2} - \frac{65536}{654885}t^{11/2}$$

$$\vdots$$

(26)

Next, we will compare our results with the homotopy perturbation method (HPM). To apply the HPM, define the homotopy $H : [0, \infty] \times [0, 1] \to \mathbb{R}$ which satisfies

$$H(y, p) = (1 - p)D^{1/2}y$$

$$+ p\left(D^{1/2}y - \Gamma\left(\frac{3}{2}\right)y^2 - \Gamma\left(\frac{3}{2}\right)(1 - t)\right) = 0.$$

(27)

The basic assumption is that the solution of problem (15) can be expressed as a power series in p:

$$y = y_0 + py_1 + p^2 y_2 + \cdots,$$

(28)

where $y_i(0) = 0$ for $i \geq 0$. The approximate solution of problem (15) can be obtained as

$$y(t) = \lim_{p \to 1}\left(y_0 + py_1 + p^2 y_2 + \cdots\right).$$

(29)

The convergence of the last series has been proved in [33].

Substituting (28) into (27) and equating the coefficients of the terms with like powers of p, we have

$$p^0 : D^{1/2}y_0 = 0, \quad y_0(0) = 0,$$

$$p^1 : D^{1/2}y_1 - \Gamma\left(\frac{3}{2}\right)y_0^2 - \Gamma\left(\frac{3}{2}\right)(1 - t) = 0, \quad y_1(0) = 0,$$

$$p^2 : D^{1/2}y_2 - 2\Gamma\left(\frac{3}{2}\right)y_0 y_1 = 0, \quad y_2(0) = 0,$$

$$p^3 : D^{1/2}y_3 - \Gamma\left(\frac{3}{2}\right)\left(y_1^2 + 2y_0 y_2\right) = 0, \quad y_3(0) = 0,$$

$$\vdots$$

(30)

which implies that

$$y_0(t) = 0,$$

$$y_1(t) = \sqrt{t} - \frac{2}{3}t^{3/2},$$

$$y_2(t) = 0,$$

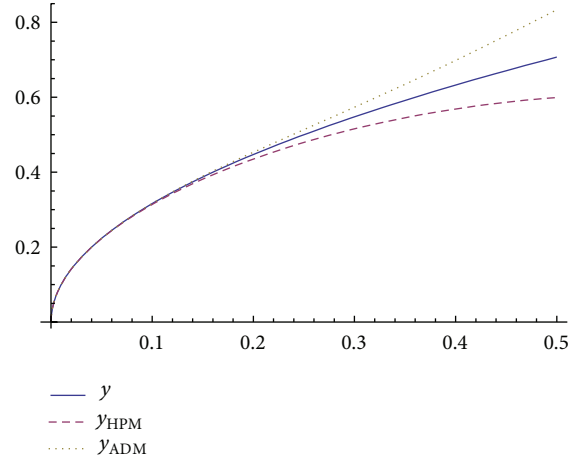

FIGURE 1: The exact and approximate solutions of Example 1 for $0 \leq t \leq 0.5$.

$$y_3(t) = \frac{2}{3}t^{3/2} - \frac{32}{45}t^{5/2} + \frac{64}{315}t^{7/2},$$

$$y_4(t) = 0,$$

$$y_5(t) = \frac{32}{45}t^{5/2} - \frac{1664}{1575}t^{7/2} + \frac{32768}{59535}t^{9/2} - \frac{65536}{654885}t^{11/2}.$$

(31)

The approximate solution is

$$y_{\text{HPM}}(t) = \sqrt{t} - \frac{64}{75}t^{7/2} + \frac{32768}{59535}t^{9/2} - \frac{65536}{654885}t^{11/2} + \cdots.$$

(32)

Figure 1 depicts the exact solution and the approximate solutions $y_{\text{ADM}}(t) = \sum_{k=0}^{4} y_k(t)$ and $y_{\text{HPM}}(t) = \sum_{k=0}^{4} y_k(t)$ obtained by the Adomian decomposition method and the homotopy perturbation method, respectively.

Example 2. Consider the fractional initial value problem

$$D_t^{2/3}y = \frac{1}{2\Gamma(4/3)}t^{1/3}\left(-3\sqrt{y} - 1\right), \quad y(0) = 1, \quad (33)$$

with $y(t) = (t - 1)^2$ being the exact solution.

Here $\alpha = 2/3$; hence $p = 2$ and $q = 3$. The solution assumes the form $y = \sum_{n=0}^{\infty} a_n t^{n/3}$ with $a_0 = y(0) = 1$. Then, according to the previous section, we have for $k \geq 0$

$$a_{k+1} = \frac{1}{2\Gamma(4/3)s_{k+1}k!}$$

$$\times \left[\frac{\partial^k}{\partial w^k}w^2\left(-3\left[\sum_{m=0}^{\infty}a_m w^m\right]^{1/2} - 1\right)\right]_{w=0}.$$

(34)

Numerical computation of (34) gives $a_3 = -2$, $a_6 = 1$, and all other coefficients are zero. Thus, our procedure produces the solution

$$y(t) = a_0 + a_3 t + a_6 t^2 = 1 - 2t + t^2 = (t - 1)^2, \quad (35)$$

which is the exact solution.

TABLE 1: Error for various t values and different values of n for Example 3.

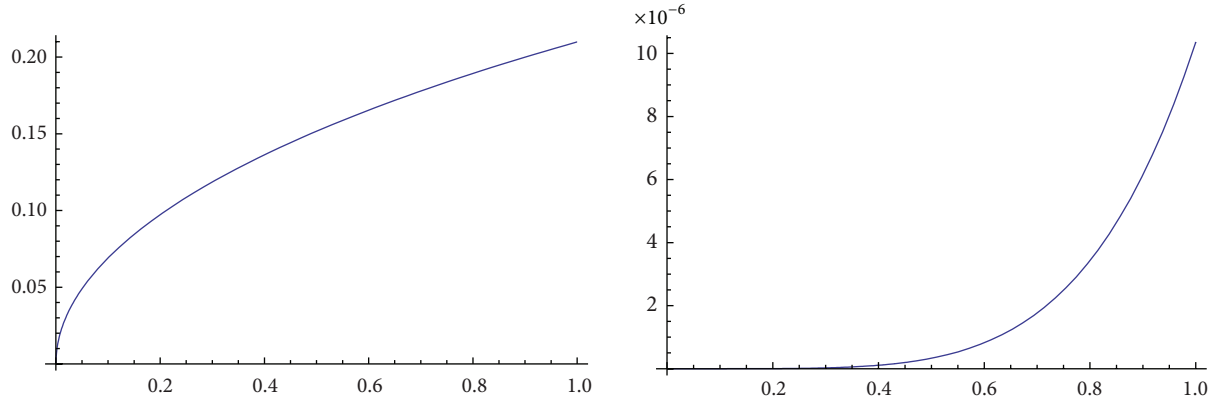| n	$t = 0.2$	$t = 0.4$	$t = 0.6$	$t = 0.7$	$t = 1.0$
5	0.001798	0.0136945	0.0443229	0.101588	0.193623
10	3.2139×10^{-6}	0.0000989578	0.000725096	0.00295596	0.00874915
15	2.70532×10^{-10}	6.70364×10^{-8}	1.66634×10^{-6}	0.000016172	0.0000938036
20	1.30212×10^{-13}	1.29792×10^{-10}	7.29593×10^{-9}	1.26453×10^{-7}	1.15074×10^{-6}

FIGURE 2: A plot of y_{10} and E_{10}, for Example 3, where $0 \le t \le 1$.

Example 3. Consider the fractional initial value problem

$$D_t^\alpha y = 1 - y^2, \quad y(0) = 0. \tag{36}$$

For $\alpha = p/q$, the solution assumes the form $y = \sum_{k=0}^\infty a_k t^{k/q}$ with $a_0 = y(0) = 0$. Then according to the previous section, we have for $k \ge 0$

$$a_{k+1} = \frac{1}{s_{k+1}k!}\left[\frac{\partial^k}{\partial w^k}w^{p-1}\left(1 - \left[\sum_{m=0}^\infty a_m w^m\right]^2\right)\right]_{w=0} \tag{37}$$

$$= \frac{1}{s_{k+1}k!}\left[\frac{\partial^k}{\partial w^k}w^{p-1}\left(1 - \sum_{m=0}^\infty C_m w^m\right)\right]_{w=0}, \tag{38}$$

where

$$C_m = \sum_{j=0}^m a_j a_{m-j}. \tag{39}$$

Simplification of (38) reveals the following recursion, where $k \ge 0$:

$$a_{k+1} = \begin{cases} 0, & \text{if } k < p-1 \\ \dfrac{\delta_{k-p+1}}{s_{k+1}} - \dfrac{1}{s_{k+1}}C_{k-p+1}, & \text{if } k \ge p-1. \end{cases} \tag{40}$$

Numerical computation of (38) gives $a_{2j} = 0$, $j \ge 0$. We consider $\alpha = 1/2$. Since the exact solution, in closed form, is not available, we define the error

$$E_n(t) = \left|D_t^{1/2}y_n - \left(1 - y_n^2\right)\right|, \tag{41}$$

where $y_n = \sum_{k=0}^n a_k t^{k/q}$. Figure 2 on the left presents the approximate solution $y_{10} = \sum_{k=0}^{10} a_k t^{k/2}$ and on the right the

error $E_{10} = |D_t^{1/2}y_{10} - (1 - y_{10}^2)|$. Table 1 depicts the absolute error E_n for different values of n at various values of t. From the results presented, it is clear that the series converges and sufficient accuracy is achieved with few terms. However, more terms would be needed for larger values of t which is expected for any initial value problem. We note that, for $\alpha = 1$, we get $s_k = k$ and $a_1 = 1$, $a_3 = -1/3$, $a_5 = 2/15$, $a_7 = -17/315$, $a_9 = 62/2835, \ldots$, and the obtained series solution is

$$y(t) = t - \frac{1}{3}t - \frac{1}{3}t^3 + \frac{2}{15}t^5 - \frac{17}{315}t^7 + \frac{62}{2835}t^9 + \cdots \tag{42}$$

which coincides with the Taylor series expansion of the exact solution $y(t) = (e^{2t} - 1)/(e^{2t} + 1)$.

Example 4. Consider the fractional initial value problem

$$D_t^\alpha u = u_{xx} + \frac{1}{10}u(1 - u), \quad u(x,0) = x. \tag{43}$$

For this example we take $\alpha = 1/2$; hence $p = 1$ and $q = 2$. The solution assumes the form $y = \sum_{k=0}^\infty a_k(x)t^{k/2}$ with $a_0(x) = u(x,0) = x$. Then from (12), we have, for $k \ge 0$,

$$a_{k+1}(x) = \frac{1}{s_{k+1}}a''_{k-p+1}(x) + \frac{1}{s_{k+1}k!}$$

$$\times \left[\frac{\partial^k}{\partial w^k}\left[w^{p-1}h\left(w^q, x, \sum_{m=0}^\infty a_m(x)w^m\right)\right]\right]_{w=0},$$

TABLE 2: Error for various x and t values and $n = 5$.

x	$t = 0.2$	$t = 0.4$	$t = 0.6$	$t = 0.8$
0	8.3726110×10^{-6}	0.0000474301	0.000130878	0.000269023
0.5	2.38184×10^{-9}	3.53911×10^{-8}	1.69571×10^{-7}	5.09316×10^{-7}
1	8.2742×10^{-6}	0.000046751	0.000128695	0.000263925

TABLE 3: Error for various x and t values and $n = 15$.

x	$t = 0.2$	$t = 0.4$	$t = 0.6$	$t = 0.8$
0	3.59581×10^{-11}	6.5209×10^{-9}	1.36682×10^{-7}	1.18428×10^{-6}
0.5	1.3542×10^{-12}	2.39819×10^{-10}	4.93086×10^{-9}	4.19941×10^{-8}
1	3.53848×10^{-11}	6.40568×10^{-9}	1.34033×10^{-7}	1.15927×10^{-6}

$$= \frac{1}{s_{k+1}} a_k''(x) + \frac{1}{10 s_{k+1} k!}$$

$$\times \left[\frac{\partial^k}{\partial w^k} \left[\sum_{m=0}^{\infty} a_m(x) w^m - \sum_{m=0}^{\infty} C_m(x) w^m \right] \right]_{w=0},$$

$$= \frac{1}{s_{k+1}} a_k''(x) + \frac{1}{10 s_{k+1}} \left(a_k(x) - C_k(x) \right),$$

$$(44)$$

where

$$C_k(x) = \sum_{j=0}^{k} a_j(x) a_{k-j}(x). \tag{45}$$

The first few terms of the series solution are

$$u(x,t) = x + \frac{\sqrt{t}}{\sqrt{\pi}} \left(\frac{262144}{4849845} x - \frac{262144}{4849845} x^2 \right)$$

$$+ \frac{t}{\pi} \left(-\frac{274877906944}{4704199304805} + \frac{68719476736}{23520996524025} x \right.$$

$$- \frac{68719476736}{7840332174675} x^2$$

$$\left. + \frac{137438953472}{23520996524025} x^3 \right)$$

$$+ \left(\frac{t}{\pi} \right)^{3/2} \left(-\frac{288230376151711744}{22814637477412005225} \right.$$

$$+ \frac{414331165718085632}{16296169626722860875} x$$

$$- \frac{36028797018963968}{38024395795686675375} x^2$$

$$+ \frac{36028797018963968}{22814637477412005225} x^3$$

$$\left. - \frac{18014398509481984}{22814637477412005225} x^4 \right) + \cdots. \tag{46}$$

Tables 2 and 3 present the error

$$E_n(x,t) = \left| D_t^{1/2} u_n - u_{nxx} - \frac{1}{10} u_n (1 - u_n) \right| \tag{47}$$

for various values of t and x and $n = 5, 15$, where $u_n(x,t) = \sum_{k=0}^{n} a_k(x) t^{k/2}$. The presented data indicate the accuracy of the series solutions obtained.

4. Concluding Remarks

We have presented a new algorithm for obtaining a series solution for a class of fractional differential equations. The algorithm is developed for a class of fractional partial differential equations of the Caputo type. We have applied the new algorithm to different examples. Accurate numerical solutions have been obtained as well as exact solutions for certain problems. The new algorithm is compared with the two well-known methods, the Adomian decomposition method (ADM) and the homotopy perturbation method (HPM), for one example. The exact solution is obtained after one step in the current method and after getting a telescoping sum by the HAM, where an approximate solution is obtained by the ADM. The idea of the new algorithm can be generalized to deal with various types of fractional functional equations.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

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