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Research Article

General Vertex-Distinguishing Total Coloring of Graphs

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The general vertex-distinguishing total chromatic number of a graph G is the minimum integer k, for which the vertices and edges of G are colored using k colors such that any two vertices have distinct sets of colors of them and their incident edges. In this paper, we figure out the exact value of this chromatic number of some special graphs and propose a conjecture on the upper bound of this chromatic number.

1. Introduction

All graphs considered in this paper are simple and finite. For a graph G, we denote by $V(G)$, $E(G)$, $\Delta(G)$, and $\delta(G)$ the sets of vertices, edges, maximum degree, and minimum degree of G, respectively. For a vertex v of G, $d_G(v)$ is the degree of v in G. For any $V' \subseteq V(G)$, we use $G[V']$ to denote the subgraph induced by V'. For any undefined terms, the reader is referred to the book [1].

The coloring problem of graphs is one of the classical research areas in graph theory. It has been widely applied to various fields, such as large scheduling [2], assignment of radio frequency [3], and separating combustible chemical combinations [4]. Due to its extensive application, many new variants of colorings have been studied [5].

Recall that a *k-edge coloring* of a graph G is a mapping $f: E(G) \rightarrow C$, where C is a set of k colors. An edge coloring is *proper* if adjacent edges receive distinct colors. In 1985, Harary and Plantholt [6] first considered point-distinguishing chromatic index, which is a variant of edge coloring. After that, many other variants of edge coloring were introduced, such as vertex-distinguishing proper edge coloring [7], adjacent vertex-distinguishing edge coloring [8], and general adjacent vertex-distinguishing edge coloring [9].

A total k-coloring of a graph G is a coloring of $V(G) \cup E(G)$ using k colors. A total k-coloring is *proper* if no two adjacent or incident elements receive the same color. The minimum number of colors required for a proper total coloring of G is called the total chromatic number of G and is denoted by $\chi_t(G)$. Behzad [10] and Vizing [11] independently made the conjecture that, for any graph G,

$$\chi_t(G) \leq (G) + 2. \quad (1)$$

This is known as the total coloring conjecture (TCC) and is still unproven.

Let f be a total k-coloring of G. The *total color set* (with respect to f) of a vertex $v \in V(G)$ is the set, denoted by $C_f(v)$, of colors of v and its incident edges. We denote by $\mathscr{C}_f(G)$ the set of total color sets of all vertices of G. Furthermore, let S be a subset of $V(G) \cup E(G)$; we use $C_f(S)$ to denote the set of colors of elements of S.

Like edge coloring, total coloring also has some variants. In 2005, Zhang et al. [12] added a restriction to the definition of total coloring and proposed a new type of coloring defined as follows.

Definition 1. Let f be a proper total k-coloring of a graph G. If, for all $u, v \in V(G)$, $C_f(u) \neq C_f(v)$, then f is called an adjacent vertex-distinguishing total k-coloring of G, or a k-AVDTC of G for short. The minimum number k for which G has a k-AVDTC is the adjacent vertex-distinguishing total chromatic number of G, denoted by $\chi_{at}(G)$.

Zhang et al. [12] conjectured that, for any graph G, it has

$$\chi_{at}(G) \leq \Delta(G) + 3. \quad (2)$$

In [13–15], authors proved that there exists a 6-AVDTC of graphs with $\Delta = 3$, which indicates conjecture (2) holds for such graphs. For further research on adjacent vertex-distinguishing total chromatic number, one may refer to [16–23].

For a k-AVDTC f of a graph G, if $C_f(u) \neq C_f(v)$ is required for any two distinct vertices u, v, then f is called a *vertex-distinguishing total k-coloring* of G, abbreviated as k-VDTC. The minimum number k such that G has a k-VDTC is called the vertex-distinguishing total chromatic number, denoted by $\chi_{vt}(G)$ [24]. Zhang et al. conjectured in [24] that, for any graph G, it follows that

$$\mu_t(G) \leq \chi_{vt}(G) \leq \mu_t(G) + 1, \tag{3}$$

where $\mu_t(G) = \min\{k \mid \binom{k}{i+1} \geq n_i,\ \delta \leq i \leq \Delta\}$.

In this paper, we introduce a variant of vertex-distinguishing total coloring of a graph G, which relaxes the restriction that the coloring is proper. We now present the detailed definition as follows.

Definition 2. Let G be a graph and k be a positive integer. A total coloring f of G using k colors is called a general vertex-distinguishing total k-coloring of G (or k-GVDTC of G briefly) if, for all $u, v \in V(G)$, $C_f(u) \neq C_f(v)$. The minimum number k for which G has a k-GVDTC is the general vertex-distinguishing total chromatic number, denoted by $\chi_{gvt}(G)$.

It is evident that $\chi_{gvt}(G)$ does exist for any graph G. In this paper, we study the general vertex-distinguishing total coloring of some special classes of graphs and obtain the exact value of the general vertex-distinguishing total chromatic number of these graphs. Furthermore, we propose a conjecture on the upper bound of general vertex-distinguishing total chromatic number of a graph.

2. Main Results

We first present a trivial lower bound on the general vertex-distinguishing total chromatic number of a graph.

Theorem 3. *Let G be a graph on n vertices. Then*

$$\chi_{gvt}(G) \geq \lceil \log_2(n+1) \rceil. \tag{4}$$

Proof. Let $\chi_{gvt}(G) = k$. It follows that $n \leq \binom{k}{1} + \binom{k}{2} + \cdots + \binom{k}{k} = 2^k - 1$, so $k \geq \lceil \log_2(n+1) \rceil$. □

Notice that the lower bound of Theorem 3 can be attained in graphs, such as the n-vertex path P_n for $n = 1, 2, \ldots, 7$. One can readily check that $\chi_{gvt(P_1)} = 1$ and $\chi_{gvt(P_n)} = 2$ for $n = 2, 3$ and $\chi_{gvt(P_n)} = 3$ for $n = 4, 5, 6, 7$.

Theorem 4. *Let G be a graph without isolated vertices and isolated edges. Then*

$$\chi_{gvt}(G) \leq \chi'_{gvd}(G). \tag{5}$$

Proof. Suppose that f is a k-GVDEC of G. For any $u \in V(G)$, let $f(u) = f(uv)$, where $uv \in E(G)$. Obviously, f is a k-GVDTC of G. □

We now turn to investigating the general vertex-distinguishing total chromatic number of an n-vertex path.

Theorem 5. *Let P_n be a path on n vertices, $n \geq 1$. Then*

$$\chi_{gvt}(P_n) = \left\lceil \sqrt[3]{3n + \sqrt{9n^2 + \frac{125}{27}}} + \sqrt[3]{3n - \sqrt{9n^2 + \frac{125}{27}}} \right\rceil. \tag{6}$$

Proof. Denote by $P_n = v_1 v_2 \cdots v_n$ a path P_n with vertex set $\{v_1, v_2, \ldots, v_n\}$ and edge set $\{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n\}$. Let $\chi_{gvt}(P_n) = k$, and let f be a k-GVDTC of P_n. Let $\alpha_k = \binom{k-1}{1} + \binom{k-1}{2} + \binom{k-1}{3}$, $\beta_k = \binom{k}{1} + \binom{k}{2} + \binom{k}{3}$, $\gamma_k = 1 + \binom{k-1}{1} + \binom{k-1}{2}$, and

$$k^* = \left\lceil \sqrt[3]{3n + \sqrt{9n^2 + \frac{125}{27}}} + \sqrt[3]{3n - \sqrt{9n^2 + \frac{125}{27}}} \right\rceil. \tag{7}$$

Evidently, $|C_f(v_i)| \leq 3$, and $n \leq \beta_k$ (which implies $k \geq k^*$). In order to prove the conclusion, $k = k^*$, it suffices to give a k^*-GVDTC of P_n. When $n \leq 7$, it is not hard to construct the corresponding general vertex-distinguishing total colorings. Let $n \geq 8$. We first construct a k^*-GVDTC f' of P_{β_k} recursively. Note that when $n = \beta_k$, it has $k^* = k$.

Procedure 1. Construct a 4-GVDTC f_4 of P_{β_4} (i.e., P_{14}) as follows: the vertices $v_1, v_2, \ldots, v_{14}$ are colored by 1, 3, 4, 2, 3, 1, 4, 2, 3, 4, 3, 4, 4, 1, respectively; the edges $v_1v_2, v_2v_3, v_3v_4, \ldots, v_{13}v_{14}$ are colored by 1, 3, 1, 1, 2, 4, 2, 2, 2, 3, 3, 4, 4, respectively. It is easy to see that f_4 is a 4-GVDTC of P_{14}.

Procedure 2. Construct a k-GVDTC f_k of P_{β_k} based on a $(k-1)$-GVDTC f_{k-1} of $P_{\beta_{k-1}}$. Let f_k be

$$\begin{gathered} f_k(v_{i+\gamma_k}) = f_{k-1}(v_i) + 1, \quad i = 1, 2, \ldots, \alpha_k - 1; \\ f_k(v_{\alpha_k+\gamma_k}) = f_{k-1}(v_{\alpha_k}) = 1; \\ f_k(v_{i+\gamma_k} v_{i+1+\gamma_k}) = f_{k-1}(v_i v_{i+1}) + 1, \quad i = 1, 2, \ldots, \alpha_k - 1; \\ f_k(v_1) = 1, \qquad f_k(v_{\gamma_k}) = k; \end{gathered} \tag{8}$$

$v_2, v_3, \ldots, v_{\gamma_k - 1}$ are colored by

$$\underbrace{k-1, k}_{2 \text{ elements}}, \underbrace{k-2, k-1, k}_{3 \text{ elements}}, \ldots, \underbrace{2, 3, \ldots, k}_{k-1 \text{ elements}}; \tag{9}$$

when k is even, $v_1v_2, v_2v_3, v_3v_4, \ldots, v_{\gamma_k} v_{\gamma_{k+1}}$ are colored by

$$\underbrace{1}_{1\text{ element}}, \underbrace{k-1,1}_{2\text{ elements}}, \underbrace{1,k-2,1}_{3\text{ elements}}, \underbrace{k-3,1,k-3,1}_{4\text{ elements}}, \underbrace{1,k-4,1,k-4,1}_{5\text{ elements}},$$
$$\vdots,$$
$$\underbrace{1,k-(k-4),\ldots,1,k-(k-4),1}_{k-3\text{ elements}}, \underbrace{k-(k-3),1,\ldots,k-(k-3),1}_{k-2\text{ elements}}, \tag{10}$$
$$\underbrace{2,1,\ldots,2,1}_{k-2\text{ elements}}, \underbrace{2,2}_{2\text{ elements}};$$

when k is odd, $v_1v_2, v_2v_3, v_3v_4, \ldots, v_{\gamma_k}v_{\gamma_{k+1}}$ are colored by

$$\underbrace{1}_{1\text{ element}}, \underbrace{k-1,1}_{2\text{ elements}}, \underbrace{1,k-2,1}_{3\text{ elements}}, \underbrace{k-3,1,k-3,1}_{4\text{ elements}}, \underbrace{1,k-4,1,k-4,1}_{5\text{ elements}},$$
$$\vdots,$$
$$\underbrace{k-(k-4),1,\ldots,k-(k-4),1}_{k-3\text{ elements}}, \underbrace{1,k-(k-3),\ldots,1,k-(k-3),1}_{k-2\text{ elements}}, \tag{11}$$
$$\underbrace{1,2,\ldots,1,2,1}_{k-2\text{ elements}}, \underbrace{2,2}_{2\text{ elements}}.$$

It should be pointed out that when j is odd for $j \in \{1,2,\ldots,k-3\}$, the colors' form is $k-j,1,k-j,1,\ldots,k-j,1$ with totally $j+1$ elements, and when j is even for $j \in \{1,2,\ldots,k-3\}$, the colors' form is $1,k-j,1,k-j,\ldots,1,k-j,1$ with $j+1$ elements in total.

According to f_{k-1}, we can see that, for any $i,j = \gamma_k+1,\gamma_k+2,\ldots,\beta_k-1$, and $i \neq j$, it follows that $\{2,k\}$ is not a total color set of vertices v_i, $1 \notin C_{f_k}(v_i)$, and $C_{f_k}(v_i) \neq C_{f_k}(v_j)$. In addition, $C_1, C_2, \ldots, C_{\gamma-1}$ are as follows: $\{1\}$ (1 item), $\{\{1,k-1\},\{1,k-1,k\}\}$ (2 items), $\{\{1,k-2\},\{1,k-2,k-1\},\{1,k-2,k\}\}$ (3 items),...,$\{\{1,k-j\},\{1,k-j,k-j+1\},\{1,k-j,k-j+2\},\ldots,\{1,k-j,k\}\}$ ($j+1$ items),..., and $\{\{1,2\},\{1,2,3\},\{1,2,4\},\ldots,\{1,2,k\}\}$ ($k-1$ items). And $C_{f_k}(v_{\gamma_k}) = \{2,k\}$. So, f_k is a k-GVDTC of P_{β_k}. We now show that P_n also has a k-GVDTC based on a k-GVDTC of P_{β_k}, for $\binom{k-1}{1}+\binom{k-1}{2}+\binom{k-1}{3} < n < \beta_k$.

Let $r = \beta - n$, and let f be a k-GVDTC of P_{β_k} constructed by Procedures 1 and 2. We first delete r vertices $v_1, v_2, \ldots, v_r$ from P_{β_k}. Obviously, the resulting graph, denoted by $v_{r+1}v_{r+2}\cdots v_{\beta_k}$, is isomorphic to P_n. Let f' be $f'(v_iv_{i+1}) = f(v_iv_{i+1})$ for $i = r+1, r+2, \ldots, \beta_k-1$; $f'(v_i) = f(v_i)$ for $i = r+2,\ldots,\beta$; and $f'(v_{r+1}) = 1$. Then f' is a k-GVDTC of P_n.

All the above show that the conclusion holds. □

According to Theorem 5, we have the same conclusion on cycles. Let $C_n = v_1v_2\cdots v_nv_1$ be an n-vertex cycle with vertex set $\{v_1,v_2,\ldots,v_n\}$ and edge set $\{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\}$.

Corollary 6. *For any cycle C_n $(n \geq 3)$, one has*

$$\chi_{gvt}(C_n) = \left\lceil \sqrt[3]{3n+\sqrt{9n^2+\frac{125}{27}}} + \sqrt[3]{3n-\sqrt{9n^2+\frac{125}{27}}} \right\rceil. \tag{12}$$

Proof. Let $C_n = v_1v_2\cdots v_nv_1$ and $P_n = C_n \setminus v_1v_n$; let also

$$k^* = \left\lceil \sqrt[3]{3n+\sqrt{9n^2+\frac{125}{27}}} + \sqrt[3]{3n-\sqrt{9n^2+\frac{125}{27}}} \right\rceil \tag{13}$$

and let f be a k^*-GVETC of P_n, constructed by the method of Theorem 5. Then we can extend f to a k^*-GVETC of C_n by assigning color 1 to edge v_1v_n. So, the conclusion holds. □

In the following (Theorem 7 to Theorem 9), we discuss the general vertex-distinguishing total chromatic number of some kinds of special trees. A *star* S_n is the complete bipartite graph $K_{1,n}$ $(n \geq 1)$. A *double star* $S_{m,n}$ is a tree containing exactly two vertices that are not leaves (which are necessarily adjacent). A *tristar* $S_{p,q,r}$ is a tree with vertex set $V(S_{p,q,r}) = \{u_i \mid i = 0,1,\ldots,p\} \cup \{v_i \mid i = 0,1,\ldots,q\} \cup \{w_i \mid i = 0,1,\ldots,\ell\}$ and edge set $E(S_{p,q,r}) = \{u_0u_i \mid i = 1,2,\ldots,p\} \cup \{v_0v_i \mid i = 1,2,\ldots,q\} \cup \{w_0w_i \mid i = 1,2,\ldots,r\} \cup \{u_0v_0, v_0w_0\}$, where p,q,r are positive integers.

Theorem 7. *For a star S_n $(n \geq 1)$, one has*

$$\chi_{gvt}(S_n) = \begin{cases} 2, & n = 1,2; \\ \left\lceil \dfrac{\sqrt{8n+1}-1}{2} \right\rceil, & n \geq 3. \end{cases} \tag{14}$$

Proof. When $n = 1, 2$, the conclusion is trivial. For $n \geq 3$, let $\chi_{gvt}(S_n) = k$, and $\lceil (\sqrt{8n+1} - 1)/2 \rceil = k'$. Since, for any $u \in V(S_n) \setminus u_0$ (u_0 is the vertex with $d_{S_n}(u_0) \geq 2$), $|C_f(u)| \leq 2$ for any k-GVDTCf of S_n, it follows that $n \leq \binom{k}{1} + \binom{k}{2}$; that is, $k \geq k'$. In order to prove $k = k'$, we need to show that there exists a k'-GVDTC of S_n. Otherwise, let S_{n^*} be the graph with minimum n^* such that S_{n^*} does not have a k'-GVDTC, where $n^* \leq n$. Let u be a vertex of degree 1 in S_{n^*}. Consider the graph $G' = S_{n^*} - u$, obtained from S_{n^*} by deleting the vertex u and its incident edge. By the assumption of S_{n^*}, G' has a k'-GVDTC, denoted by f'. In addition, by interchanging the colors of some vertex and its incident edge appropriately, we can assume $|C'_f(u_0)| \geq 2$. Since $n^* - 1 < \binom{k'}{1} + \binom{k'}{2}$, there is at least one set $\{a, b\}$, for $a, b \in \{1, 2, \ldots, k'\}$, which is not the total color set of the vertices of G'. So, on the basis of f', in S_{n^*} we can color u and its incident edge uu_0 by a and b, respectively. Obviously, the resulting coloring is a k'-GVDTC of S_{n^*}. □

For two vertices u, v of a graph G, to *identify* these two vertices is to replace them by a single vertex (denoted by u-v in this paper) incident to all the edges which were incident in G to either u or v. The resulting graph is denoted by $G/\{u, v\}$. In what follows, we denoted by $[1, k]$ the set of $\{1, 2, \ldots, k\}$.

Theorem 8. *Let $S_{m,n}$ ($m \geq n \geq 1$) be a double star, and $\ell = m + n$. Then*

$$\chi_{gvt}(S_{m,n}) = \begin{cases} 3, & \ell = 2, 3, 4, 5; \\ 4, & \ell = 6; \\ \left\lceil \dfrac{\sqrt{8\ell+1} - 1}{2} \right\rceil, & \ell \geq 7. \end{cases} \quad (15)$$

Proof. When $\ell \leq 6$, the results are easy to be proved. When $\ell \geq 7$, let u, v be two vertices with degree more than 1, and $G' = S_{m,n}/\{u, v\}$. Evidently, the graph G' is isomorphic to the star S_ℓ. Let $k' = \lceil (\sqrt{8\ell+1} - 1)/2 \rceil$. Since $|C_f(x)| \leq 2$ for any $x \in V(S_{m,n}) \setminus \{u, v\}$, we have $\chi_{gvt}(S_{m,n}) \geq k'$.

By Theorem 7, G' contains a k'-GVDTCf'. Evidently, $|C_{f'}(x)| \leq 2$ for any $x \in V(G') \setminus \{u-v\}$ and $|C_{f'}(u-v)| \leq k'$. If $|C_{f'}(u-v)| \leq 2$, then we can extend f' to a k'-GVDTC of $S_{m,n}$ by coloring vertices u, v and edge uv with any three different colors in $[1, k] \setminus \{f'(u-v)\}$; if $|C_{f'}(u-v)| = \ell' \geq 3$, we without loss of generality assume $C_{f'}(u-v) \setminus f'(u-v) = [1, \ell']$. Let E_u (resp., E_v) be the set of edges (except edge uv) incident to u (resp., v) in $S_{m,n}$. We now extend f' to a k'-GVDTC of $S_{m,n}$ as follows. By the fact that there remain vertices u, v and edge uv uncolored in $S_{m,n}$ when f' is restricted to $S_{m,n}$, we consider the following two cases. First, one of u, v, say u, satisfies that $C_{f'}(E_u)$ contains at most two elements. Assume $C_{f'}(E_u) \subseteq \{1, 2\}$; we then color u, uv, v by $2, 3, c$, respectively, where $c = 4$ when $C_{f'}(E_u) \neq \{3, 4\}$ and $c = 2$ when $C_{f'}(E_u) = \{3, 4\}$. The resulting coloring of $S_{m,n}$ is also denoted by f'. Then it follows in $S_{m,n}$ that $|C_{f'}(u)| \geq 3$, $|C_{f'}(v)| \geq 3$, and $4 \notin C_{f'}(u)$ and $4 \in C_{f'}(v)$. So, f' is a k'-GVDTC of $S_{m,n}$. Second, $|C_{f'}(E_v)| \geq 3$ and $|C_{f'}(E_u)| \geq 3$; then we will further discuss two subcases.

(1) Consider $|C'_f(E_u)| = |C'_f(E_v)| = k'$. Suppose that V_u (and V_v) is the set of vertices, except v (or u), adjacent to u (and v) in $S_{m,n}$. Because f' is a k'-GVDTC of G', either V_u or V_v contains no vertices with total color set $\{i\}$, for some $i \in \{1, 2, \ldots, k'\}$. Without loss of generality we assume that there is no vertex $x \in V_u$ with $C'_f(x) = \{i\}$. For any vertex y in V_u such that $C'_f(y) = \{i, j\}$ and $f'(yu) = i$, interchange the two colors of y and yu. The resulting coloring, still denoted by f', satisfies that $C_{f'}(E_u)$ does not contain color i. Then we color u, uv, v by any three colors in $\{1, 2, \ldots, k'\} \setminus \{i\}$ and obtain a k'-GVDTC of $S_{m,n}$.

(2) Consider $|C'_f(E_u)| < k'$ or $|C'_f(E_v)| < k'$; assume $|C'_f(E_u)| < k'$ here. Let $i \notin C'_f(E_u)$. Color v by i and color u, uv by any two colors in $[1, k'] \setminus \{i\}$. Obviously, the resulting coloring is a k'-GVDTC of $S_{m,n}$.

All the above show that $\chi_{gvt}(S_{m,n}) \leq k'$. So, the conclusion holds. □

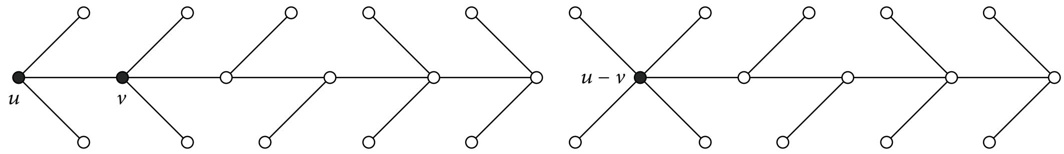
Theorem 9. *Let $S_{p,q,\ell}$ be a tristar defined as above, and $\ell = p + q + r$. Then*

$$\chi_{gvt}(S_{p,q,r}) = \begin{cases} 3, & \ell = 3, 4; \\ 4, & \ell = 5, 6; \\ \left\lceil \left(\sqrt{8r+1} - 1\right)/2 \right\rceil, & \ell \geq 7. \end{cases} \quad (16)$$

Proof. When $\ell = 3, 4$, the conclusion is easy to be checked; when $\ell = 5$ or 6, since $|V(S_{p,q,r})| \geq 8$ and $\binom{3}{1} + \binom{3}{2} + \binom{3}{3} = 7$, it follows that $\chi_{gvt}(S_{p,q,r}) > 3$. In addition, it is not hard to give a 4-GVDTC of $V(S_{p,q,r})$ in each case of $\ell = 5$ or 6, so $\chi_{gvt}(S_{p,q,r}) = 4$; when $\ell \geq 7$, let $k' = \lceil (\sqrt{8r+1} - 1)/2 \rceil$. Identify vertices u_0 and v_0 in $S_{p,q,r}$ and let $G' = S_{p,q,r}/\{u, v\}$. By Theorem 8, G' has a k'-GVDTCf'. With the analogous analysis method of Theorem 8, we can also extend f' to a k'-GVDTC of $S_{p,q,r}$. This shows $\chi_{gvt}(S_{p,q,r}) \leq k'$. On the other hand, for any k-GVDTCf of $S_{p,q,r}$, it has that $\ell \leq \binom{k}{1} + \binom{k}{2}$; that is, $k \geq k'$. So, the result holds. □

In the above, we construct a k-GVDTC of a graph G by extending a k-GVDTC of graph G', where G' is the resulting graph of identifying two vertices of degree more than 1 in G. But this method does not always work. For instance, the graph $G/\{u, v\}$ shown in Figure 1(b) has a 4-GVDTC, but the graph G shown in Figure 1(a) does not contain any 4-GVDTC. So any 4-GVDTC of $G/\{u, v\}$ can not be extended to a 4-GVDTC of G.

In the following we are devoted to the study of the general vertex-distinguishing chromatic number of fan graph F_n, wheel graph W_n, and complete graph K_n. Let G, H be two graphs such that $V(G) \cap V(H) = \emptyset$. The join $G + H$ of G and H is a graph with vertex set $V(G + H) = V(G) \cup V(H)$ and edge set $E(G + H) = E(G) \cup E(H) \cup \{uv \mid u \in V(G), v \in V(H)\}$. A *fan graph* F_n is defined as the join of a path of n vertices and an isolated vertex. A *wheel graph* W_n is defined as the join of a cycle of n vertices and an isolated vertex.



(a) (b)

FIGURE 1: (a) A graph, G; (b) $G/\{u,v\}$.

Theorem 10. *Let F_n be a fan, $n \geq 2$; then*

$$\chi_{gvt}(F_n) = \begin{cases} 3, & n = 2,3,\ldots,6; \\ 4, & n = 7,8,\ldots,14; \\ k, & n \geq 15, \end{cases} \quad (17)$$

where $\binom{k-1}{1}+\binom{k-1}{2}+\binom{k-1}{3}+\binom{k-1}{4} < n \leq \binom{k}{1}+\binom{k}{2}+\binom{k}{3}+\binom{k}{4}$.

Proof. Let $V(F_n) = \{v_i \mid i = 0,1,\ldots,n\}$ and $E(F_n) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n\} \cup \{v_0v_i \mid i = 1,2,\ldots,n\}$. When $n \leq 14$, the conclusion is easy to show. We now consider the case of $n \geq 15$ (which implies $k \geq 5$). Since $\binom{k-1}{1}+\binom{k-1}{2}+\binom{k-1}{3}+\binom{k-1}{4} < n$ and $|C_f(v_i)| \leq 4$ for $i \in [1,n]$, we can easily deduce that $\chi_{gvt}(F_n) \geq k$. So, it suffices to show that F_n contains a k-GVDTC. In particular, we prove that F_n contains a k-GVDTC such that the total color set of v_0 contains at least 5 elements. By induction on n. When $n = 15$, it is not hard to construct such a $k(\geq 5)$-GVDTC of F_n. Suppose that, for any $F_{n'}$, $n' < n$, there exists a k-GVDTC of $F_{n'}$. Consider the fan graph F_{n-1}, and let f be a k-GVDTC of F_{n-1}. Anyway, we can assume that, for any color $x \in [1,k]$, there is an edge v_iv_{i+1} for some $i \in [1,n-1]$ such that $f(v_iv_{i+1}) = x$ (If not, we can permutate x and $f(v_iv_{i+1})$).

Note that for any edge v_iv_{i+1} of F_{n-1}, $i \in [1,k]$, if we replace this edge by a vertex u and connect this vertex to v_0, v_i, and v_{i+1}, then the resulting graph, denoted by $F_{n-1}^{v_i,u,v_{i+1}}$, is isomorphic to F_n. We will use this to construct a k-GVDTC of F_n based on f. It is obvious that there remain only 4 uncolored elements, v_iu, uv_{i+1}, uv_0, and u in $F_{n-1}^{v_i,u,v_{i+1}}$, if we restrict f to $F_{n-1}^{v_i,u,v_{i+1}}$. We need to consider the following 2 cases.

Case 1. There exist colors $x,y,z \in [1,k]$ such that $\{x,y,z\} \notin \mathscr{C}_f(F_{n-1})$. Let v_iv_{i+1} be the edge with $f(v_iv_{i+1}) = x$, $i \in [1,n-1]$. In $F_{n-1}^{v_i,u,v_{i+1}}$, let $f(uv_i) = f(uv_{i+1}) = x$, $f(uv_0) = y$, and $f(u) = z$, and the resulting coloring is still denoted by f. Evidently, in $F_{n-1}^{v_i,u,v_{i+1}}$, $C_f(u) = \{x,y,z\}$; meanwhile $C_f(v_i)$ and $C_f(v_{i+1})$ are the same as those in F_{n-1}, and $|C_f(v_0)| \geq 5$. So, f a k-GVDTC of $F_{n-1}^{v_i,u,v_{i+1}}$.

Case 2. Four different colors $x,y,z,w \in [1,k]$ such that $\{x,y,z,w\} \notin \mathscr{C}_f(F_{n-1})$. Select an edge v_iv_{i+1}, $i \in [1,n-1]$, for which $f(v_iv_{i+1}) = x$. Since f is a k-GVDTC of F_{n-1}, $C_f(v_i) \setminus C_f(v_{i+1})$ contains at least one element (here we assume $|C_f(v_i)| \geq |C_f(v_{i+1})|$), say c. Obviously, $c \neq x$, $c \in C_f(v_i)$, and $c \notin C_f(v_{i+1})$ in F_{n-1}. We can permutate the colors so that $c \in \{y,x,w\}$ and $f(v_i) = c$ in F_{n-1}, say $c = y$. Then, in $F_{n-1}^{v_i,u,v_{i+1}}$, erase the color of vertex v_i and recolor it by color x, and let $f(uv_i) = y$, $f(uv_{i+1}) = x$, $f(uv_0) = z$, and $f(u) = w$. Obviously, in $F_{n-1}^{v_i,u,v_{i+1}}$, it follows that $C_f(u) = \{x,y,z,w\}$, $C_f(v_i)$ and $C_f(v_{i+1})$ are the same as those in F_{n-1}, and $|C_f(v_0)| \geq 5$. So, f a k-GVDTC of $F_{n-1}^{v_i,u,v_{i+1}}$.

All of the above show that F_n has a k-GVDTC. □

Theorem 11. *Let W_n be a wheel graph, $n \geq 2$; then*

$$\chi_{gvt}(W_n) = \begin{cases} 3, & n = 2,3,\ldots,6; \\ 4, & n = 7,8,\ldots,14; \\ k, & n \geq 15, \end{cases} \quad (18)$$

where $\binom{k-1}{1}+\binom{k-1}{2}+\binom{k-1}{3}+\binom{k-1}{4} < n \leq \binom{k}{1}+\binom{k}{2}+\binom{k}{3}+\binom{k}{4}$.

We omit the proof for Theorem 11, since it is analogous to that of Theorem 10.

Theorem 12. *For a complete graph K_n, $n \geq 1$, one has*

$$\chi_{gvt}(K_n) = 1 + \lceil \log_2 n \rceil. \quad (19)$$

Proof. When $n < 10$ the conclusion is easy to show. So we assume $n \geq 10$.

Denote by $V(K_n) = \{v_1, v_2, \ldots, v_n\}$ and $\{1,2,\ldots,k\}$ the set of k colors. For integer $\ell = \lceil n/2 \rceil$, we construct a ℓ-GVDTC f' of K_n as follows: let $f'(v_i) = i$ for any $i \in [1,\ell]$; $f'(v_iv_j) = 1$ for $i \neq j$, $i \in [1,\ell]$, and $j \in [1,n]$; $f(v_i) = i - \ell + 2$ for $i \in [\ell+1, n-2]$; $f'(v_iv_j) = 2$ for $i \neq j$, $i \in [\ell+1,n-2]$, and $j \in [\ell+1,n]$; $f'(v_{n-1}v_n) = 3$; $f'(v_{n-1}) = 4$; $f'(v_n) = 5$. One can readily check that $C'_f(v_1) = \{1\}$, $C'_f(v_i) = \{1,i\}$ for $i = 1,2,\ldots,\ell$; $C'_f(v_i) = \{1,2,i-\ell+2\}$ for $i = \ell+1, \ell+2, \ldots, n-2$; $C'_f(v_{n-1}) = \{1,2,3,4\}$; and $C'_f(v_{n-1}) = \{1,2,3,5\}$. Thus, f' is a ℓ-GVDTC of K_n, which shows $\chi_{gvt}(K_n) \leq \ell$.

Suppose that $\chi_{gvt}(K_n) = k$ and f is a k-GVDTC of K_n. Since for any two vertices v_i, v_j $(i \neq j \in [1,k])$, $C_f(v_i) \cap C_f(v_j) \neq \emptyset$, one can see that there is at most one vertex whose total color set contains only one color. If there is a vertex v with $|C(v)| = 1$, without loss of generality assume $C(v) = \{1\}$; then the total color set of each vertex contains color 1, which indicates

$$n \leq 1 + \binom{k-1}{1} + \binom{k-1}{2} + \cdots + \binom{k-1}{k-1} = 2^{k-1}. \quad (20)$$

If there is no vertex whose total color set contains only one color, then for each v_i it has $|C(v_i)| \geq 2$. Since, for any vertex $v_j \neq v_i$, $|C(v_i) \cap C(v_j)| \geq 1$, it follows that $C(v)$ and $[1,k] \setminus C(v)$ can not be two total color sets with respect to f. This implies $n \leq 2^{k-1}$. So $k \geq 1 + \lceil \log_2 n \rceil$.

To prove $k = 1 + \lceil \log_2 n \rceil$, we need to show that K_n has a $(1+\lceil \log_2 n \rceil)$-GVDTC. In particular, we show that any K_n has a $(1 + \lceil \log_2 n \rceil)$-GVDTC such that for each color $c \in [1, 1 + \lceil \log_2 n \rceil]$ there is at least one vertex in K_n being colored by c.

We prove this by induction on n. When $n = 10$, 5-GVDTC is the f' defined above for $n = 10, \ell = 5$, where $C'_f(v_1) = \{1\}$. Consider the graph $K_n - v_n$ obtained from K_n by deleting vertex v_n and its incident edges. Obviously, $K_n - v_n$ is isomorphic to K_{n-1}. By the induction hypothesis, K_{n-1} has a $(1 + \lceil \log_2 n - 1 \rceil)$-GVDTC, say f, such that for each color $c \in [1, 1 + \lceil \log_2 n - 1 \rceil]$ there is at least one vertex of K_{n-1} being colored by c. Since $n \le 2^{k-1}$, there must be some set C (denoted by $\{c_1, c_2, \ldots, c_\ell\}$) $\notin \mathscr{C}_f(K_{n-1})$. We consider the following two cases.

(1) Consider $\lceil \log_2 n - 1 \rceil = \lceil \log_2 n \rceil$. By the induction hypothesis, each color $c_i \in [1, 1 + \lceil \log_2 n - 1 \rceil]$ appears at a vertex. Without loss of generality assume $f(v_i) = c_i$ for $i = 1, 2, \ldots, \ell$. Then, f is extended to a $(1 + \lceil \log_2 n - 1 \rceil)$-GVDTC of K_n via coloring v_n by one of the colors in $[c_1, c_2, \ldots, c_\ell]$; coloring $v_n v_i$ for $i = 1, 2, \ldots, \ell$ by c_i; and coloring $v_n v_j$ for $j = \ell+1, \ldots, n-1$ by one of the colors in $C_f(v_j)$ $(v_j \in V(K_{n-1}))$.

(2) Consider $\lceil \log_2 n - 1 \rceil = \lceil \log_2 n \rceil - 1$. Then on the basis of f, we only need to color v_n and all of its incident edges in K_n by color $1 + \lceil \log_2 n - 1 \rceil$.

One can readily check that the resulting coloring of K_n in the above two cases is $(1 + \lceil \log_2 n \rceil)$-GVDTCs of K_n such that each color in $[1, 1 + \lceil \log_2 n \rceil]$ appears at a vertex of K_n. Hence, K_n has a $(1+\lceil \log_2 n \rceil)$-GVDTC, and the conclusion holds. □

In the following, we present a trivial upper bound of the general vertex-distinguishing total chromatic number of the join graph of two graphs.

Theorem 13. *Suppose G, H are two simple graphs and $G \cap H = \emptyset$. Then*

$$\chi_{gvt}(G + H) \le \chi_{gvt}(G) + \chi_{gvt}(H). \tag{21}$$

Proof. Let $V(G) = \{u_i \mid i = 1, 2, \ldots, m\}$ and $V(H) = \{v_i \mid i = 1, 2, \ldots, n\}$. Suppose that f_1 is a $\chi_{gvt}(G)$-GVDTC of G and f_2 is a $\chi_{gvt}(H)$-GVDTC of H, where the sets of colors of f_1 and f_2 are C_1 and C_2 $(C_1 \cap C_2 = \emptyset)$, respectively.

Define f as $f(u_i v_j) = f_1(u_i)$ (or $f_2(v_j)$), $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$.

Combining colorings f_1, f_2, f, we can obtain a $(\chi_{gvt}(G) + \chi_{gvt}(H))$-GVDTC of $G + H$. □

3. Remarks

Based on the above results, we propose two conjectures as follows.

Conjecture 14. *Let G be a graph without isolated vertices. Then*

$$\chi_{gvt}(G) \le \left\lceil \frac{n}{2} \right\rceil. \tag{22}$$

Conjecture 15. *Let G be a connected graph on n vertices. Then*

$$\chi_{gvt}(G) \le 1 + \lceil \log_2 n \rceil. \tag{23}$$

Note that if Conjecture 15 is true, then Conjecture 14 is true. On the other hand, if Conjecture 14 is true, then the upper bound cannot be improved. For instance, the graph G contains exactly three K_2 components. It is easy to show that $\chi_{gvt}(G) = 3$.

In addition, there is a very interesting observation about the general vertex-distinguishing total chromatic number.

Observation 1. Let H be a subgraph of a graph G. Then it possibly follows that

$$\chi_{gvt}(H) > \chi_{gvt}(G). \tag{24}$$

As an illustration of this observation, we consider the path P_{15} and the fan graph F_{14}. P_{15} is a subgraph of F_{14}, while $\chi_{gvt}(P_{15}) (= 5) > \chi_{gvt}(F_{14}) (= 4)$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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