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Research Article

Linearization of Nonautonomous Impulsive System with Nonuniform Exponential Dichotomy

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This paper gives a version of Hartman-Grobman theorem for the impulsive differential equations. We assume that the linear impulsive system has a nonuniform exponential dichotomy. Under some suitable conditions, we proved that the nonlinear impulsive system is topologically conjugated to its linear system. Indeed, we do construct the topologically equivalent function (the transformation). Moreover, the method to prove the topological conjugacy is quite different from those in previous works (e.g., see Barreira and Valls, 2006).

1. Introduction

A basic contribution to the linearization problem for autonomous differential equations is the famous Hartman-Grobman theorem (see [1, 2]). Then Palmer successfully generalized the standard Hartman-Grobman theorem to nonautonomous differential equations (see [3]). Then Fenner and Pinto [4] generalized Hartman-Grobman theorem to impulsive differential equations. However, they did not discuss the Hölder regularity of the topologically equivalent function $H(t, x)$. Then Xia et al. [5] gave a rigorous proof of the Hölder regularity. Xia et al. [6, 7] gave a version of the generalized Hartman-Grobman theorem for dynamic systems on measure chains. It should be noted that the above mentioned works are based on the linear differential equations with uniform exponential dichotomy. Recently, Barreira and Valls have introduced the notion of nonuniform exponential dichotomies and have developed the corresponding theory in a systematic way [8–11]. So, a version of the Hartman-Grobman theorem is also given for differential equations with nonuniform hyperbolicity (see [12]). However, they did not discuss the impulsive systems with nonuniform hyperbolicity. For this reason, in this paper, we considered the linearization of impulsive differential equations with nonuniform hyperbolicity. Moreover, our method to prove the topological conjugacy used in this paper is completely different from that in [12]. We divided the proof into several lemmas and constructed a concrete topologically equivalent function.

2. Definitions

Consider the linear nonautonomous system with impulses at times $\{t_k\}_{k\in\mathbb{Z}}$ as follows:

$$\begin{aligned} \dot{x}(t) &= A(t)\,x, \quad t \neq t_k, \\ \Delta x(t_k) &= \widetilde{A}(t_k)\,x(t_k), \quad k \in \mathbb{Z}, \end{aligned} \tag{1}$$

where $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, $x(t_k^-) = x(t_k)$, represents the jump of the solution $x(t)$ at $t = t_k$.

A perturbed nonautonomous system with impulsive is therefore described by

$$\begin{aligned} \dot{x}(t) &= A(t)\,x + f(t, x), \quad t \neq t_k, \\ \Delta x(t_k) &= \widetilde{A}(t_k)\,x(t_k) + \widetilde{f}(t_k, x(t_k)), \quad k \in \mathbb{Z}, \end{aligned} \tag{2}$$

where, in systems (1) and (2), $x \in \mathbb{R}^n$, $A(t)$ and $\widetilde{A}(t)$ are $n \times n$ matrixes.

Definition 1 (see [11, 12]). The impulsive system (1) is said to be a nonuniform exponential dichotomy in $\mathbb{R}$, if there exist a projection $P(t)$ and positive constants α, k, and $\varepsilon \geq 0$, such that

$$\begin{aligned} \|T(t,s)P(s)\| &\leq k\exp\{-\alpha(t-s)+\varepsilon|s|\}, \quad t \geq s, \\ \|T(t,s)Q(s)\| &\leq k\exp\{-\alpha(s-t)+\varepsilon|s|\}, \quad t \leq s, \end{aligned} \tag{3}$$

where $Q(t) = \mathrm{Id} - P(t)$ is the complementary projection and $T(t,s)$ is the evolution operator of the impulsive system (1), which satisfies $T(t,s)P(s) = P(t)T(t,s)$, $t, s \in \mathbb{R}$.

Definition 2 (see [5, 7]). Suppose that there exists a function $H : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ such that

(i) for each fixed t, $H(t,\cdot)$ is a homeomorphism of $\mathbb{R}^n$ into $\mathbb{R}^n$;

(ii) $\|H(t,x) - x\|$ uniformly bounded with respect to t;

(iii) $G(t,\cdot) = H^{-1}(t,\cdot)$ has property (ii) also;

(iv) if $x(t)$ is a solution of system (2), then $H(t,x(t))$ is a solution of system (1).

If such a map H_t exists, then system (2) is topologically conjugated to (1). H is an equivalent function.

3. Main Results and Proof

Theorem 3. *Suppose that the linear impulsive system* (1) *has a nonuniform exponential dichotomy (i.e., system* (1) *has an evolution operator* $T(t,s)$ *satisfying* (3)*) and, for any* $x, x_1, x_2 \in \mathbb{R}^n$ *and* $t \in \mathbb{R}$*, one assumes that*

(H_1) $\|f(t,x)\| \leq \mu\exp(-\varepsilon|t|)$,

(H_2) $\|\widetilde{f}(t,x)\| \leq \mu\exp(-\varepsilon|t|)$,

(H_3) $\|f(t,x_1) - f(t,x_2)\| \leq r\exp(-\varepsilon|t|)\|x_1 - x_2\|$,

(H_4) $\|\widetilde{f}(t,x_1) - \widetilde{f}(t,x_2)\| \leq r\exp(-\varepsilon|t|)\|x_1 - x_2\|$,

(H_5) $2kr\alpha^{-1} + 2krN[1 + (1/(1-\exp(-\alpha)))] < 1$,

where $\mu, r \geq 0$, k, α, *and* ε *are the same constants in* (3)*, and* N *is a positive integer such that the intervals* $[n, n+1)$ *contain no more than* N *terms of the sequences* $\{t_k\}_{k\in\mathbb{Z}}$*, for all* $n \in \mathbb{Z}$*. Then system* (2) *is topologically conjugated to system* (1).

We divide the proof of Theorem 3 into several lemmas.

In what follows, we always suppose that the conditions of Theorem 3 are satisfied. Denote that $X(t,t_0,x_0)$ is a solution of the system (2) satisfying the initial condition $X(t_0) = x_0$, and that $Y(t,t_0,y_0)$ is a solution of the system (1) satisfying the initial condition $Y(t_0) = y_0$.

Lemma 4. *If system* (1) *has a nonuniform exponential dichotomy, then* $x(t) = 0$ *is the unique bounded solution of system* (1).

Proof. Let $T(t,s)$ be the evolution operator satisfying $x(t) = T(t,s)x(s)$ for every $t, s \in \mathbb{R}$. Then there exists $\alpha, k > 0$, $\varepsilon \geq 0$, and a projection $P(t)$ satisfying (3). We suppose that $x(t)$ is any bounded solution of the system (1), and it satisfies the initial condition $(s, x(s))$. Therefore, $x(t)$ can be written as $x(t) = T(t,s)P(s)x(s) + T(t,s)[\mathrm{Id} - P(s)]x(s)$. Now we prove $P(s)x(s) = 0$ and $[\mathrm{Id} - P(s)]x(s) = 0$.

If $P(s)x(s) \neq 0$, considering $t \leq 0$,

$$\begin{aligned} \|x(t)\| &= \|T(t,s)P(s)x(s) + T(t,s)[\mathrm{Id} - P(s)]x(s)\| \\ &\geq |\|T(t,s)P(s)x(s)\| - \|T(t,s)[\mathrm{Id} - P(s)]x(s)\|| . \end{aligned} \tag{4}$$

It follows from the first expression of (3) that

$$\begin{aligned} \|P(s)x(s)\| &= \left\|P^2(s)x(s)\right\| \\ &= \left\|P(s)T^{-1}(t,s)T(t,s)P(s)x(s)\right\| \\ &\leq \left\|P(s)T^{-1}(t,s)\right\| \|T(t,s)P(s)x(s)\| \\ &= \|P(s)T(s,t)\| \|T(t,s)P(s)x(s)\| \\ &= \|T(s,t)P(t)\| \|T(t,s)P(s)x(s)\| \\ &\leq k\exp\{-\alpha(s-t)+\varepsilon|t|\} \|T(t,s)P(s)x(s)\| . \end{aligned} \tag{5}$$

Namely,

$$\|T(t,s)P(s)x(s)\| \geq k^{-1}\exp\{\alpha(s-t)-\varepsilon|t|\} \|P(s)x(s)\| . \tag{6}$$

On the other hand, it follows from the second expression of (3) that

$$\begin{aligned} \|T(t,s)[\mathrm{Id} - P(s)]x(s)\| &= \|T(t,s)[\mathrm{Id} - P(s)] \\ &\quad \times [\mathrm{Id} - P(s)]x(s)\| \\ &\leq \|T(t,s)[\mathrm{Id} - P(s)]\| \\ &\quad \times \|[\mathrm{Id} - P(s)]x(s)\| \\ &\leq k\exp\{-\alpha(s-t)+\varepsilon|s|\} \\ &\quad \times \|[\mathrm{Id} - P(s)]x(s)\| . \end{aligned} \tag{7}$$

From the above analysis, which implies that

$$\begin{aligned} \|x(t)\| &\geq k^{-1}\exp\{\alpha(s-t)+\varepsilon|t|\} \|P(s)x(s)\| \\ &\quad - k\exp\{-\alpha(s-t)+\varepsilon|s|\} \|[\mathrm{Id} - P(s)]x(s)\| . \end{aligned} \tag{8}$$

Then we obtain $\|x(t)\| \to +\infty$ as $t \to -\infty$. Similarly, if $[\mathrm{Id} - P(s)]x(s) \neq 0$, we obtain $\|x(t)\| \to +\infty$ as $t \to +\infty$. Consequently, $P(s)x(s) = 0$ and $[\mathrm{Id} - P(s)]x(s) = 0$. Hence, $x(t) = 0$. □

Lemma 5. *For each* (τ, ξ)*, system*

$$\begin{aligned} z' &= A(t)z - f(t, X(t,\tau,\xi)), \quad t \neq t_k, \\ \Delta z(t_k) &= \widetilde{A}(t_k)z(t_k) - f(t_k, X(t_k,\tau,\xi)), \quad k \in \mathbb{Z}, \end{aligned} \tag{9}$$

has a unique bounded solution $h(t,(\tau,\xi))$ *with* $|h(t,(\tau,\xi))| \leq 2k\mu\alpha^{-1} + 2k\mu N[1 + (1/(1-\exp(-\alpha)))]$.

Proof. For each (τ,ξ), let

$$\begin{aligned} z_0(t) = &-\int_{-\infty}^{t} T(t,s)P(s)f(s,X(s,\tau,\xi))\,ds \\ &+\int_{t}^{+\infty} T(t,s)Q(s)f(s,X(s,\tau,\xi))\,ds \\ &-\sum_{t_k\in(-\infty,t)} T\left(t,t_k^+\right)P\left(t_k^+\right)\widetilde{f}\left(t_k,X\left(t_k,\tau,\xi\right)\right) \\ &+\sum_{t_k\in[t,+\infty)} T\left(t,t_k^+\right)Q\left(t_k^+\right)\widetilde{f}\left(t_k,X\left(t_k,\tau,\xi\right)\right). \end{aligned} \tag{10}$$

Differentiating it, then $z_0(t)$ is a solution of system (9) It follows from (3), (H_1), and (H_2) that we can easily deduce

$$\begin{aligned} \left|z_0(t)\right| \le &\int_{-\infty}^{t} k\exp\{-\alpha(t-s)+\varepsilon|s|\} \\ &\times\mu\exp(-\varepsilon|s|)\,ds \\ &+\int_{t}^{+\infty} k\exp\{-\alpha(s-t)+\varepsilon|s|\} \\ &\times\mu\exp(-\varepsilon|s|)\,ds \\ &+\sum_{t_k\in(-\infty,t)} k\exp\{-\alpha\left(t-t_k\right)+\varepsilon\left|t_k\right|\} \\ &\times\mu\exp\left(-\varepsilon\left|t_k\right|\right) \\ &+\sum_{t_k\in[t,+\infty)} k\exp\{-\alpha\left(t_k-t\right)+\varepsilon\left|t_k\right|\} \\ &\times\mu\exp\left(-\varepsilon\left|t_k\right|\right) \\ \le\ & 2k\mu\alpha^{-1}+2k\mu N\left[1+\frac{1}{1-\exp(-\alpha)}\right]. \end{aligned} \tag{11}$$

It is easy to show that $z_0(t)$ is a bounded solution of (9). On the other hand, for each (τ,ξ), the linear part of system (9) has a nonuniform exponential dichotomy, by Lemma 4, then system (9) has a unique bounded solution $z_0(t)$, we denote $h(t,(\tau,\xi))$ and $|h(t,(\tau,\xi))| \le 2k\mu\alpha^{-1}+2k\mu N[1+(1/(1-\exp(-\alpha)))]$. □

Lemma 6. *For each (τ,ξ), the system*

$$\begin{aligned} Z' &= A(t)Z + f(t,Y(t,\tau,\xi)+Z), \quad t\ne t_k, \\ \Delta Z\left(t_k\right) &= \widetilde{A}\left(t_k\right)Z\left(t_k\right)+\widetilde{f}\left(t_k,Y\left(t_k,\tau,\xi\right)+Z\left(t_k\right)\right), \quad k\in\mathbb{Z}, \end{aligned} \tag{12}$$

has a unique bounded solution $g(t,(\tau,\xi))$ and $|g(t,(\tau,\xi))| \le 2k\mu\alpha^{-1}+2k\mu N[1+(1/(1-\exp(-\alpha)))]$.

Proof. Let $\mathbb{B}$ be the set of all the continuous bounded functions $z(t)$ with $|z(t)| \le 2k\mu\alpha^{-1}+2k\mu N[1+(1/(1-\exp(-\alpha)))]$. For each (τ,ξ) and any $z(t)\in\mathbb{B}$, define the mapping T as follows:

$$\begin{aligned} Tz(t) = &\int_{-\infty}^{t} T(t,s)P(s)f(s,Y(s,\tau,\xi)+z(s))\,ds \\ &-\int_{t}^{+\infty} T(t,s)Q(s)f(s,Y(s,\tau,\xi)+z(s))\,ds \\ &+\sum_{t_k\in(-\infty,t)} T\left(t,t_k^+\right)P\left(t_k^+\right)\widetilde{f}\left(t_k,Y\left(t_k,\tau,\xi\right)+z\left(t_k\right)\right) \\ &-\sum_{t_k\in[t,+\infty)} T\left(t,t_k^+\right)Q\left(t_k^+\right)\widetilde{f}\left(t_k,Y\left(t_k,\tau,\xi\right)+z\left(t_k\right)\right). \end{aligned} \tag{13}$$

It follows from (3), (H_1), and (H_2) that

$$\begin{aligned} |Tz(t)| \le &\int_{-\infty}^{t} k\exp\{-\alpha(t-s)+\varepsilon|s|\} \\ &\times\mu\exp(-\varepsilon|s|)\,ds \\ &+\int_{t}^{+\infty} k\exp\{-\alpha(s-t)+\varepsilon|s|\} \\ &\times\mu\exp(-\varepsilon|s|)\,ds \\ &+\sum_{t_k\in(-\infty,t)} k\exp\{-\alpha\left(t-t_k\right)+\varepsilon\left|t_k\right|\} \\ &\times\mu\exp\left(-\varepsilon\left|t_k\right|\right) \\ &+\sum_{t_k\in[t,+\infty)} k\exp\{-\alpha\left(t_k-t\right)+\varepsilon\left|t_k\right|\} \\ &\times\mu\exp\left(-\varepsilon\left|t_k\right|\right) \\ \le\ & 2k\mu\alpha^{-1}+2k\mu N\left[1+\frac{1}{1-\exp(-\alpha)}\right] \\ \triangleq\ & B, \end{aligned} \tag{14}$$

which implies that T is a self-map of a sphere with radius B. For any $z_1(t), z_2(t)\in\mathbb{B}$, and it follows from (3), (H_4), and (H_5), then we have

$$\begin{aligned} \left|Tz_1(t)-Tz_2(t)\right| \le &\int_{-\infty}^{t} k\exp\{-\alpha(t-s)+\varepsilon|s|\} \\ &\times r\exp(-\varepsilon|s|)\left|z_1(s)-z_2(s)\right|ds \\ &+\int_{t}^{+\infty} k\exp\{-\alpha(s-t)+\varepsilon|s|\} \\ &\times r\exp(-\varepsilon|s|)\left|z_1(s)-z_2(s)\right|ds \\ &+\sum_{t_k\in(-\infty,t)} k\exp\{-\alpha\left(t-t_k\right)+\varepsilon\left|t_k\right|\} \\ &\times r\exp\left(-\varepsilon\left|t_k\right|\right)\left|z_1(s)-z_2(s)\right| \end{aligned}$$

$$
\begin{aligned}
&+ \sum_{t_k \in [t,+\infty)} k \exp\left\{-\alpha\left(t_k - t\right) + \varepsilon\left|t_k\right|\right\} \\
&\qquad \times r \exp\left(-\varepsilon\left|t_k\right|\right)\left|z_1(s) - z_2(s)\right| \\
&\le \left(2kr\alpha^{-1} + 2krN\left[1 + \frac{1}{1 - \exp(-\alpha)}\right]\right) \\
&\quad \times \left\|z_1 - z_2\right\|.
\end{aligned} \tag{15}
$$

And together with (H_5), T has a unique fixed point, namely, $z_0(t)$, and

$$
\begin{aligned}
z_0(t) &= \int_{-\infty}^{t} T(t,s) P(s) f\left(s, Y(s,\tau,\xi) + z_0(s)\right) ds \\
&\quad - \int_{t}^{+\infty} T(t,s) Q(s) f\left(s, Y(s,\tau,\xi) + z_0(s)\right) ds \\
&\quad + \sum_{t_k \in (-\infty,t)} T\left(t, t_k^+\right) P\left(t_k^+\right) \widetilde{f}\left(t_k, Y\left(t_k,\tau,\xi\right) + z_0\left(t_k\right)\right) \\
&\quad - \sum_{t_k \in [t,+\infty)} T\left(t, t_k^+\right) Q\left(t_k^+\right) \widetilde{f}\left(t_k, Y\left(t_k,\tau,\xi\right) + z_0\left(t_k\right)\right).
\end{aligned} \tag{16}
$$

It is easy to show that $z_0(t)$ is a bounded solution of (12). Now we are going to show that the bounded solution is unique. For this purpose, we assume that there is another bounded solution $z_1(t)$ of (12). Thus $z_1(t)$ can be written as follows:

$$
\begin{aligned}
z_1(t) &= T(t,0) x_0 + \int_0^t T(t,s) f\left(s, Y(s,\tau,\xi) + z_1(s)\right) ds \\
&\quad + \sum_{t_k \in [0,t)} T\left(t, t_k^+\right) \widetilde{f}\left(t_k, Y\left(t_k,\tau,\xi\right) + z_1\left(t_k\right)\right) \\
&= T(t,0) x_0 + \int_0^t T(t,s) \left[P(s) + (\mathrm{Id} - P(s))\right] \\
&\qquad \times f\left(s, Y(s,\tau,\xi) + z_1(s)\right) ds \\
&\quad + \sum_{t_k \in [0,t)} T\left(t, t_k^+\right) \left[P\left(t_k^+\right) + \left(\mathrm{Id} - P\left(t_k^+\right)\right)\right] \\
&\qquad \times \widetilde{f}\left(t_k, Y\left(t_k,\tau,\xi\right) + z_1\left(t_k\right)\right) \\
&= T(t,0) x_0 + \int_{-\infty}^t T(t,s) P(s) \\
&\qquad \times f\left(s, Y(s,\tau,\xi) + z_1(s)\right) ds \\
&\quad - \int_{-\infty}^0 T(t,s) P(s) f\left(s, Y(s,\tau,\xi) + z_1(s)\right) ds \\
&\quad + \int_0^{+\infty} T(t,s) Q(s) f\left(s, Y(s,\tau,\xi) + z_1(s)\right) ds \\
&\quad - \int_t^{+\infty} T(t,s) Q(s) f\left(s, Y(s,\tau,\xi) + z_1(s)\right) ds \\
&\quad + \sum_{t_k \in (-\infty,t)} T\left(t, t_k^+\right) P\left(t_k^+\right) \widetilde{f}\left(t_k, Y\left(t_k,\tau,\xi\right) + z_1\left(t_k\right)\right) \\
&\quad - \sum_{t_k \in (-\infty,0)} T\left(t, t_k^+\right) P\left(t_k^+\right) \widetilde{f}\left(t_k, Y\left(t_k,\tau,\xi\right) + z_1\left(t_k\right)\right) \\
&\quad + \sum_{t_k \in [0,+\infty)} T\left(t, t_k^+\right) Q\left(t_k^+\right) \widetilde{f}\left(t_k, Y\left(t_k,\tau,\xi\right) + z_1\left(t_k\right)\right) \\
&\quad - \sum_{t_k \in [t,+\infty)} T\left(t, t_k^+\right) Q\left(t_k^+\right) \widetilde{f}\left(t_k, Y\left(t_k,\tau,\xi\right) + z_1\left(t_k\right)\right).
\end{aligned} \tag{17}
$$

Note that

$$
\begin{aligned}
&\int_{-\infty}^0 T(t,s) P(s) f\left(s, Y(s,\tau,\xi) + z_1(s)\right) ds \\
&\quad = T(t,0) \int_{-\infty}^0 T(0,s) P(s) f\left(s, Y(s,\tau,\xi) + z_1(s)\right) ds \\
&\quad \triangleq T(t,0) x_1.
\end{aligned} \tag{18}
$$

And together with (3) and (H_1), we have

$$
\begin{aligned}
\left|x_1\right| &= \left|\int_{-\infty}^0 T(0,s) P(s) f\left(s, Y(s,\tau,\xi) + z_1(s)\right) ds\right| \\
&\le \left|\int_{-\infty}^0 \left\|T(0,s) P(s)\right\| \left\|f\left(s, Y(s,\tau,\xi) + z_1(s)\right)\right\| ds\right| \\
&\le \int_{-\infty}^0 k \exp\left\{-\alpha(0 - s) + \varepsilon|s|\right\} \mu \exp\left(-\varepsilon|s|\right) ds \\
&= k\mu\alpha^{-1}.
\end{aligned} \tag{19}
$$

Similarly,

$$
\begin{aligned}
&\int_0^{+\infty} T(t,s) Q(s) f\left(s, Y(s,\tau,\xi) + z_1(s)\right) ds \triangleq T(t,0) x_2, \\
&\left|x_2\right| \le k\mu\alpha^{-1}.
\end{aligned} \tag{20}
$$

On the other hand,

$$
\begin{aligned}
&\sum_{t_k \in (-\infty,0)} T\left(t, t_k^+\right) P\left(t_k^+\right) \widetilde{f}\left(t_k, Y\left(t_k,\tau,\xi\right) + z_1\left(t_k\right)\right) \\
&\quad = T(t,0) \sum_{t_k \in (-\infty,0)} T\left(0, t_k^+\right) P\left(t_k^+\right) \\
&\qquad \times \widetilde{f}\left(t_k, Y\left(t_k,\tau,\xi\right) + z_1\left(t_k\right)\right) \\
&\quad \triangleq T(t,0) x_3.
\end{aligned} \tag{21}
$$

And together with (3) and (H_2), we have

$$\begin{aligned}
|x_3| &= \Bigg| \sum_{t_k\in(-\infty,0)} T\left(0,t_k^+\right) P\left(t_k^+\right) \\
&\qquad \times \widetilde{f}\left(t_k, Y\left(t_k,\tau,\xi\right)+z_1\left(t_k\right)\right)\Bigg| \\
&\le \Bigg| \sum_{t_k\in(-\infty,0)} \left\|T\left(0,t_k^+\right) P\left(t_k^+\right)\right\| \\
&\qquad \times \left\|\widetilde{f}\left(t_k, Y\left(t_k,\tau,\xi\right)+z_1\left(t_k\right)\right)\right\|\Bigg| \\
&\le \sum_{t_k\in(-\infty,0)} k\exp\left\{-\alpha\left(0-t_k\right)+\varepsilon\left|t_k\right|\right\} \\
&\qquad \times \mu\exp\left(-\varepsilon\left|t_k\right|\right) \\
&= k\mu \sum_{t_k\in(-\infty,0)} \exp\left(\alpha t_k\right) \\
&\le k\mu N\left[1+\frac{1}{1-\exp\left(-\alpha\right)}\right].
\end{aligned} \tag{22}$$

Similarly,

$$\begin{aligned}
&\sum_{t_k\in[0,+\infty)} T\left(t,t_k^+\right) Q\left(t_k^+\right) \widetilde{f}\left(t_k, Y\left(t_k,\tau,\xi\right)+z_1\left(t_k\right)\right) \\
&\quad \triangleq T\left(t,0\right) x_4, \\
&\quad \left|x_4\right| \le k\mu N\left[1+\frac{1}{1-\exp\left(-\alpha\right)}\right].
\end{aligned} \tag{23}$$

Therefore, it follows from the expression of $z_1(t)$ and it can be written as follows:

$$\begin{aligned}
z_1\left(t\right) &= T\left(t,0\right)\left(x_0-x_1+x_2-x_3+x_4\right) \\
&\quad + \int_{-\infty}^{t} T\left(t,s\right) P\left(s\right) f\left(s, Y\left(s,\tau,\xi\right)+z_1\left(s\right)\right) ds \\
&\quad - \int_{t}^{+\infty} T\left(t,s\right) Q\left(s\right) f\left(s, Y\left(s,\tau,\xi\right)+z_1\left(s\right)\right) ds \\
&\quad + \sum_{t_k\in(-\infty,t)} T\left(t,t_k^+\right) P\left(t_k^+\right) \widetilde{f}\left(t_k, Y\left(t_k,\tau,\xi\right)+z_1\left(t_k\right)\right) \\
&\quad - \sum_{t_k\in[t,+\infty)} T\left(t,t_k^+\right) Q\left(t_k^+\right) \widetilde{f}\left(t_k, Y\left(t_k,\tau,\xi\right)+z_1\left(t_k\right)\right).
\end{aligned} \tag{24}$$

Noticing that $z_1(t)$ is bounded, hence, $T(t,0)(x_0-x_1+x_2-x_3+x_4)$ is bounded. And it is a solution of system (1). By Lemma 4, we can obtain that $T(t,0)(x_0-x_1+x_2-x_3+x_4)=0$. It follows that

$$\begin{aligned}
z_1\left(t\right) &= \int_{-\infty}^{t} T\left(t,s\right) P\left(s\right) f\left(s, Y\left(s,\tau,\xi\right)+z_1\left(s\right)\right) ds \\
&\quad - \int_{t}^{+\infty} T\left(t,s\right) Q\left(s\right) f\left(s, Y\left(s,\tau,\xi\right)+z_1\left(s\right)\right) ds \\
&\quad + \sum_{t_k\in(-\infty,t)} T\left(t,t_k^+\right) P\left(t_k^+\right) \widetilde{f}\left(t_k, Y\left(t_k,\tau,\xi\right)+z_1\left(t_k\right)\right) \\
&\quad - \sum_{t_k\in[t,+\infty)} T\left(t,t_k^+\right) Q\left(t_k^+\right) \widetilde{f}\left(t_k, Y\left(t_k,\tau,\xi\right)+z_1\left(t_k\right)\right).
\end{aligned} \tag{25}$$

Simple calculation shows

$$\begin{aligned}
\left|z_1\left(t\right)-z_0\left(t\right)\right| &\le \int_{-\infty}^{t} k\exp\left\{-\alpha\left(t-s\right)+\varepsilon\left|s\right|\right\} \\
&\qquad \times r\exp\left(-\varepsilon\left|s\right|\right)\left|z_1\left(s\right)-z_0\left(s\right)\right| ds \\
&\quad + \int_{t}^{+\infty} k\exp\left\{-\alpha\left(s-t\right)+\varepsilon\left|s\right|\right\} \\
&\qquad \times r\exp\left(-\varepsilon\left|s\right|\right)\left|z_1\left(s\right)-z_0\left(s\right)\right| ds \\
&\quad + \sum_{t_k\in(-\infty,t)} k\exp\left\{-\alpha\left(t-t_k\right)+\varepsilon\left|t_k\right|\right\} \\
&\qquad \times r\exp\left(-\varepsilon\left|t_k\right|\right)\left|z_1\left(t_k\right)-z_0\left(t_k\right)\right| \\
&\quad + \sum_{t_k\in[t,+\infty)} k\exp\left\{-\alpha\left(t_k-t\right)+\varepsilon\left|t_k\right|\right\} \\
&\qquad \times r\exp\left(-\varepsilon\left|t_k\right|\right)\left|z_1\left(t_k\right)-z_0\left(t_k\right)\right| \\
&\le \left(2kr\alpha^{-1}+2krN\left[1+\frac{1}{1-\exp\left(-\alpha\right)}\right]\right) \\
&\quad \times \left\|z_1-z_0\right\|.
\end{aligned} \tag{26}$$

It follows from (H_5) that we can obtain $z_1(t) \equiv z_0(t)$. This implies that the bounded solution of (12) is unique. We denote it as $g(t(\tau,\xi))$. From the above proof, it is easy to see that $|g(t,(\tau,\xi))| \le 2k\mu\alpha^{-1}+2k\mu N[1+(1/(1-\exp(-\alpha)))]$. □

Lemma 7. *Let $x(t)$ be any solution of the system (2); then $z(t)=0$ is the unique bounded solution of system*

$$\begin{aligned}
z' &= A\left(t\right) Z + f\left(t, x\left(t\right)+Z\right) - f\left(t, x\left(t\right)\right), \quad t\ne t_k, \\
\Delta Z\left(t_k\right) &= \widetilde{A}\left(t_k\right) Z\left(t_k\right) + \widetilde{f}\left(t_k, x\left(t_k\right)+Z\left(t_k\right)\right) \\
&\quad - \widetilde{f}\left(t_k, x\left(t_k\right)\right), \quad k\in\mathbb{Z}.
\end{aligned} \tag{27}$$

Proof. Obviously, $z\equiv 0$ is a bounded solution of system (27). We show that the bounded solution is unique; if not, there

is another bounded solution $z_1(t)$, which can be written as follows:

$$\begin{aligned} z_1(t) &= T(t,0)\, z_1(0) \\ &\quad + \int_0^t T(t,s)\left[f\left(s, x(s) + z_1(s)\right) - f(s, x(s))\right] ds \\ &\quad + \sum_{t_k \in [0,t)} T\left(t, t_k^+\right)\left[\widetilde{f}\left(t_k, x\left(t_k\right) + z_1\left(t_k\right)\right)\right. \\ &\qquad \left. - \widetilde{f}\left(t_k, x\left(t_k\right)\right)\right]. \end{aligned} \tag{28}$$

By Lemma 6, we can get

$$\begin{aligned} z_1(t) &= \int_{-\infty}^t T(t,s)\, P(s)\left[f\left(s, x(s) + z_1(s)\right)\right. \\ &\qquad \left. - f(s, x(s))\right] ds \\ &\quad - \int_t^{+\infty} T(t,s)\, Q(s)\left[f\left(s, x(s) + z_1(s)\right)\right. \\ &\qquad \left. - f(s, x(s))\right] ds \\ &\quad + \sum_{t_k \in (-\infty, t)} T\left(t, t_k^+\right) P\left(t_k^+\right)\left[\widetilde{f}\left(t_k, x\left(t_k\right) + z_1\left(t_k\right)\right)\right. \\ &\qquad \left. - \widetilde{f}\left(t_k, x\left(t_k\right)\right)\right] \\ &\quad - \sum_{t_k \in [t, +\infty)} T\left(t, t_k^+\right) Q\left(t_k^+\right)\left[\widetilde{f}\left(t_k, x\left(t_k\right) + z_1\left(t_k\right)\right)\right. \\ &\qquad \left. - \widetilde{f}\left(t_k, x\left(t_k\right)\right)\right]. \end{aligned} \tag{29}$$

Then it follows from (3), (H_3), and (H_4) that

$$\begin{aligned} \left|z_1(t)\right| &\le \int_{-\infty}^t k \exp\left\{-\alpha(t-s) + \varepsilon|s|\right\} \\ &\qquad \times r \exp\left(-\varepsilon|s|\right)\left|z_1(s)\right| ds \\ &\quad + \int_t^{+\infty} k \exp\left\{-\alpha(s-t) + \varepsilon|s|\right\} \\ &\qquad \times r \exp\left(-\varepsilon|s|\right)\left|z_1(s)\right| ds \\ &\quad + \sum_{t_k \in (-\infty, t)} k \exp\left\{-\alpha\left(t - t_k\right) + \varepsilon\left|t_k\right|\right\} \\ &\qquad \times r \exp\left(-\varepsilon\left|t_k\right|\right)\left|z_1\left(t_k\right)\right| \\ &\quad + \sum_{t_k \in [t, +\infty)} k \exp\left\{-\alpha\left(t_k - t\right) + \varepsilon\left|t_k\right|\right\} \\ &\qquad \times r \exp\left(-\varepsilon\left|t_k\right|\right)\left|z_1\left(t_k\right)\right| \\ &\le \left[2kr\alpha^{-1} + 2krN\left[1 + \frac{1}{1 - \exp(-\alpha)}\right]\right]\left\|z_1\right\|, \end{aligned} \tag{30}$$

And, together with (H_5), consequently, $z_1(t) \equiv 0$. This completes the proof of Lemma 7. □

Now we define two functions as follows:

$$H(t,x) = x + h(t,(t,x)), \tag{31}$$

$$G(t,x) = y + g(t,(t,y)). \tag{32}$$

Lemma 8. *For any fixed (t_0, x_0), $H(t, X(t, t_0, x_0))$ is a solution of system* (1).

Proof. Replace (τ, ξ) by $(t, X(t, \tau, \xi))$ in (9); system (9) is not changed. Due to the uniqueness of the bounded solution of (9), we can get that $h(t, (t, X(t, t_0, x_0))) = h(t, (t_0, x_0))$. Thus

$$H\left(t, X\left(t, t_0, x_0\right)\right) = X\left(t, t_0, x_0\right) + h\left(t, \left(t_0, x_0\right)\right). \tag{33}$$

Differentiating it and noticing that $X(t, t_0, x_0)$ and $h(t, (t_0, x_0))$ are the solutions of (2) and (9), respectively, we can obtain

$$\begin{aligned} \left[H\left(t, X\left(t, t_0, x_0\right)\right)\right]' &= A(t)\, X\left(t, t_0, x_0\right) + f\left(t, X\left(t, t_0, x_0\right)\right) \\ &\quad + A(t)\, h\left(t, \left(t_0, x_0\right)\right) \\ &\quad - f\left(t, X\left(t, t_0, x_0\right)\right) \\ &= A(t)\left(X\left(t, t_0, x_0\right) + h\left(t, \left(t_0, x_0\right)\right)\right) \\ &= A(t)\, H\left(t, X\left(t, t_0, x_0\right)\right) \\ \Delta H\left(t_k, X\left(t_k, t_0, x_0\right)\right) &= \widetilde{A}\left(t_k\right) X\left(t_k, t_0, x_0\right) \\ &\quad + \widetilde{f}\left(t_k, X\left(t_k, t_0, x_0\right)\right) \\ &\quad + \widetilde{A}\left(t_k\right) h\left(t_k, \left(t_k, X\left(t_k, t_0, x_0\right)\right)\right) \\ &\quad - \widetilde{f}\left(t_k, X\left(t_k, t_0, x_0\right)\right) \\ &= \widetilde{A}\left(t_k\right)\left(X\left(t_k, t_0, x_0\right) + h\left(t_k, \left(t_0, x_0\right)\right)\right) \\ &= \widetilde{A}\left(t_k\right) H\left(t_k, X\left(t_k, t_0, x_0\right)\right). \end{aligned} \tag{34}$$

It indicates that $H(t, X(t, t_0, x_0))$ is the solution of system (1). □

Lemma 9. *For any fixed (t_0, y_0), $G(t, Y(t, t_0, y_0))$ is a solution of the system* (2).

Proof. The proof is similar to Lemma 8. □

Lemma 10. *For any $t \in \mathbb{R}$, $y \in \mathbb{R}^n$, $H(t, G(t, y)) \equiv y$.*

Proof. Let $y(t)$ be any solution of system (1). By Lemma 9, $G(t, y(t))$ is a solution of system (2). Then by Lemma 8, we see that $H(t, G(t, y(t)))$ is a solution of system (1) written as $y_1(t)$. Denote $J(t) = y_1(t) - y(t)$. Differentiating it, we have

$$\begin{aligned} J'(t) &= y_1'(t) - y'(t) = A(t)\, y_1(t) - A(t)\, y(t) \\ &= A(t)\, J(t), \\ \Delta J(t) &= \Delta y_1(t) - \Delta y(t) = \widetilde{A}\left(t_k\right) y_1\left(t_k\right) - \widetilde{A}\left(t_k\right) y\left(t_k\right) \\ &= \widetilde{A}\left(t_k\right) J\left(t_k\right), \end{aligned} \tag{35}$$

which implies that $J(t)$ is a solution of system (1). On the other hand, following the definition of H, G, and Lemmas 5 and 6, we can obtain

$$\begin{aligned} |J(t)| &= \left|H\left(t, G\left(t, y\left(t\right)\right)\right) - y\left(t\right)\right| \\ &\leq \left|H\left(t, G\left(t, y\left(t\right)\right)\right) - G\left(t, y\left(t\right)\right)\right| \\ &\quad + \left|G\left(t, y\left(t\right)\right) - y\left(t\right)\right| \\ &= \left|h\left(t, \left(t, G\left(t, y\left(t\right)\right)\right)\right)\right| + \left|g\left(t, \left(t, y\right)\right)\right| \\ &\leq 4k\mu\alpha^{-1} + 4k\mu N\left[1 + \frac{1}{1 - \exp\left(-\alpha\right)}\right]. \end{aligned} \tag{36}$$

This implies that $J(t)$ is a bounded solution of system (1). However, by Lemma 4, system (1) has only one zero solution. Hence $J(t) \equiv 0$; consequently, $y_1(t) \equiv y(t)$; that is, $H(t, G(t, y)) = y(t)$. Since $y(t)$ is any solution of the system (1), then Lemma 10 follows. □

Lemma 11. *For any $t \in \mathbb{R}$, $x \in \mathbb{R}^n$, $G(t, H(t, x)) \equiv x$.*

Proof. The proof is similar to Lemma 10. □

Now we are in a position to prove the main result.

Proof of Theorem 3. We are going to show that $H(t, \cdot)$ satisfies the four conditions of Definition 2 in the following.

Proof of condition (i) for any fixed t, it follows from Lemmas 10 and 11 that $H(t, \cdot)$ is homeomorphism and $G(t, \cdot) = H^{-1}(t, \cdot)$.

Proof of condition (ii) it follows from (31) and Lemma 5 that $|H(t, x) - x|$ is bounded uniformly with respect to t.

Proof of condition (iii) it follows from (32) and Lemma 6 that $|G(t, y) - y|$ is bounded uniformly with respect to t.

Proof of condition (iv) it follows from Lemma 8 and Lemma 9 that we easily prove that condition (iv) is true.

Hence, system (2) is topologically conjugated to system (1). This completes the proof of Theorem 3. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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