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Research Article

Complete Moment Convergence for Arrays of Rowwise φ-Mixing Random Variables

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We investigate the complete moment convergence for maximal partial sum of arrays of rowwise φ-mixing random variables under some more general conditions. The results obtained in the paper generalize and improve some known ones.

1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of random variables defined on a fixed probability space $(\Omega, \mathscr{F}, P)$. Let n and m be positive integers. Write $\mathscr{F}_n^m = \sigma(X_i, n \leq i \leq m)$. Given σ-algebras $\mathscr{B}, \mathscr{R}$ in $\mathscr{F}$, let

$$\varphi(\mathscr{B}, \mathscr{R}) = \sup_{A\in\mathscr{B}, B\in\mathscr{R}, P(A)>0} |P(B \mid A) - P(B)|. \tag{1}$$

Define the φ-mixing coefficients by

$$\varphi(n) = \sup_{k\geq 1} \varphi\left(\mathscr{F}_1^k, \mathscr{F}_{k+n}^\infty\right), \quad n \geq 0. \tag{2}$$

A random variable sequence $\{X_n, n \geq 1\}$ is said to be φ-mixing if $\varphi(n) \downarrow 0$ as $n \to \infty$. $\varphi(n)$ is called mixing coefficient. A triangular array of random variables $\{X_{nk}, k \geq 1, n \geq 1\}$ is said to be an array of rowwise φ-mixing random variables if, for every $n \geq 1$, $\{X_{nk}, k \geq 1\}$ is a φ-mixing sequence of random variables. The notion of φ-mixing random variables was introduced by Dobrushin [1] and many applications have been found. See, for example, Utev [2] for central limit theorem, Gan and Chen [3] for limit theorem, Peligrad [4] for weak invariance principle, Shao [5] for almost sure invariance principles, Chen and Wang [6], Shen et al. [7, 8], Wu [9], and Wang et al. [10] for complete convergence, Hu and Wang [11] for large deviations, and so forth. When these are compared with corresponding results of independent random variable sequences, there still remains much to be desired.

Definition 1. A sequence of random variables $\{U_n, n \geq 1\}$ is said to converge completely to a constant a if, for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P\left(\left|U_n - a\right| > \varepsilon\right) < \infty. \tag{3}$$

In this case, one writes $U_n \to a$ completely. This notion was given first by Hsu and Robbins [12].

Definition 2. Let $\{Z_n, n \geq 1\}$ be a sequence of random variables and $a_n > 0$, $b_n > 0$, and $q > 0$. If

$$\sum_{n=1}^{\infty} a_n E\left\{b_n^{-1}\left|Z_n\right| - \varepsilon\right\}_+^q < \infty \quad \forall \varepsilon > 0, \tag{4}$$

then the above result was called the complete moment convergence by Chow [13].

Let $\{X_{nk}, k \geq 1, n \geq 1\}$ be an array of rowwise φ-mixing random variables with mixing coefficients $\{\varphi(n), n \geq 1\}$ in each row, let $\{a_n, n \geq 1\}$ be a sequence of positive real numbers such that $a_n \uparrow \infty$, and let $\{\Psi_k(t), k \geq 1\}$ be a sequence of positive even functions such that

$$\frac{\Psi_k(|t|)}{|t|^q} \uparrow, \quad \frac{\Psi_k(|t|)}{|t|^p} \downarrow \quad \text{as } |t| \uparrow \tag{5}$$

for some $1 \le q < p$ and each $k \ge 1$. In order to prove our results, we mention the following conditions:

$$EX_{nk} = 0, \quad k \ge 1, \ n \ge 1, \tag{6}$$

$$\sum_{n=1}^{\infty} \sum_{k=1}^{n} E \frac{\Psi_k (X_{nk})}{\Psi_k (a_n)} < \infty, \tag{7}$$

$$\sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} E \left(\frac{X_{nk}}{a_n} \right)^2 \right)^{v/2} < \infty, \tag{8}$$

where $v \ge p$ is a positive integer.

The following are examples of function $\Psi_k(t)$ satisfying assumption (5): $\Psi_k(t) = |t|^\beta$ for some $q < \beta < p$ or $\Psi_k(t) = |t|^q \log(1 + |t|^{p-q})$ for $t \in (-\infty, +\infty)$. Note that these functions are nonmonotone on $t \in (-\infty, +\infty)$, while it is simple to show that, under condition (5), the function $\Psi_k(t)$ is an increasing function for $t > 0$. In fact, $\Psi_k(t) = (\Psi_k(t)/|t|^q) \cdot |t|^q, t > 0$, and $|t|^q \uparrow$ as $|t| \uparrow$; then we have $\Psi_k(t) \uparrow$.

Recently Gan et al. [14] obtained the following complete convergence for φ-mixing random variables.

Theorem A. *Let $\{X_n, n \ge 1\}$ be a sequence of φ-mixing mean zero random variables with $\sum_{n=1}^{\infty} \varphi^{1/2}(n) < \infty$, let $\{a_n, n \ge 1\}$ be a sequence of positive real numbers with $a_n \uparrow \infty$, and let $\{\Psi_n(t), n \ge 1\}$ be a sequence of nonnegative even functions such that $\Psi_n(t) > 0$ as $t > 0$ and $(\Psi_n(|t|)/|t|) \uparrow$ and $(\Psi_n(|t|)/|t|^p) \downarrow$ as $|t| \uparrow \infty$, where $p \ge 2$. If the following conditions are satisfied:*

$$\sum_{n=1}^{\infty} \sum_{k=1}^{n} E \frac{\Psi_k (X_k)}{\Psi_k (a_n)} < \infty, \tag{9}$$

$$\sum_{n=1}^{\infty} \left[\sum_{k=1}^{n} \frac{E|X_k|^r}{a_n^r} \right]^s < \infty, \tag{10}$$

where $0 < r \le 2$, $s > 0$, then

$$\frac{1}{a_n} \max_{1 \le j \le n} \left| \sum_{k=1}^{j} X_k \right| \longrightarrow 0 \quad \text{completely}. \tag{11}$$

For more details about this type of complete convergence, one can refer to Gan and Chen [3], Wu et al. [15], Wu [16], Huang et al. [17], Shen [18], Shen et al. [19, 20], and so on. The purpose of this paper is extending Theorem A to the complete moment convergence, which is a more general version of the complete convergence, and making some improvements such that the conditions are more general. In this work, the symbol C always stands for a generic positive constant, which may vary from one place to another.

2. Preliminary Lemmas

In this section, we give the following lemma which will be used to prove our main results.

Lemma 3 (cf. Wang et al. [10]). *Let $\{X_n, n \ge 1\}$ be a sequence of φ-mixing random variables satisfying $\sum_{n=1}^{\infty} \varphi^{1/2}(n) < \infty$, $p \ge 2$. Assume that $EX_n = 0$, and $E|X_n|^p < \infty$, for each $n \ge 1$. Then there exists a constant C depending only on p and $\varphi(\cdot)$ such that*

$$E \left(\max_{1 \le j \le n} \left| \sum_{i=a+1}^{a+j} X_i \right|^p \right) \le C \left[\sum_{i=a+1}^{a+n} E|X_i|^p + \left(\sum_{i=a+1}^{a+n} EX_i^2 \right)^{p/2} \right], \tag{12}$$

for every $a \ge 0$ and $n \ge 1$. In particular, one has

$$E \left(\max_{1 \le j \le n} \left| \sum_{i=1}^{j} X_i \right|^p \right) \le C \left[\sum_{i=1}^{n} E|X_i|^p + \left(\sum_{i=1}^{n} EX_i^2 \right)^{p/2} \right], \tag{13}$$

for every $n \ge 1$.

3. Main Results and Their Proofs

Let $\{X_{nk}, k \ge 1, n \ge 1\}$ be an array of rowwise φ-mixing random variables and let $\varphi_n(\cdot)$ be the mixing coefficient of $\{X_{nk}, k \ge 1\}$ for any $n \ge 1$. Our main results are as follows.

Theorem 4. *Let $\{X_{nk}, k \ge 1, n \ge 1\}$ be an array of rowwise φ-mixing random variables satisfying $\sup_{n\ge1} \sum_{k=1}^{\infty} \varphi_n^{1/2}(k) < \infty$ and let $\{a_n, n \ge 1\}$ be a sequence of positive real numbers such that $a_n \uparrow \infty$. Also, let $\{\Psi_k(t), k \ge 1\}$ be a positive even function satisfying (5) for $1 \le q < p \le 2$. Then under conditions (6) and (7), one has*

$$\sum_{n=1}^{\infty} a_n^{-q} E \left\{ \max_{1 \le j \le n} \left| \sum_{k=1}^{j} X_{nk} \right| - \varepsilon a_n \right\}_+^q < \infty, \quad \forall \varepsilon > 0. \tag{14}$$

Proof. Firstly, let us prove the following statements from conditions (5) and (7).

(i) For $r \ge 1$, $0 < u \le q$,

$$\begin{aligned} &\sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} \frac{E|X_{nk}|^u I (|X_{nk}| > a_n)}{a_n^u} \right)^r \\ &\le \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} \frac{E|X_{nk}|^q I (|X_{nk}| > a_n)}{a_n^q} \right)^r \\ &\le \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} E \frac{\Psi_k (X_{nk})}{\Psi_k (a_n)} \right)^r \\ &\le \left(\sum_{n=1}^{\infty} \sum_{k=1}^{n} E \frac{\Psi_k (X_{nk})}{\Psi_k (a_n)} \right)^r < \infty. \end{aligned} \tag{15}$$

(ii) For $v \ge p$,

$$\begin{aligned} &\sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{E|X_{nk}|^v I (|X_{nk}| \le a_n)}{a_n^v} \\ &\le \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{E|X_{nk}|^p I (|X_{nk}| \le a_n)}{a_n^p} \\ &\le \sum_{n=1}^{\infty} \sum_{k=1}^{n} E \frac{\Psi_k (X_{nk})}{\Psi_k (a_n)} < \infty. \end{aligned} \tag{16}$$

For $n \geq 1$, denote $M_n(X) = \max_{1\leq j\leq n}|\sum_{k=1}^{j} X_{nk}|$. It is easy to check that

$$\begin{aligned}
&\sum_{n=1}^{\infty} a_n^{-q} E\{M_n(X) - \varepsilon a_n\}_+^q \\
&\quad = \sum_{n=1}^{\infty} a_n^{-q} \int_0^{\infty} P\left\{M_n(X) - \varepsilon a_n > t^{1/q}\right\} dt \\
&\quad = \sum_{n=1}^{\infty} a_n^{-q} \left(\int_0^{a_n^q} P\left\{M_n(X) > \varepsilon a_n + t^{1/q}\right\} dt \right. \\
&\qquad\qquad \left. + \int_{a_n^q}^{\infty} P\left\{M_n(X) > \varepsilon a_n + t^{1/q}\right\} dt \right) \\
&\quad \leq \sum_{n=1}^{\infty} P\left\{M_n(X) > \varepsilon a_n\right\} \\
&\qquad + \sum_{n=1}^{\infty} a_n^{-q} \int_{a_n^q}^{\infty} P\left\{M_n(X) > t^{1/q}\right\} dt \doteq I_1 + I_2.
\end{aligned} \tag{17}$$

To prove (14), it suffices to prove that $I_1 < \infty$ and $I_2 < \infty$. Now let us prove them step by step. Firstly, we prove that $I_1 < \infty$.

For all $n \geq 1$, define

$$X_k^{(n)} = X_{nk} I\left(|X_{nk}| \leq a_n\right), \qquad T_j^{(n)} = \frac{1}{a_n}\sum_{k=1}^{j}\left(X_k^{(n)} - EX_k^{(n)}\right), \tag{18}$$

then for all $\varepsilon > 0$, it is easy to have

$$\begin{aligned}
&P\left(\max_{1\leq j\leq n}\left|\frac{1}{a_n}\sum_{k=1}^{j} X_{nk}\right| > \varepsilon\right) \\
&\quad \leq P\left(\max_{1\leq j\leq n}|X_{nk}| > a_n\right) \\
&\qquad + P\left(\max_{1\leq j\leq n}\left|T_j^{(n)}\right| > \varepsilon - \max_{1\leq j\leq n}\left|\frac{1}{a_n}\sum_{k=1}^{j} EX_k^{(n)}\right|\right).
\end{aligned} \tag{19}$$

By (5), (6), (7), and (15) we have

$$\begin{aligned}
&\max_{1\leq j\leq n}\left|\frac{1}{a_n}\sum_{k=1}^{j} EX_k^{(n)}\right| \\
&\quad = \max_{1\leq j\leq n}\left|\frac{1}{a_n}\sum_{k=1}^{j} EX_{nk} I\left(|X_{nk}| \leq a_n\right)\right| \\
&\quad = \max_{1\leq j\leq n}\left|\frac{1}{a_n}\sum_{k=1}^{j} EX_{nk} I\left(|X_{nk}| > a_n\right)\right| \\
&\quad \leq \sum_{k=1}^{n} \frac{E|X_{nk}| I\left(|X_{nk}| > a_n\right)}{a_n} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{20}$$

From (19) and (20), it follows that, for n large enough,

$$\begin{aligned}
&P\left(\max_{1\leq j\leq n}\left|\frac{1}{a_n}\sum_{k=1}^{j} X_{nk}\right| > \varepsilon\right) \\
&\quad \leq \sum_{k=1}^{n} P\left(|X_{nk}| > a_n\right) + P\left(\max_{1\leq j\leq n}\left|T_j^{(n)}\right| > \frac{\varepsilon}{2}\right).
\end{aligned} \tag{21}$$

Hence we only need to prove that

$$\begin{aligned}
I &\doteq \sum_{n=1}^{\infty}\sum_{k=1}^{n} P\left(|X_{nk}| > a_n\right) < \infty, \\
II &\doteq \sum_{n=1}^{\infty} P\left(\max_{1\leq j\leq n}\left|T_j^{(n)}\right| > \frac{\varepsilon}{2}\right) < \infty.
\end{aligned} \tag{22}$$

For I, it follows by (15) that

$$\begin{aligned}
I &= \sum_{n=1}^{\infty}\sum_{k=1}^{n} EI\left(|X_{nk}| > a_n\right) \\
&\leq \sum_{n=1}^{\infty}\sum_{k=1}^{n} \frac{E|X_{nk}|^q I\left(|X_{nk}| > a_n\right)}{a_n^q} < \infty.
\end{aligned} \tag{23}$$

For II, take $r \geq 2$. Since $p \leq 2$, $r \geq p$, we have by Markov inequality, Lemma 3, C_r-inequality, and (16) that

$$\begin{aligned}
II &\leq \sum_{n=1}^{\infty}\left(\frac{\varepsilon}{2}\right)^{-r} E\max_{1\leq j\leq n}\left|T_j^{(n)}\right|^r \\
&\leq C\sum_{n=1}^{\infty}\left(\frac{\varepsilon}{2}\right)^{-r}\frac{1}{a_n^r}\left[\sum_{k=1}^{n} E\left|X_k^{(n)}\right|^r + \left(\sum_{k=1}^{n} E\left|X_k^{(n)}\right|^2\right)^{r/2}\right] \\
&\leq C\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{E\left|X_k^{(n)}\right|^r}{a_n^r} + C\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{E\left|X_k^{(n)}\right|^2}{a_n^2}\right)^{r/2} \\
&\leq C\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{E|X_{nk}|^p I\left(|X_{nk}| \leq a_n\right)}{a_n^p} \\
&\quad + C\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{E|X_{nk}|^p I\left(|X_{nk}| \leq a_n\right)}{a_n^p}\right)^{r/2} \\
&\leq C\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{E|X_{nk}|^p I\left(|X_{nk}| \leq a_n\right)}{a_n^p} \\
&\quad + C\left(\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{E|X_{nk}|^p I\left(|X_{nk}| \leq a_n\right)}{a_n^p}\right)^{r/2} < \infty.
\end{aligned} \tag{24}$$

Next we prove that $I_2 < \infty$. Denote $Y_{nk} = X_{nk} I(|X_{nk}| \leq t^{1/q})$, $Z_{nk} = X_{nk} - Y_{nk}$, and $M_n(Y) = \max_{1\leq j\leq n}|\sum_{k=1}^{j} Y_{nk}|$. Obviously,

$$\begin{aligned}
&P\left\{M_n(X) > t^{1/q}\right\} \\
&\quad \leq \sum_{k=1}^{n} P\left\{|X_{nk}| > t^{1/q}\right\} + P\left\{M_n(Y) > t^{1/q}\right\}.
\end{aligned} \tag{25}$$

Hence,

$$\begin{aligned} I_2 &\leq \sum_{n=1}^{\infty}\sum_{k=1}^{n} a_n^{-q}\int_{a_n^q}^{\infty} P\left\{|X_{nk}| > t^{1/q}\right\} dt \\ &\quad + \sum_{n=1}^{\infty} a_n^{-q}\int_{a_n^q}^{\infty} P\left\{M_n(Y) > t^{1/q}\right\} dt \\ &\doteq I_3 + I_4. \end{aligned} \tag{26}$$

For I_3, by (15), we have

$$\begin{aligned} I_3 &= \sum_{n=1}^{\infty}\sum_{k=1}^{n} a_n^{-q}\int_{a_n^q}^{\infty} P\left\{|X_{nk}| I(|X_{nk}| > a_n) > t^{1/q}\right\} dt \\ &\leq \sum_{n=1}^{\infty}\sum_{k=1}^{n} a_n^{-q}\int_{0}^{\infty} P\left\{|X_{nk}| I(|X_{nk}| > a_n) > t^{1/q}\right\} dt \\ &= \sum_{n=1}^{\infty}\sum_{k=1}^{n} \frac{E|X_{nk}|^q I(|X_{nk}| > a_n)}{a_n^q} < \infty. \end{aligned} \tag{27}$$

Now let us prove that $I_4 < \infty$. Firstly, it follows by (6) and (15) that

$$\begin{aligned} &\max_{t\geq a_n^q}\max_{1\leq j\leq n} t^{-1/q}\left|\sum_{k=1}^{j} EY_{nk}\right| \\ &\quad = \max_{t\geq a_n^q}\max_{1\leq j\leq n} t^{-1/q}\left|\sum_{k=1}^{j} EZ_{nk}\right| \\ &\quad \leq \max_{t\geq a_n^q} t^{-1/q}\sum_{k=1}^{n} E|X_{nk}| I\left(|X_{nk}| > t^{1/q}\right) \\ &\quad \leq \sum_{k=1}^{n} a_n^{-1} E|X_{nk}| I(|X_{nk}| > a_n) \\ &\quad \leq \sum_{k=1}^{n} \frac{E|X_{nk}|^q I(|X_{nk}| > a_n)}{a_n^q} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{28}$$

Therefore, for n sufficiently large,

$$\max_{1\leq j\leq n}\left|\sum_{k=1}^{j} EY_{nk}\right| \leq \frac{t^{1/q}}{2}, \quad t \geq a_n^q. \tag{29}$$

Then for n sufficiently large,

$$\begin{aligned} &P\left\{M_n(Y) > t^{1/q}\right\} \\ &\quad \leq P\left\{\max_{1\leq j\leq n}\left|\sum_{k=1}^{j}(Y_{nk} - EY_{nk})\right| > \frac{t^{1/q}}{2}\right\}, \quad t \geq a_n^q. \end{aligned} \tag{30}$$

Let $d_n = [a_n] + 1$. By (30), Lemma 3, and C_r-inequality, we can see that

$$\begin{aligned} I_4 &\leq C\sum_{n=1}^{\infty} a_n^{-q}\int_{a_n^q}^{\infty} t^{-2/q} E\left(\max_{1\leq j\leq n}\left|\sum_{k=1}^{j}(Y_{nk} - EY_{nk})\right|\right)^2 dt \\ &\leq C\sum_{n=1}^{\infty} a_n^{-q}\int_{a_n^q}^{\infty} t^{-2/q}\sum_{k=1}^{n} E(Y_{nk} - EY_{nk})^2 dt \\ &\leq C\sum_{n=1}^{\infty}\sum_{k=1}^{n} a_n^{-q}\int_{a_n^q}^{\infty} t^{-2/q} EY_{nk}^2 dt \\ &= C\sum_{n=1}^{\infty}\sum_{k=1}^{n} a_n^{-q}\int_{a_n^q}^{\infty} t^{-2/q} EX_{nk}^2 I(|X_{nk}| \leq d_n) dt \\ &\quad + C\sum_{n=1}^{\infty}\sum_{k=1}^{n} a_n^{-q}\int_{d_n^q}^{\infty} t^{-2/q} EX_{nk}^2 I\left(d_n < |X_{nk}| \leq t^{1/q}\right) dt \\ &\doteq I_{41} + I_{42}. \end{aligned} \tag{31}$$

For I_{41}, since $q < 2$, we have

$$\begin{aligned} I_{41} &= C\sum_{n=1}^{\infty}\sum_{k=1}^{n} a_n^{-q} EX_{nk}^2 I(|X_{nk}| \leq d_n)\int_{a_n^q}^{\infty} t^{-2/q} dt \\ &\leq C\sum_{n=1}^{\infty}\sum_{k=1}^{n} \frac{EX_{nk}^2 I(|X_{nk}| \leq d_n)}{a_n^2} \\ &= C\sum_{n=1}^{\infty}\sum_{k=1}^{n} \frac{EX_{nk}^2 I(|X_{nk}| \leq a_n)}{a_n^2} \\ &\quad + C\sum_{n=1}^{\infty}\sum_{k=1}^{n} \frac{EX_{nk}^2 I(a_n < |X_{nk}| \leq d_n)}{a_n^2} \\ &\doteq I'_{41} + I''_{41}. \end{aligned} \tag{32}$$

Since $p \leq 2$, by (16), it implies $I'_{41} < \infty$. Now we prove that $I''_{41} < \infty$. Since $q < 2$ and $(a_n + 1)/a_n \to 1$ as $n \to \infty$, by (15) we have

$$\begin{aligned} I''_{41} &\leq C\sum_{n=1}^{\infty}\sum_{k=1}^{n} \frac{d_n^{2-q}}{a_n^2} E|X_{nk}|^q I(a_n < |X_{nk}| \leq d_n) \\ &\leq C\sum_{n=1}^{\infty}\sum_{k=1}^{n} \left(\frac{a_n + 1}{a_n}\right)^{2-q} \frac{E|X_{nk}|^q I(|X_{nk}| > a_n)}{a_n^q} \\ &\leq C\sum_{n=1}^{\infty}\sum_{k=1}^{n} \frac{E|X_{nk}|^q I(|X_{nk}| > a_n)}{a_n^q} < \infty. \end{aligned} \tag{33}$$

Let $t = u^q$ in I_{42}. Note that, for $q < 2$,

$$\begin{aligned}
&\int_{d_n}^{\infty} u^{q-3} EX_{nk}^2 I\left(d_n < |X_{nk}| \le u\right) du \\
&\quad = \int_{d_n}^{\infty} u^{q-3} EX_{nk}^2 I\left(|X_{nk}| > d_n\right) \cdot I\left(|X_{nk}| \le u\right) du \\
&\quad = E\left[X_{nk}^2 I\left(|X_{nk}| > d_n\right) \int_{|X_{nk}|}^{\infty} u^{q-3} I\left(|X_{nk}| \le u\right) du\right] \\
&\quad = E\left[X_{nk}^2 I\left(|X_{nk}| > d_n\right) \int_{|X_{nk}|}^{\infty} u^{q-3} du\right] \\
&\quad \le CE|X_{nk}|^q I\left(|X_{nk}| > d_n\right).
\end{aligned} \tag{34}$$

Then by (15) and $d_n > a_n$, we have

$$\begin{aligned}
I_{42} &= C\sum_{n=1}^{\infty}\sum_{k=1}^{n} a_n^{-q} \int_{d_n}^{\infty} u^{q-3} EX_{nk}^2 I\left(d_n < |X_{nk}| \le u\right) du \\
&\le C\sum_{n=1}^{\infty}\sum_{k=1}^{n} a_n^{-q} E|X_{nk}|^q I\left(|X_{nk}| > a_n\right) < \infty.
\end{aligned} \tag{35}$$

This completes the proof of Theorem 4. □

Theorem 5. *Let $\{X_{nk}, k \ge 1, n \ge 1\}$ be an array of rowwise φ-mixing random variables satisfying $\sup_{n\ge1}\sum_{k=1}^{\infty}\varphi_n^{1/2}(k) < \infty$ and let $\{a_n, n \ge 1\}$ be a sequence of positive real numbers such that $a_n \uparrow \infty$. Also, let $\{\Psi_k(t), k \ge 1\}$ be a positive even function satisfying* (5) *for $1 \le q < p$ and $p > 2$. Then conditions* (6)–(8) *imply* (14).

Proof. Following the notation, by a similar argument as in the proof of Theorem 4, we can easily prove that $I_1 < \infty$, $I_3 < \infty$ and that (19) and (20) hold. To complete the proof, we only need to prove that $I_4 < \infty$.

Let $\delta \ge p$ and $d_n = [a_n] + 1$. By (30), Markov inequality, Lemma 3, and the C_r-inequality we can get

$$\begin{aligned}
I_4 &\le C\sum_{n=1}^{\infty} a_n^{-q} \int_{a_n^q}^{\infty} t^{-\delta/q} E\max_{1\le j\le n}\left|\sum_{k=1}^{j}\left(Y_{nk} - EY_{nk}\right)\right|^{\delta} dt \\
&\le C\sum_{n=1}^{\infty} a_n^{-q} \int_{a_n^q}^{\infty} t^{-\delta/q} \left[\sum_{k=1}^{n} E|Y_{nk}|^{\delta} + \left(\sum_{k=1}^{n} EY_{nk}^2\right)^{\delta/2}\right] dt \\
&= C\sum_{n=1}^{\infty}\sum_{k=1}^{n} a_n^{-q} \int_{a_n^q}^{\infty} t^{-\delta/q} E|Y_{nk}|^{\delta} dt \\
&\quad + C\sum_{n=1}^{\infty} a_n^{-q} \int_{a_n^q}^{\infty} t^{-\delta/q} \left(\sum_{k=1}^{n} EY_{nk}^2\right)^{\delta/2} dt \\
&\doteq I_{43} + I_{44}.
\end{aligned} \tag{36}$$

For I_{43}, we have

$$\begin{aligned}
I_{43} &= C\sum_{n=1}^{\infty}\sum_{k=1}^{n} a_n^{-q} \int_{a_n^q}^{\infty} t^{-\delta/q} E|X_{nk}|^{\delta} I\left(|X_{nk}| \le d_n\right) dt \\
&\quad + C\sum_{n=1}^{\infty}\sum_{k=1}^{n} a_n^{-q} \int_{d_n^q}^{\infty} t^{-\delta/q} E|X_{nk}|^{\delta} I\left(d_n < |X_{nk}| \le t^{1/q}\right) dt \\
&\doteq I'_{43} + I''_{43}.
\end{aligned} \tag{37}$$

By a similar argument as in the proof of $I_{41} < \infty$ and $I_{42} < \infty$ (replacing the exponent 2 by δ), we can get $I'_{43} < \infty$ and $I''_{43} < \infty$.

For I_{44}, since $\delta > 2$, we can see that

$$\begin{aligned}
I_{44} &= C\sum_{n=1}^{\infty} a_n^{-q} \int_{a_n^q}^{\infty} t^{-\delta/q} \left(\sum_{k=1}^{n} EX_{nk}^2 I\left(|X_{nk}| \le a_n\right)\right. \\
&\qquad \left. + \sum_{k=1}^{n} EX_{nk}^2 I\left(a_n < |X_{nk}| \le t^{1/q}\right)\right)^{\delta/2} dt \\
&\le C\sum_{n=1}^{\infty} a_n^{-q} \int_{a_n^q}^{\infty} t^{-\delta/q} \left(\sum_{k=1}^{n} EX_{nk}^2 I\left(|X_{nk}| \le a_n\right)\right)^{\delta/2} dt \\
&\quad + C\sum_{n=1}^{\infty} a_n^{-q} \int_{a_n^q}^{\infty} \left(t^{-2/q}\sum_{k=1}^{n} EX_{nk}^2 I\left(a_n < |X_{nk}| \le t^{1/q}\right)\right)^{\delta/2} dt \\
&\doteq I'_{44} + I''_{44}.
\end{aligned} \tag{38}$$

Since $\delta \ge p > q$, from (8) we have

$$\begin{aligned}
I'_{44} &= C\sum_{n=1}^{\infty} a_n^{-q} \left(\sum_{k=1}^{n} EX_{nk}^2 I\left(|X_{nk}| \le a_n\right)\right)^{\delta/2} \int_{a_n^q}^{\infty} t^{-\delta/q} dt \\
&\le C\sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} \frac{EX_{nk}^2 I\left(|X_{nk}| \le a_n\right)}{a_n^2}\right)^{\delta/2} \\
&\le C\sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} \frac{EX_{nk}^2}{a_n^2}\right)^{\delta/2} < \infty.
\end{aligned} \tag{39}$$

Next we prove that $I''_{44} < \infty$. To start with, we consider the case $1 \le q \le 2$. Since $\delta > 2$, by (15), we have

$$\begin{aligned}
I''_{44} &\le C\sum_{n=1}^{\infty} a_n^{-q} \int_{a_n^q}^{\infty} \left(t^{-1}\sum_{k=1}^{n} E|X_{nk}|^q I\left(a_n < |X_{nk}| \le t^{1/q}\right)\right)^{\delta/2} dt \\
&\le C\sum_{n=1}^{\infty} a_n^{-q} \int_{a_n^q}^{\infty} \left(t^{-1}\sum_{k=1}^{n} E|X_{nk}|^q I\left(|X_{nk}| > a_n\right)\right)^{\delta/2} dt
\end{aligned}$$

$$= C\sum_{n=1}^{\infty} a_n^{-q}\left(\sum_{k=1}^{n} E\left|X_{nk}\right|^q I\left(\left|X_{nk}\right| > a_n\right)\right)^{\delta/2} \int_{a_n^q}^{\infty} t^{-\delta/2} dt$$
$$\leq C\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{E\left|X_{nk}\right|^q I\left(\left|X_{nk}\right| > a_n\right)}{a_n^q}\right)^{\delta/2} < \infty. \tag{40}$$

Finally, we prove that $I''_{44} < \infty$ in the case $2 < q < p$. Since $\delta > q$ and $\delta > 2$, we have by (15) that

$$I''_{44} \leq C\sum_{n=1}^{\infty} a_n^{-q}\int_{a_n^q}^{\infty}\left(t^{-2/q}\sum_{k=1}^{n} EX_{nk}^2 I\left(\left|X_{nk}\right| > a_n\right)\right)^{\delta/2} dt$$
$$= C\sum_{n=1}^{\infty} a_n^{-q}\left(\sum_{k=1}^{n} EX_{nk}^2 I\left(\left|X_{nk}\right| > a_n\right)\right)^{\delta/2}\int_{a_n^q}^{\infty} t^{-\delta/q} dt \tag{41}$$
$$\leq C\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{EX_{nk}^2 I\left(\left|X_{nk}\right| > a_n\right)}{a_n^2}\right)^{\delta/2} < \infty.$$

Thus we get the desired result immediately. The proof is completed. □

Corollary 6. *Let $\{X_{nk}, k \geq 1, n \geq 1\}$ be an array of rowwise φ-mixing mean zero random variables with $\sup_{n\geq 1}\sum_{k=1}^{\infty}\varphi_n^{1/2}(k) < \infty$, $q \geq 1$. If, for some $\alpha > 0$ and $v \geq 2$,*

$$\max_{1\leq k\leq n} E\left|X_{nk}\right|^{v} = O\left(n^{\alpha}\right), \tag{42}$$

where $(v/q) - \alpha > \max\{v/2, 2\}$, $v \geq 2$, then, for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{-1} E\left\{\max_{1\leq j\leq n}\left|\sum_{k=1}^{j} X_{nk}\right| - \varepsilon n^{1/q}\right\}_{+}^{q} < \infty. \tag{43}$$

Proof. Put $\Psi_k(|t|) = |t|^{v}$, $p = v + \delta$, $\delta > 0$, and $a_n = n^{1/q}$.

Since $v \geq 2$, $(v/q) - \alpha > \max\{v/r, 2\}$, then

$$\frac{\Psi_k\left(|t|\right)}{|t|^q} = |t|^{v-q}\uparrow, \quad \frac{\Psi_k\left(|t|\right)}{|t|^p} = \frac{|t|^{v}}{|t|^p} = \frac{1}{|t|^{\delta}}\downarrow \quad \text{as } |t| \uparrow \infty. \tag{44}$$

It follows by (42) and $(v/q) - \alpha > 2$ that

$$\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{E\Psi_k\left(X_{nk}\right)}{\Psi_k\left(a_n\right)} = \sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{E\left|X_{nk}\right|^{v}}{n^{v/q}} \leq C\sum_{n=1}^{\infty}\frac{1}{n^{(v/q)-\alpha-1}} < \infty. \tag{45}$$

Since $v \geq 2$, by Jensen's inequality it follows that

$$\sum_{k=1}^{n}\frac{E\left|X_{nk}\right|^2}{n^{2/q}} \leq \sum_{k=1}^{n}\frac{\left(E\left|X_{nk}\right|^{v}\right)^{2/v}}{n^{2/q}} \leq C\frac{1}{n^{(2/q)-(2\alpha/v)-1}}. \tag{46}$$

Clearly $(2/q) - (2\alpha/v) - 1 > 0$. Take $s > p$ such that $(s/2)((2/q) - (2\alpha/v) - 1) > 1$. Therefore,

$$\sum_{n=1}^{\infty}\left[\sum_{k=1}^{n}\frac{E\left|X_{nk}\right|^2}{n^{2/q}}\right]^{s/2} < \infty. \tag{47}$$

Combining Theorem 5 and (45)–(47), we can prove Corollary 6 immediately. □

Remark 7. Noting that in this paper we consider the case $1 \leq q \leq p$, which has a more wide scope than the case $q = 1$, $p \geq 2$ in Gan et al. [14]. In addition, compared with φ-mixing random variables, the arrays of φ-mixing random variables not only have many related properties, but also have a wide range of application. So it is very significant to study it.

Remark 8. Under the condition of Theorem 4, we have

$$\infty > \sum_{n=1}^{\infty} a_n^{-q} E\left\{\max_{1\leq j\leq n}\left|\sum_{k=1}^{j} X_{nk}\right| - \varepsilon a_n\right\}_{+}^{q}$$
$$= \sum_{n=1}^{\infty} a_n^{-q}\int_0^{\infty} P\left\{\max_{1\leq j\leq n}\left|\sum_{k=1}^{j} X_{nk}\right| - \varepsilon a_n > t^{1/q}\right\} dt$$
$$\geq \sum_{n=1}^{\infty} a_n^{-q}\int_0^{\varepsilon^q a_n^q} P\left\{\max_{1\leq j\leq n}\left|\sum_{k=1}^{j} X_{nk}\right| - \varepsilon a_n > \varepsilon a_n\right\} dt \tag{48}$$
$$= \varepsilon^q\sum_{n=1}^{\infty} P\left\{\max_{1\leq j\leq n}\left|\sum_{k=1}^{j} X_{nk}\right| > 2\varepsilon a_n\right\}.$$

Then we can obtain (11) directly. In this case, condition (10) is not needed. Especially, for $p = 2$, the conditions of Theorem 4 are weaker than Theorem A. So Theorem 4 generalizes and improves it.

Remark 9. Note that Theorem A only considers $q = 1$, while Theorem 5 considers $q \geq 1$. In addition, (14) implies (11), so Theorem 5 generalizes the corresponding result of Theorem A.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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