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*Research Article*

# Least Squares Pure Imaginary Solution and Real Solution of the Quaternion Matrix Equation $AXB + CXD = E$ with the Least Norm

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Using the Kronecker product of matrices, the Moore-Penrose generalized inverse, and the complex representation of quaternion matrices, we derive the expressions of least squares solution with the least norm, least squares pure imaginary solution with the least norm, and least squares real solution with the least norm of the quaternion matrix equation $AXB + CXD = E$, respectively.

## 1. Introduction

Quaternions were introduced by Irish mathematician Sir William Rowan Hamilton in 1843. The family of quaternions is a skew field or noncommutative division algebra, since the characteristic property of quaternions is their noncommutativity under multiplication. A quaternion $q$ can be uniquely expressed as $q = q_0 + q_1 i + q_2 j + q_3 k$ with real coefficients $q_0$, $q_1$, $q_2$, $q_3$, $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, and $q$ can be uniquely expressed as $q = c_1 + c_2 j$, where $c_1$ and $c_2$ are complex numbers. Thus, every pure imaginary quaternion $q'$ can be uniquely expressed as $q' = q'_1 i + q'_2 j + q'_3 k$.

Throughout this paper, let $\mathrm{Q}$, $\mathrm{R}^{m\times n}$, $\mathrm{C}^{m\times n}$, $\mathrm{Q}^{m\times n}$, and $\mathrm{IQ}^{m\times n}$ be the skew field of quaternions, the set of all $m \times n$ real matrices, the set of all $m \times n$ complex matrices, the set of all $m \times n$ quaternion matrices, and the set of all $m \times n$ pure imaginary quaternion matrices, respectively. For $A \in \mathrm{C}^{m\times n}$, $\mathrm{Re}(A)$ and $\mathrm{Im}(A)$ denote the real part and the imaginary part of matrix $A$, respectively. For $A \in \mathrm{Q}^{m\times n}$, $\overline{A}$, $A^T$, $A^H$, and $A^+$ denote the conjugate matrix, the transpose matrix, the conjugate transpose matrix, and the Moore-Penrose generalized inverse matrix of matrix $A$, respectively.

For any $A \in \mathrm{Q}^{m\times n}$, $A$ can be uniquely expressed as $A = A_1 + A_2 j$, where $A_1, A_2 \in \mathrm{C}^{m\times n}$. The complex representation matrix of $A = A_1 + A_2 j \in \mathrm{Q}^{m\times n}$ is denoted by

$$f(A) = \begin{bmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{bmatrix} \in \mathrm{C}^{2m\times 2n}. \tag{1}$$

Notice that $f(A)$ is uniquely determined by $A$. For $A \in \mathrm{Q}^{m\times n}$, $B \in \mathrm{Q}^{n\times s}$, we have $f(AB) = f(A)f(B)$ (see [1]). Denote the trace of a square matrix $A = (a_{ij}) \in \mathrm{Q}^{n\times n}$ by $\mathrm{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn}$. We define the inner product $\langle A, B\rangle = \mathrm{tr}(B^H A)$ for all $A, B \in \mathrm{Q}^{m\times n}$. Then $\mathrm{Q}^{m\times n}$ is a Hilbert inner product space and the norm of a matrix generated by this inner product is the quaternion matrix Frobenius norm $\|\cdot\|$. The 2-norm of the vector $x$ is denoted by $\|x\|$.

Various aspects of the solutions of matrix equations such as $AXB = C$, $AX + XB = C$, $AXB + CYD = E$, $(AXB, CXD) = (E, F)$ have been investigated. See, for example, [2–35]. For the matrix equation

$$AXB + CXD = E, \tag{2}$$

if $B$ and $C$ are identity matrices, then the matrix equation (2) reduces to the well-known Sylvester equation [5, 36]. If $C$ and $D$ are identity matrices, then the matrix equation (2) reduces to the well-known Stein equation [37]. There are many important results about the matrix equation (2). For example, Hernández and Gassó [38] obtained the explicit solution of the matrix equation (2). Mansour [11] considered the solvability condition of the matrix equation (2) in the operator algebra. Mitra [39] studied the solvability conditions of matrix equation (2). For the quaternion matrix equation (2). Huang [40] obtained necessary and sufficient conditions for the existence of a solution or a unique solution using the method of complex representation of quaternion matrices.

Note that some authors have investigated the real and pure imaginary solutions to the quaternion matrix equations. For example, Au-Yeung and Cheng [2] considered the pure imaginary quaternionic solutions of the Hurwitz matrix equations. Wang et al. [41] studied the quaternion matrix equation $AXB = C$ and obtained necessary and sufficient conditions for the existence of a real solution or pure imaginary solution of the quaternion matrix equation $AXB = C$. Using the complex representation of quaternion matrices and the Moore-Penrose generalized inverse, Yuan et al. [42] derived the expressions of the least squares solution with the least norm, the least squares pure imaginary solution with the least norm, and the least squares real solution with the least norm for the quaternion matrix equation $AX = B$, respectively. Motivated by the work mentioned above, in this paper, we will consider the related problem of quaternion matrix equation (2).

*Problem 1.* Given $A \in Q^{m\times n}$, $B \in Q^{k\times s}$, $C \in Q^{m\times n}$, $D \in Q^{k\times s}$, $E \in Q^{m\times s}$, let

$$H_L = \left\{ X \mid X \in Q^{n\times k}, \|AXB + CXD - E\| = \min_{X_0 \in Q^{n\times k}} \|AX_0B + CX_0D - E\| \right\}. \tag{3}$$

Find $X_H \in H_L$ such that

$$\|X_H\| = \min_{X\in H_L} \|X\|. \tag{4}$$

*Problem 2.* Given $A \in Q^{m\times n}$, $B \in Q^{k\times s}$, $C \in Q^{m\times n}$, $D \in Q^{k\times s}$, $E \in Q^{m\times s}$, let

$$J_L = \left\{ X \mid X \in IQ^{n\times k}, \|AXB + CXD - E\| = \min_{X_0 \in IQ^{n\times k}} \|AX_0B + CX_0D - E\| \right\}. \tag{5}$$

Find $X_J \in J_L$ such that

$$\|X_J\| = \min_{X\in J_L} \|X\|. \tag{6}$$

*Problem 3.* Given $A \in Q^{m\times n}$, $B \in Q^{k\times s}$, $C \in Q^{m\times n}$, $D \in Q^{k\times s}$, $E \in Q^{m\times s}$, let

$$A_L = \left\{ X \mid X \in R^{n\times k}, \|AXB + CXD - E\| = \min_{X_0 \in R^{n\times k}} \|AX_0B + CX_0D - E\| \right\}. \tag{7}$$

Find $X_A \in A_L$ such that

$$\|X_A\| = \min_{X\in A_L} \|X\|. \tag{8}$$

The solution $X_H$ of Problem 1 is called the least squares solution with the least norm; the solution $X_J$ of Problem 2 is called the least squares pure imaginary solution with the least norm; and the solution $X_A$ of Problem 3 is called the least squares real solution with the least norm for matrix equation (2) over the skew field of quaternions.

This paper is organized as follows. In Section 2, we derive the explicit expression for the solution of Problem 1. In Section 3, we derive the explicit expression for the solution of Problem 2. In Section 4, we derive the explicit expression for the solution of Problem 3. Finally, in Section 5, we report numerical algorithms and numerical examples to illustrate our results.

## 2. The Solution of Problem 1

To study Problem 1, we begin with the following lemmas.

**Lemma 4** (see [43]). *The matrix equation $Ax = b$, with $A \in R^{m\times n}$ and $b \in R^n$, has a solution $x \in R^n$ if and only if*

$$AA^+b = b; \tag{9}$$

*in this case it has the general solution*

$$x = A^+b + (I - A^+A)\, y, \tag{10}$$

*where $y \in R^n$ is an arbitrary vector.*

**Lemma 5** (see [43]). *The least squares solutions of the matrix equation $Ax = b$, with $A \in R^{m\times n}$ and $b \in R^n$, can be represented as*

$$x = A^+b + (I - A^+A)\, y, \tag{11}$$

*where $y \in R^n$ is an arbitrary vector, and the least squares solution of the matrix equation $Ax = b$ with the least norm is $x = A^+b$.*

We identify $q \in Q$ with a complex vector $\vec{q} \in C^2$ and denote such an identification by the symbol $\cong$, that is;

$$c_1 + c_2 j = q \cong \vec{q} = (c_1, c_2). \tag{12}$$

For $A = A_1 + A_2 j \in Q^{m\times n}$, we have $A \cong \Phi_A = (A_1, A_2)$ and

$$\|A\| = \|\Phi_A\| = \sqrt{\|\mathrm{Re}\, A_1\|^2 + \|\mathrm{Im}\, A_1\|^2 + \|\mathrm{Re}\, A_2\|^2 + \|\mathrm{Im}\, A_2\|^2}. \tag{13}$$

We denote $\vec{A} = (\operatorname{Re} A_1, \operatorname{Im} A_1, \operatorname{Re} A_2, \operatorname{Im} A_2)$,

$$\operatorname{vec}(\vec{A}) = \begin{bmatrix} \operatorname{vec}(\operatorname{Re} A_1) \\ \operatorname{vec}(\operatorname{Im} A_1) \\ \operatorname{vec}(\operatorname{Re} A_2) \\ \operatorname{vec}(\operatorname{Im} A_2) \end{bmatrix}. \tag{14}$$

Notice that $\|\Phi_A\| = \|\vec{A}\|$. In particular, for $A = A_1 + A_2 i \in \mathbb{C}^{m\times n}$ with $A_1, A_2 \in \mathbb{R}^{m\times n}$, we have $A \cong \vec{A} = (A_1, A_2)$, and

$$\operatorname{vec}(A_1) + \operatorname{vec}(A_2)\, i = \operatorname{vec}(A) \cong \operatorname{vec}(\vec{A}) = \begin{bmatrix} \operatorname{vec}(A_1) \\ \operatorname{vec}(A_2) \end{bmatrix}. \tag{15}$$

Addition of two quaternion matrices $A = A_1 + A_2 j$ and $B = B_1 + B_2 j$ is defined by

$$\begin{aligned} (A_1 + B_1) + (A_2 + B_2)\, j &= (A + B) \\ &\cong \Phi_{A+B} = (A_1 + B_1, A_2 + B_2), \end{aligned} \tag{16}$$

whereas multiplication is defined as

$$\begin{aligned} AB &= (A_1 + A_2 j)(B_1 + B_2 j) \\ &= \left(A_1 B_1 - A_2 \overline{B_2}\right) + \left(A_1 B_2 + A_2 \overline{B_1}\right) j. \end{aligned} \tag{17}$$

So $\Phi_{A+B} = \Phi_A + \Phi_B$, $AB \cong \Phi_{AB}$; moreover, $\Phi_{AB}$ can be expressed as

$$\begin{aligned} \Phi_{AB} &= \left(A_1 B_1 - A_2 \overline{B_2}, A_1 B_2 + A_2 \overline{B_1}\right) \\ &= (A_1, A_2) \begin{bmatrix} B_1 & B_2 \\ -\overline{B_2} & \overline{B_1} \end{bmatrix} \\ &= \Phi_A f(B). \end{aligned} \tag{18}$$

**Lemma 6** (see [34]). *Let $A = A_1 + A_2 j \in \mathbb{Q}^{m\times n}$, $B = B_1 + B_2 j \in \mathbb{Q}^{n\times s}$, and $C = C_1 + C_2 j \in \mathbb{Q}^{s\times t}$ be given. Then*

$$\operatorname{vec}(\Phi_{ABC}) = \left(f(C)^T \otimes A_1, f(Cj)^H \otimes A_2\right) \begin{bmatrix} \operatorname{vec}(\Phi_B) \\ \operatorname{vec}(-\Phi_{jBj}) \end{bmatrix}. \tag{19}$$

**Lemma 7.** *For $X = X_1 + X_2 j \in \mathbb{Q}^{n\times k}$, let*

$$K = \begin{bmatrix} I_{nk} & iI_{nk} & 0 & 0 \\ 0 & 0 & I_{nk} & iI_{nk} \\ I_{nk} & -iI_{nk} & 0 & 0 \\ 0 & 0 & I_{nk} & -iI_{nk} \end{bmatrix}. \tag{20}$$

*Then*

$$\begin{bmatrix} \operatorname{vec}(\Phi_X) \\ \operatorname{vec}(-\Phi_{jXj}) \end{bmatrix} = K \operatorname{vec}(\vec{X}). \tag{21}$$

*Proof.* For $X = X_1 + X_2 j \in \mathbb{Q}^{n\times k}$, we have

$$\begin{aligned} \begin{bmatrix} \operatorname{vec}(\Phi_X) \\ \operatorname{vec}(-\Phi_{jXj}) \end{bmatrix} &= \begin{bmatrix} \operatorname{vec}(X_1) \\ \operatorname{vec}(X_2) \\ \operatorname{vec}(\overline{X_1}) \\ \operatorname{vec}(\overline{X_2}) \end{bmatrix} \\ &= \begin{bmatrix} I_{nk} & iI_{nk} & 0 & 0 \\ 0 & 0 & I_{nk} & iI_{nk} \\ I_{nk} & -iI_{nk} & 0 & 0 \\ 0 & 0 & I_{nk} & -iI_{nk} \end{bmatrix} \begin{bmatrix} \operatorname{vec}(\operatorname{Re}(X_1)) \\ \operatorname{vec}(\operatorname{Im}(X_1)) \\ \operatorname{vec}(\operatorname{Re}(X_2)) \\ \operatorname{vec}(\operatorname{Im}(X_2)) \end{bmatrix} \\ &= K \operatorname{vec}(\vec{X}). \end{aligned} \tag{22}$$

□

By Lemmas 6 and 7, we have the following.

**Lemma 8.** *If $A = A_1 + A_2 j \in \mathbb{Q}^{m\times n}$, $X = X_1 + X_2 j \in \mathbb{Q}^{n\times k}$, and $B = B_1 + B_2 j \in \mathbb{Q}^{k\times s}$, then*

$$\operatorname{vec}(\Phi_{AXB}) = \left(f(B)^T \otimes A_1, f(Bj)^H \otimes A_2\right) K \operatorname{vec}(\vec{X}). \tag{23}$$

Based on our earlier discussions, we now turn our attention to Problem 1. The following notations are necessary for deriving the solutions of Problem 1. For $A = A_1 + A_2 j \in \mathbb{Q}^{m\times n}$, $B \in \mathbb{Q}^{k\times s}$, $C = C_1 + C_2 j \in \mathbb{Q}^{m\times n}$, $D \in \mathbb{Q}^{k\times s}$, $E \in \mathbb{Q}^{m\times s}$, set

$$\begin{aligned} P &= \left(f(B)^T \otimes A_1 + f(D)^T \otimes C_1, \right. \\ &\qquad \left. f(Bj)^H \otimes A_2 + f(Dj)^H \otimes C_2\right) K, \\ P_1 &= \operatorname{Re}(P), \qquad P_2 = \operatorname{Im}(P), \\ e &= \begin{bmatrix} \operatorname{vec}(\operatorname{Re}(\Phi_E)) \\ \operatorname{vec}(\operatorname{Im}(\Phi_E)) \end{bmatrix}, \end{aligned} \tag{24}$$

$$\begin{aligned} R &= \left(I_{4nk} - P_1^+ P_1\right) P_2^T, \\ H &= R^+ + \left(I_{2ms} - R^+ R\right) Z P_2 P_1^+ P_1^{+T} \left(I_{4nk} - P_2^T R^+\right), \\ Z &= \left(I_{2ms} + \left(I_{2ms} - R^+ R\right) P_2 P_1^+ P_1^{+T} P_2^T \left(I_{2ms} - R^+ R\right)\right)^{-1}, \\ S_{11} &= I_{2ms} - P_1 P_1^+ + P_1^{+T} P_2^T Z \left(I_{2ms} - R^+ R\right) P_2 P_1^+, \\ S_{12} &= -P_1^{+T} P_2^T \left(I_{2ms} - R^+ R\right) Z, \\ S_{22} &= \left(I_{2ms} - R^+ R\right) Z. \end{aligned} \tag{25}$$

From the results in [33], one has

$$\begin{aligned} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}^+ &= \left(P_1^+ - H^T P_2 P_1^+, H^T\right), \\ \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}^+ \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} &= P_1^+ P_1 + R R^+, \\ I_{4nk} - \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}^+ &= \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix}. \end{aligned} \tag{26}$$

**Theorem 9.** *Let $A \in \mathrm{Q}^{m\times n}$, $B \in \mathrm{Q}^{k\times s}$, $C \in \mathrm{Q}^{m\times n}$, $D \in \mathrm{Q}^{k\times s}$, and $E \in \mathrm{Q}^{m\times s}$, and let $P_1$, $P_2$, $e$ be as in (24). Then*

$$\begin{aligned} H_L = \Big\{X \mid \operatorname{vec}\left(\vec{X}\right) = \left(P_1^{+} - H^T P_2 P_1^{+}, H^T\right) e \\ + \left(I - P_1^{+} P_1 - RR^{+}\right) y\Big\}, \end{aligned} \tag{27}$$

*where $y$ is an arbitrary vector of appropriate order.*

*Proof.* By Lemmas 5 and 8, we can get

$$\begin{aligned} &\|AXB + CXD - E\|^2 \\ &\quad = \left\|\Phi_{AXB+CXD-E}\right\|^2 \\ &\quad = \left\|\operatorname{vec}\left(\Phi_{AXB+CXD-E}\right)\right\|^2 \\ &\quad = \left\|\operatorname{vec}\left(\Phi_{AXB} + \Phi_{CXD} - \Phi_E\right)\right\|^2 \\ &\quad = \left\|\operatorname{vec}\left(\Phi_{AXB}\right) + \operatorname{vec}\left(\Phi_{CXD}\right) - \operatorname{vec}\left(\Phi_E\right)\right\|^2 \\ &\quad = \left\|P \operatorname{vec}\left(\vec{X}\right) - \operatorname{vec}\left(\Phi_E\right)\right\|^2 \\ &\quad = \Big\|\left(P_1 + iP_2\right) \operatorname{vec}\left(\vec{X}\right) \\ &\qquad - \left[\operatorname{vec}\left(\operatorname{Re}\left(\Phi_E\right)\right) + i \operatorname{vec}\left(\operatorname{Im}\left(\Phi_E\right)\right)\right] \Big\|^2 \\ &\quad = \left\|\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \operatorname{vec}\left(\vec{X}\right) - e\right\|^2. \end{aligned} \tag{28}$$

By Lemma 5, it follows that

$$\operatorname{vec}\left(\vec{X}\right) = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}^{+} e + \left[I_{4nk} - \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}^{+} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}\right] y; \tag{29}$$

thus

$$\operatorname{vec}\left(\vec{X}\right) = \left(P_1^{+} - H^T P_2 P_1^{+}, H^T\right) e + \left(I_{4nk} - P_1^{+} P_1 - RR^{+}\right) y. \tag{30}$$

The proof is completed. □

By Lemma 4 and Theorem 9, we get the following conclusion.

**Corollary 10.** *The quaternion matrix equation (2) has a solution $X \in \mathrm{Q}^{n\times k}$ if and only if*

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix} e = 0. \tag{31}$$

*In this case, denote by $H_E$ the solution set of (2). Then*

$$\begin{aligned} H_E = \Big\{X \mid \operatorname{vec}\left(\vec{X}\right) = \left(P_1^{+} - H^T P_2 P_1^{+}, H^T\right) e \\ + \left(I - P_1^{+} P_1 - RR^{+}\right) y\Big\}, \end{aligned} \tag{32}$$

*where $y$ is an arbitrary vector of appropriate order.*

*Furthermore, if (31) holds, then the quaternion matrix equation (2) has a unique solution $X \in H_E$ if and only if*

$$\operatorname{rank} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = 4nk. \tag{33}$$

*In this case,*

$$H_E = \left\{X \mid \operatorname{vec}\left(\vec{X}\right) = \left(P_1^{+} - H^T P_2 P_1^{+}, H^T\right) e\right\}. \tag{34}$$

**Theorem 11.** *Problem 1 has a unique solution $X_H \in H_L$. This solution satisfies*

$$\operatorname{vec}\left(\vec{X}_H\right) = \left(P_1^{+} - H^T P_2 P_1^{+}, H^T\right) e. \tag{35}$$

*Proof.* From (27), it is easy to verify that the solution set $H_L$ is nonempty and is a closed convex set. Hence, Problem 1 has a unique solution $X_H \in H_L$.

We now prove that the solution $X_H$ can be expressed as (35).

From (27), we have

$$\min_{X \in H_L} \|X\| = \min_{X \in H_L} \left\|\operatorname{vec}\left(\vec{X}\right)\right\|; \tag{36}$$

by Lemma 5 and (27),

$$\operatorname{vec}\left(\vec{X}\right) = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}^{+} e. \tag{37}$$

Thus,

$$\operatorname{vec}\left(\vec{X}\right) = \left(P_1^{+} - H^T P_2 P_1^{+}, H^T\right) e. \tag{38}$$

Thus we have completed the proof. □

**Corollary 12.** *The least norm problem*

$$\left\|X_H\right\| = \min_{X \in H_E} \|X\| \tag{39}$$

*has a unique solution $X_H \in H_E$ and $X_H$ can be expressed as (35).*

## 3. The Solution of Problem 2

We now discuss the solution of Problem 2. For $X = X_1 + X_2 j \in \mathrm{IQ}^{n\times k}$, we have $\operatorname{Re}(X_1) = 0$.

For $A = A_1 + A_2 j \in \mathrm{Q}^{m\times n}$, $B \in \mathrm{Q}^{k\times s}$, $C = C_1 + C_2 j \in \mathrm{Q}^{m\times n}$, $D \in \mathrm{Q}^{k\times s}$, $E \in \mathrm{Q}^{m\times s}$, set

$$\begin{aligned} Q = &\Big(f(B)^T \otimes A_1 + f(D)^T \otimes C_1, \\ &\quad f(Bj)^H \otimes A_2 + f(Dj)^H \otimes C_2\Big) \\ &\times \begin{bmatrix} iI_{nk} & 0 & 0 \\ 0 & I_{nk} & iI_{nk} \\ -iI_{nk} & 0 & 0 \\ 0 & I_{nk} & -iI_{nk} \end{bmatrix}, \end{aligned} \tag{40}$$

$$\begin{aligned} Q_1 &= \operatorname{Re}(Q), \qquad Q_2 = \operatorname{Im}(Q), \\ e &= \begin{bmatrix} \operatorname{vec}(\operatorname{Re}(\Phi_E)) \\ \operatorname{vec}(\operatorname{Im}(\Phi_E)) \end{bmatrix}, \end{aligned} \tag{41}$$

$$\begin{aligned} R_1 &= \left(I_{3nk} - Q_1^+ Q_1\right) Q_2^T, \\ H_1 &= R_1^+ + \left(I_{2ms} - R_1^+ R_1\right) \\ &\quad \times Z_1 Q_2 Q_1^+ Q_1^{+T} \left(I_{3nk} - Q_2^T R_1^+\right), \\ Z_1 &= \left(I_{2ms} + \left(I - R_1^+ R_1\right)\right. \\ &\quad \left.\times Q_2 Q_1^+ Q_1^{+T} Q_2^T \left(I_{2ms} - R_1^+ R_1\right)\right)^{-1}, \\ \Delta_{11} &= I_{2ms} - Q_1 Q_1^+ + Q_1^{+T} Q_2^T Z_1 \\ &\quad \times \left(I_{2ms} - R_1^+ R_1\right) Q_2 Q_1^+, \\ \Delta_{12} &= -Q_1^{+T} Q_2^T \left(I_{2ms} - R_1^+ R_1\right) Z_1, \\ \Delta_{22} &= \left(I_{2ms} - R_1^+ R_1\right) Z_1. \end{aligned} \tag{42}$$

Thus we have

$$\begin{aligned} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}^+ &= \left(Q_1^+ - H_1^T Q_2 Q_1^+, H_1^T\right), \\ \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}^+ \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} &= Q_1^+ Q_1 + R_1 R_1^+, \\ I_{3nk} - \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}^+ &= \begin{bmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{12}^T & \Delta_{22} \end{bmatrix}. \end{aligned} \tag{43}$$

We now study Problem 2. Since the methods are the same as in Section 2, we only describe the following results using Lemmas 4 and 5 and Theorem 9 and omit their detailed proofs.

**Theorem 13.** *Let $A \in Q^{m\times n}$, $B \in Q^{k\times s}$, $C \in Q^{m\times n}$, $D \in Q^{k\times s}$, and $E \in Q^{m\times s}$; let $Q_1$, $Q_2$, $E$ be as in* (41). *Then the set $J_L$ of Problem 2 can be expressed as*

$$J_L = \left\{ X \mid \begin{bmatrix} \operatorname{vec}(\operatorname{Im}(X_1)) \\ \operatorname{vec}(\operatorname{Re}(X_2)) \\ \operatorname{vec}(\operatorname{Im}(X_2)) \end{bmatrix} = \left(Q_1^+ - H_1^T Q_2 Q_1^+, H_1^T\right) e \right. \\ \left. + \left(I_{3nk} - Q_1^+ Q_1 - R_1 R_1^+\right) y \right\}, \tag{44}$$

*where $y$ is an arbitrary vector of appropriate order.*

**Corollary 14.** *The quaternion matrix equation* (2) *has a solution $X \in IQ^{m\times n}$ if and only if*

$$\begin{bmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{12}^T & \Delta_{22} \end{bmatrix} e = 0. \tag{45}$$

*In this case, denote by $J_E$ the pure imaginary solution set of* (2). *Then*

$$J_E = \left\{ X \mid \begin{bmatrix} \operatorname{vec}(\operatorname{Im}(X_1)) \\ \operatorname{vec}(\operatorname{Re}(X_2)) \\ \operatorname{vec}(\operatorname{Im}(X_2)) \end{bmatrix} = \left(Q_1^+ - H_1^T Q_2 Q_1^+, H_1^T\right) e \right. \\ \left. + \left(I_{3nk} - Q_1^+ Q_1 - R_1 R_1^+\right) y \right\}, \tag{46}$$

*where $y$ is an arbitrary vector of appropriate order.*

*Furthermore, if* (45) *holds, then the quaternion matrix equation* (2) *has a unique solution $X \in J_E$ if and only if*

$$\operatorname{rank} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = 3nk. \tag{47}$$

*In this case,*

$$J_E = \left\{ X \mid \begin{bmatrix} \operatorname{vec}(\operatorname{Im}(X_1)) \\ \operatorname{vec}(\operatorname{Re}(X_2)) \\ \operatorname{vec}(\operatorname{Im}(X_2)) \end{bmatrix} = \left(Q_1^+ - H_1^T Q_2 Q_1^+, H_1^T\right) e \right\}. \tag{48}$$

**Theorem 15.** *Problem 2 has a unique solution $X_J = \operatorname{Im} X_{J1} i + \operatorname{Re} X_{J2} j + \operatorname{Im} X_{J2} k \in J_L$. This solution satisfies*

$$\begin{bmatrix} \operatorname{vec}(\operatorname{Im}(X_{J1})) \\ \operatorname{vec}(\operatorname{Re}(X_{J2})) \\ \operatorname{vec}(\operatorname{Im}(X_{J2})) \end{bmatrix} = \left(Q_1^+ - H_1^T Q_2 Q_1^+, H_1^T\right) e. \tag{49}$$

**Corollary 16.** *The least norm problem*

$$\|X_J\| = \min_{X \in J_E} \|X\| \tag{50}$$

*has a unique solution $X_J = \operatorname{Im} X_{J1} i + \operatorname{Re} X_{J2} j + \operatorname{Im} X_{J2} k \in J_E$ and $X_J$ can be expressed as* (49).

## 4. The Solution of Problem 3

We now discuss the solution of Problem 3. For $X = X_1 + X_2 j \in R^{n\times k}$, we have $\operatorname{Im}(X_1) = \operatorname{Re}(X_2) = \operatorname{Im}(X_2) = 0$, and $X = \operatorname{Re}(X_1)$. Thus, we have the following lemmas.

For $A = A_1 + A_2 j \in Q^{m\times n}$, $B \in Q^{k\times s}$, $C = C_1 + C_2 j \in Q^{m\times n}$, $D \in Q^{k\times s}$, $E \in Q^{m\times s}$, set

$$T = \left(f(B)^T \otimes A_1 + f(D)^T \otimes C_1, \right. \\ \left. f(Bj)^H \otimes A_2 + f(Dj)^H \otimes C_2\right) \begin{bmatrix} I_{nk} \\ 0 \\ I_{nk} \\ 0 \end{bmatrix}, \tag{51}$$

$$T_1 = \operatorname{Re}(T), \quad T_2 = \operatorname{Im}(T), \quad e = \begin{bmatrix} \operatorname{vec}(\operatorname{Re}(\Phi_E)) \\ \operatorname{vec}(\operatorname{Im}(\Phi_E)) \end{bmatrix}, \tag{52}$$

$$\begin{aligned} R_2 &= (I_{nk} - T_1^+ T_1) T_2^T, \\ H_2 &= R_2^+ + (I_{2ms} - R_2^+ R_2) Z_2 T_2 T_1^+ T_1^{+T} \left(I_{nk} - T_2^T R_2^+\right), \\ Z_2 &= \left(I_{2ms} + (I_{2ms} - R_2^+ R_2)\right. \\ &\qquad \left.\times T_2 T_1^+ T_1^{+T} T_2^T (I_{2ms} - R_2^+ R_2)\right)^{-1}, \end{aligned} \tag{53}$$

$$\begin{aligned} \Lambda_{11} &= I_{2ms} - T_1 T_1^+ + T_1^{+T} T_2^T Z_2 (I_{2ms} - R_1^+ R_1) T_2 T_1^+, \\ \Lambda_{12} &= -T_1^{+T} T_2^T (I_{2ms} - R_2^+ R_2) Z_2, \\ \Lambda_{22} &= (I_{2ms} - R_2^+ R_2) Z_2. \end{aligned}$$

We have

$$\begin{aligned} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}^+ &= \left(T_1^+ - H_2^T T_2 T_1^+, H_2^T\right), \\ \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}^+ \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} &= T_1^+ T_1 + R_2 R_2^+, \\ I_{nk} - \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}^+ &= \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{12}^T & \Lambda_{22} \end{bmatrix}. \end{aligned} \tag{54}$$

By Lemma 5, we can easily get the following results for Problem 3.

**Theorem 17.** *Let $A \in Q^{m\times n}$, $B \in Q^{k\times s}$, $C \in Q^{m\times n}$, $D \in Q^{k\times s}$, and $E \in Q^{m\times s}$; let $T_1$, $T_2$, $e$ be as in (52). Then the set $A_L$ of Problem 3 can be expressed as*

$$\begin{aligned} A_L = \Big\{X \mid \operatorname{vec}(\operatorname{Re}(X_1)) &= \left(T_1^+ - H_2^T T_2 T_1^+, H_2^T\right) e \\ &\quad + \left(I_{nk} - T_1^+ T_1 - R_2 R_2^+\right) y\Big\}, \end{aligned} \tag{55}$$

*where $y$ is an arbitrary vector of appropriate order.*

**Corollary 18.** *The quaternion matrix equation (2) has a solution $X \in R^{m\times n}$ if and only if*

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{12}^T & \Lambda_{22} \end{bmatrix} e = 0. \tag{56}$$

*In this case, denote by $A_E$ the real solution set of (2). Then*

$$\begin{aligned} A_E = \Big\{X \mid \operatorname{vec}(\operatorname{Re}(X_1)) &= \left(T_1^+ - H_2^T T_2 T_1^+, H_2^T\right) e \\ &\quad + \left(I_{nk} - T_1^+ T_1 - R_2 R_2^+\right) y\Big\}, \end{aligned} \tag{57}$$

*where $y$ is an arbitrary vector of appropriate order.*

*Furthermore, if (56) holds, then the quaternion matrix equation (2) has a unique solution $X \in A_E$ if and only if*

$$\operatorname{rank} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = nk. \tag{58}$$

*In this case,*

$$A_E = \left\{X \mid \operatorname{vec}(\operatorname{Re}(X_1)) = \left(T_1^+ - H_2^T T_2 T_1^+, H_2^T\right) e\right\}. \tag{59}$$

**Theorem 19.** *Problem 3 has a unique solution $X_A \in A_L$. This solution satisfies*

$$\operatorname{vec}(\operatorname{Re} X_A) = \left(T_1^+ - H_2^T T_2 T_1^+, H_2^T\right) e. \tag{60}$$

**Corollary 20.** *The least norm problem*

$$\|X_A\| = \min_{X\in A_E} \|X\| \tag{61}$$

*has a unique solution $X_A \in A_E$ and $X_A$ can be expressed as* (60).

## 5. Numerical Verification

Based on the discussions in Sections 2, 3, and 4, we report numerical tests in this section. We give three numerical algorithms and four numerical examples to find the solutions of Problems 1, 2, and 3.

Algorithms 21, 22, and 23 provide the methods to find the solutions of Problems 1, 2, and 3. If the consistent conditions for matrix equation (2) hold, Examples 24 and 25 consider the numerical solutions of Problem 1 for $X \in Q^{n\times k}$. In Examples 26 and 27, if the consistent conditions for matrix equation (2) are not satisfied, we can compute the least squares solution with the least norm in Problems 2 and 3 by Algorithms 22 and 23, respectively. For demonstration purpose and avoiding the matrices with large norm to interrupt the solutions of Problems 1 and 2, we only consider the coefficient matrices of small sizes in numerical experiments.

*Algorithm 21* (for Problem 1). We have the following.

(1) Input $A$, $B$, $C$, $D$, and $E$ ($A \in Q^{m\times n}$, $B \in Q^{k\times s}$, $C \in Q^{m\times n}$, $D \in Q^{k\times s}$, and $E \in Q^{m\times s}$).

(2) Compute $P_1, P_2, R, H, Z, S_{11}, S_{12}, S_{22}, e$.

(3) If (31) and (33) hold, then calculate $X_H$ ($X_H \in H_E$) according to (34).

(4) If (31) holds, then calculate $X_H$ ($X_H \in H_E$) according to (35). Otherwise go to next step.

(5) Calculate $X_H$ ($X_H \in H_L$) according to (35).

*Algorithm 22* (for Problem 2). We have the following.

(1) Input $A$, $B$, $C$, $D$, and $E$ ($A \in Q^{m\times n}$, $B \in Q^{k\times s}$, $C \in Q^{m\times n}$, $D \in Q^{k\times s}$, and $E \in Q^{m\times s}$).

(2) Compute $Q_1, Q_2, R_1, Z_1, H_1, \Delta_{11}, \Delta_{12}, \Delta_{22}, e$.

(3) If (45) and (47) hold, then calculate $X_J$ ($X_A \in J_E$) according to (48).

(4) If (45) holds, then calculate $X_J$ ($X_J \in J_E$) according to (49). Otherwise go to next step.

(5) Calculate $X_A$ ($X_J \in J_L$) according to (49).

*Algorithm 23* (for Problem 3). We have the following.

(1) Input $A$, $B$, $C$, $D$, and $E$ ($A \in Q^{m\times n}$, $B \in Q^{k\times s}$, $C \in Q^{m\times n}$, $D \in Q^{k\times s}$, and $E \in Q^{m\times s}$).

(2) Compute $T_1, T_2, R_2, Z_2, H_2, \Lambda_{11}, \Lambda_{12}, \Lambda_{22}, e$.

(3) If (56) and (58) hold, then calculate $X_A$ ($X_A \in A_E$) according to (59).

(4) If (56) holds, then calculate $X_A$ ($X_A \in A_E$) according to (60). Otherwise go to next step.

(5) Calculate $X_A$ ($X_A \in A_L$) according to (60).

*Example 24.* Let $m = 6, n = 6, k = 5, s = 5$,

$$A = A_1 + A_2 j, \qquad B = B_1 + B_2 j,$$
$$C = C_1 + C_2 j, \qquad D = D_1 + D_2 j, \qquad X = X_1 + X_2 j, \tag{62}$$

$E = AXB + CXD$, where

$$\begin{aligned}
A_1 &= \text{eye}(m) + \text{ones}(n)\, i,\\
A_2 &= \text{magic}(m) + \text{ones}(n)\, i,\\
B_1 &= \text{ones}(k) + \text{eye}(k)\, i,\\
B_2 &= \text{eye}(s) + \text{magic}(s)\, i,\\
C_1 &= \text{eye}(m) + \text{magic}(n)\, i,\\
C_2 &= \text{magic}(m) + \text{eye}(n)\, i,\\
D_1 &= \text{ones}(k) + \text{ones}(k)\, i,\\
D_2 &= \text{eye}(s) + \text{ones}(s)\, i,\\
X_1 &= \text{rand}(n,k) + \text{randn}(n,k)\, i,\\
X_2 &= \text{randn}(n,k) + \text{rand}(n,k)\, i.
\end{aligned} \tag{63}$$

Let

$$\Phi_A = (A_1, A_2), \qquad \Phi_B = (B_1, B_2),$$
$$\Phi_C = (C_1, C_2), \qquad \Phi_D = (D_1, D_2),$$
$$\Phi_X = (X_1, X_2), \qquad \Phi_E = \Phi_A f(X)\, f(B) + \Phi_C f(X)\, f(D). \tag{64}$$

By using matlab 7.7 and Algorithm 21, we obtain

$$\text{rank}\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = 120,$$
$$\left\| \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix} e \right\| = 2.2435 \times 10^{-9}. \tag{65}$$

According to Algorithm 21 (3), we can see the matrix equation $AXB + CXD = E$ has a unique solution which is a unique solution with the least norm $X_H \in H_E$. We can get $\|X_H - X\| = 8.3184 \times 10^{-13}$.

*Example 25.* Let $m = 6, n = 6, k = 5, s = 5$,

$$A = A_1 + A_2 j, \qquad B = B_1 + B_2 j,$$
$$C = C_1 + C_2 j, \qquad D = D_1 + D_2 j, \tag{66}$$
$$X = X_1 + X_2 j,$$

$E = AXB + CXD$, where

$$\begin{aligned}
A_1 &= \text{ones}(n)\, i, \qquad & A_2 &= \text{zeros}(n),\\
B_1 &= \text{ones}(k), & B_2 &= \text{eye}(s)\, i,\\
C_1 &= \text{magic}(n), & C_2 &= \text{eye}(n)\, i,\\
D_1 &= \text{zeros}(k), & D_2 &= \text{ones}(s)\, i,\\
X_1 &= \text{rand}(n,k) + \text{randn}(n,k)\, i, &&\\
X_2 &= \text{randn}(n,k) + \text{rand}(n,k)\, i. &&
\end{aligned} \tag{67}$$

Let

$$\Phi_A = (A_1, A_2), \qquad \Phi_B = (B_1, B_2),$$
$$\Phi_C = (C_1, C_2), \qquad \Phi_D = (D_1, D_2),$$
$$\Phi_X = (X_1, X_2), \qquad \Phi_E = \Phi_A f(X)\, f(B) + \Phi_C f(X)\, f(D). \tag{68}$$

By using matlab 7.7 and Algorithm 21, we obtain

$$\text{rank}\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = 40, \qquad \left\| \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix} e \right\| = 1.0356 \times 10^{-10}. \tag{69}$$

According to Algorithm 21 (3), we can see the matrix equation $AXB + CXD = E$ has infinite solution and a unique solution with the least norm $X_H \in H_E$, and we can get $\|X_H - X\| = 6.8049$.

*Example 26.* Suppose $A$, $B$, $C$, $D$, $X$, $\Phi_A$, $\Phi_B$, $\Phi_C$, $\Phi_D$, $\Phi_X$ are the same as in Example 25; by using matlab 7.7 and Algorithm 22, we obtain

$$\text{rank}\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = 30, \qquad \left\| \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix} e \right\| = 1.8885 \times 10^{3}. \tag{70}$$

According to Algorithm 22 (4), we can see the matrix equation $AXB+CXD = E$ has infinite pure imaginary least squares solutions and a unique pure imaginary solution with the least norm $X_J \in J_L$ for Problem 2 and we can get $\|X_J - X\| = 8.7041$, and $X_J = \operatorname{Im} X_{J1} i + \operatorname{Re} X_{J2} j + \operatorname{Im} X_{J2} k$, where

$$\operatorname{Im} X_{J1} = \begin{bmatrix} 0.0323 & -0.1259 & -1.0004 & -0.2715 & -0.4757 & 0.6627 \\ -0.2810 & 0.4537 & 0.2955 & -0.7813 & -0.4568 & 0.0128 \\ -0.9309 & 0.1404 & 0.6728 & 0.5146 & -0.9665 & 0.0317 \\ -0.9120 & -0.5095 & 0.3595 & 0.4875 & 0.3294 & -0.4781 \\ -1.4218 & -0.4906 & -0.2904 & 0.1743 & 0.9760 & 0.8178 \end{bmatrix},$$
$$\operatorname{Re} X_{J2} = \begin{bmatrix} -0.3681 & 0.7317 & -0.1398 & 0.0302 & 0.7283 & 0.0698 \\ 0.5187 & -0.8747 & 0.2251 & -0.4052 & 0.9515 & 0.1300 \\ 0.5789 & 0.0122 & -1.1401 & -0.0403 & 0.5162 & 0.0317 \\ 0.8022 & 0.0724 & -0.2533 & -0.2188 & 0.8810 & -0.0821 \\ 0.3668 & 0.2956 & -0.1930 & 0.6681 & -0.8170 & 0.2827 \end{bmatrix}, \tag{71}$$
$$\operatorname{Im} X_{J2} = \begin{bmatrix} 1.4629 & 0.8243 & -0.3951 & -0.5003 & 0.0911 & 1.2750 \\ 1.2764 & 1.3112 & 0.6725 & -0.2591 & -0.5545 & 0.1595 \\ 0.1610 & 1.1247 & 1.4472 & 0.8086 & -0.3133 & -0.4861 \\ -0.4846 & 0.0093 & 1.2607 & 1.3930 & 0.7544 & -0.2448 \\ -0.2434 & -0.6364 & 0.1453 & 1.2065 & 1.4614 & 0.8228 \end{bmatrix}.$$

*Example 27.* Suppose $A$, $B$, $C$, $D$, $X$, $\Phi_A$, $\Phi_B$, $\Phi_C$, $\Phi_D$, $\Phi_X$ are the same as in Example 25. By using matlab 7.7 and Algorithm 22, we obtain

$$\operatorname{rank}\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = 10, \qquad \left\| \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix} e \right\| = 1.2725e + 003. \tag{72}$$

According to Algorithm 23 (5), we can see the matrix equation $AXB + CXD = E$ has infinite least squares real solutions and a unique least squares real solution with the least norm $X_A \in A_L$ for Problem 3 and we can get $\|X_A - X\| = 1.3046e + 017$, and

$$X_A = 10^{16} \times \begin{bmatrix} 3.7968 & 0.2949 & -1.9996 & -0.7850 & -1.3773 & -3.0782 \\ -2.4551 & 4.5700 & -3.1195 & -2.2379 & -4.8106 & 2.3596 \\ 0.6512 & -3.6256 & -0.8427 & -2.4444 & -0.8145 & -3.9656 \\ -1.7307 & -0.1625 & -2.6617 & 1.3151 & -1.7554 & -0.6584 \\ 1.7804 & -0.6472 & -0.1669 & -3.3724 & 0.0203 & -1.5349 \end{bmatrix}. \tag{73}$$

Examples 24, 25, 26, and 27 are used to show the feasibility of Algorithms 21, 22, and 23.

## Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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## References

[1] F. Z. Zhang, "Quaternions and matrices of quaternions," *Linear Algebra and Its Applications*, vol. 251, pp. 21–57, 1997.

[2] Y.-H. Au-Yeung and C.-M. Cheng, "On the pure imaginary quaternionic solutions of the Hurwitz matrix equations," *Linear Algebra and Its Applications*, vol. 419, no. 2-3, pp. 630–642, 2006.

[3] K.-W. E. Chu, "Singular value and generalized singular value decompositions and the solution of linear matrix equations," *Linear Algebra and Its Applications*, vol. 88-89, pp. 83–98, 1987.

[4] M. Dehghan and M. Hajarian, "Finite iterative algorithms for the reflexive and anti-reflexive solutions of the matrix equation $A_1X_1B_1 + A_2X_2B_2 = C$," *Mathematical and Computer Modelling*, vol. 49, no. 9-10, pp. 1937–1959, 2009.

[5] J. D. Gardiner, A. J. Laub, J. J. Amato, and C. B. Moler, "Solution of the Sylvester matrix equation $AXB^T + CXD^T = E$," *ACM Transactions on Mathematical Software*, vol. 18, no. 2, pp. 223–231, 1992.

[6] Z.-H. He and Q.-W. Wang, "A real quaternion matrix equation with applications," *Linear and Multilinear Algebra*, vol. 61, no. 6, pp. 725–740, 2013.

[7] N. Li and Q.-W. Wang, "Iterative algorithm for solving a class of quaternion matrix equation over the generalized $(P, Q)$-reflexive matrices," *Abstract and Applied Analysis*, vol. 2013, Article ID 831656, 15 pages, 2013.

[8] N. Li, Q.-W. Wang, and J. Jiang, "An efficient algorithm for the reflexive solution of the quaternion matrix equation $AXB + CX^HD = F$," *Journal of Applied Mathematics*, vol. 2013, Article ID 217540, 14 pages, 2013.

[9] Y.-T. Li and W.-J. Wu, "Symmetric and skew-antisymmetric solutions to systems of real quaternion matrix equations," *Computers & Mathematics with Applications*, vol. 55, no. 6, pp. 1142–1147, 2008.

[10] A.-P. Liao and Y. Lei, "Least-squares solution with the minimum-norm for the matrix equation $(AXB, GXH) = (C, D)$," *Computers & Mathematics with Applications*, vol. 50, no. 3-4, pp. 539–549, 2005.

[11] A. Mansour, "Solvability of $AXB - CXD = E$ in the operators algebra $B(H)$," *Lobachevskii Journal of Mathematics*, vol. 31, no. 3, pp. 257–261, 2010.

[12] S. K. Mitra, "Common solutions to a pair of linear matrix equations $A_1XB_1 = C_1$ and $A_2XB_2 = C_2$," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 74, pp. 213–216, 1973.

[13] A. Navarra, P. L. Odell, and D. M. Young, "A representation of the general common solution to the matrix equations $A_1XB_1 = C_1$ and $A_2XB_2 = C_2$ with applications," *Computers & Mathematics with Applications*, vol. 41, no. 7-8, pp. 929–935, 2001.

[14] Y.-X. Peng, X.-Y. Hu, and L. Zhang, "An iterative method for symmetric solutions and optimal approximation solution of the system of matrix equations $A_1XB_1 = C_1, A_2XB_2 = C_2$," *Applied Mathematics and Computation*, vol. 183, no. 2, pp. 1127–1137, 2006.

[15] Z.-H. Peng, X.-Y. Hu, and L. Zhang, "An efficient algorithm for the least-squares reflexive solution of the matrix equation $A_1XB_1 = C_1$, $A_2XB_2 = C_2$," *Applied Mathematics and Computation*, vol. 181, no. 2, pp. 988–999, 2006.

[16] Z.-Y. Peng, "Solutions of symmetry-constrained least-squares problems," *Numerical Linear Algebra with Applications*, vol. 15, no. 4, pp. 373–389, 2008.

[17] J. W. van der Woude, "On the existence of a common solution $X$ to the matrix equations $A_iXB_j = C_{ij}, (i, j) \in \Gamma$," *Linear Algebra and Its Applications*, vol. 375, pp. 135–145, 2003.

[18] Y. X. Yuan, "Two classes of best approximation problems of matrices," *Mathematica Numerica Sinica*, vol. 23, no. 4, pp. 429–436, 2001.

[19] Y. X. Yuan, "The minimum norm solutions of two classes of matrix equations," *Numerical Mathematics*, vol. 24, no. 2, pp. 127–134, 2002.

[20] Y. X. Yuan, "The optimal solution of linear matrix equations by using matrix decompositions," *Mathematica Numerica Sinica*, vol. 24, no. 2, pp. 165–176, 2002 (Chinese).

[21] T. S. Jiang, Y. H. Liu, and M. S. Wei, "Quaternion generalized singular value decomposition and its applications," *Applied Mathematics*, vol. 21, no. 1, pp. 113–118, 2006.

[22] T. S. Jiang and M. S. Wei, "Real representations of quaternion matrices and quaternion matrix equations," *Acta Mathematica Scientia A*, vol. 26, no. 4, pp. 578–584, 2006 (Chinese).

[23] Q.-W. Wang, "The general solution to a system of real quaternion matrix equations," *Computers & Mathematics with Applications*, vol. 49, no. 5-6, pp. 665–675, 2005.

[24] Q.-W. Wang, H.-X. Chang, and C.-Y. Lin, "$P$-(skew)symmetric common solutions to a pair of quaternion matrix equations," *Applied Mathematics and Computation*, vol. 195, no. 2, pp. 721–732, 2008.

[25] Q.-W. Wang and Z.-H. He, "Solvability conditions and general solution for mixed Sylvester equations," *Automatica*, vol. 49, no. 9, pp. 2713–2719, 2013.

[26] Q. Wang and Z. He, "A system of matrix equations and its applications," *Science China Mathematics*, vol. 56, no. 9, pp. 1795–1820, 2013.

[27] Q.-W. Wang, J. W. van der Woude, and H.-X. Chang, "A system of real quaternion matrix equations with applications," *Linear Algebra and Its Applications*, vol. 431, no. 12, pp. 2291–2303, 2009.

[28] Q. Wang, J. W. van der Woude, and S. W. Yu, "An equivalence canonical form of a matrix triplet over an arbitrary division ring with applications," *Science China Mathematics*, vol. 54, no. 5, pp. 907–924, 2011.

[29] Q.-W. Wang, S.-W. Yu, and C.-Y. Lin, "Extreme ranks of a linear quaternion matrix expression subject to triple quaternion matrix equations with applications," *Applied Mathematics and Computation*, vol. 195, no. 2, pp. 733–744, 2008.

[30] Q.-W. Wang and F. Zhang, "The reflexive re-nonnegative definite solution to a quaternion matrix equation," *Electronic Journal of Linear Algebra*, vol. 17, pp. 88–101, 2008.

[31] Q. W. Wang, H.-S. Zhang, and G.-J. Song, "A new solvable condition for a pair of generalized Sylvester equations," *Electronic Journal of Linear Algebra*, vol. 18, pp. 289–301, 2009.

[32] Q.-W. Wang, X. Zhang, and J. W. van der Woude, "A new simultaneous decomposition of a matrix quaternity over an arbitrary division ring with applications," *Communications in Algebra*, vol. 40, no. 7, pp. 2309–2342, 2012.

[33] J. R. Magnus, "$L$-structured matrices and linear matrix equations," *Linear and Multilinear Algebra*, vol. 14, no. 1, pp. 67–88, 1983.

[34] S. F. Yuan, A. P. Liao, and Y. Lei, "Least squares Hermitian solution of the matrix equation $(AXB, CXD) = (E, F)$ with the least norm over the skew field of quaternions," *Mathematical and Computer Modelling*, vol. 48, no. 1-2, pp. 91–100, 2008.

[35] S.-F. Yuan, Q.-W. Wang, and X. Zhang, "Least-squares problem for the quaternion matrix equation $AXB + CYD = E$ over different constrained matrices," *International Journal of Computer Mathematics*, vol. 90, no. 3, pp. 565–576, 2013.

[36] A.-G. Wu, F. Zhu, G.-R. Duan, and Y. Zhang, "Solving the generalized Sylvester matrix equation $AV + BW = EVF$ via a Kronecker map," *Applied Mathematics Letters*, vol. 21, no. 10, pp. 1069–1073, 2008.

[37] S. F. Yuan and A. P. Liao, "Least squares solution of the quaternion matrix equation $X - A\widehat{X}B = C$ with the least norm," *Linear and Multilinear Algebra*, vol. 59, no. 9, pp. 985–998, 2011.

[38] V. Hernández and M. Gassó, "Explicit solution of the matrix equation $AXB - CXD = E$," *Linear Algebra and Its Applications*, vol. 121, pp. 333–344, 1989.

[39] S. K. Mitra, "The matrix equation $AXB + CXD = E$," *SIAM Journal on Applied Mathematics*, vol. 32, no. 4, pp. 823–825, 1977.

[40] L. P. Huang, "The matrix equation $AXB - GXD = E$ over the quaternion field," *Linear Algebra and Its Applications*, vol. 234, pp. 197–208, 1996.

[41] Q. W. Wang, S. W. Yu, and W. Xie, "Extreme ranks of real matrices in solution of the quaternion matrix equation $AXB = C$ with applications," *Algebra Colloquium*, vol. 17, no. 2, pp. 345–360, 2010.

[42] S.-F. Yuan, Q.-W. Wang, and X.-F. Duan, "On solutions of the quaternion matrix equation $AX = B$ and their applications in color image restoration," *Applied Mathematics and Computation*, vol. 221, pp. 10–20, 2013.

[43] A. Ben-Israel and T. N. E. Greville, *Generalized Inverses: Theory and Applications*, John Wiley & Sons, New York, NY, USA, 1974.