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Research Article

Monotonicity of the Ratio of the Power and Second Seiffert Means with Applications

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We present the necessary and sufficient condition for the monotonicity of the ratio of the power and second Seiffert means. As applications, we get the sharp upper and lower bounds for the second Seiffert mean in terms of the power mean.

1. Introduction

Throughout this paper, we assume that $a, b > 0$ with $a \neq b$. The second Seiffert mean $T(a,b)$ and rth power mean $M_r(a,b)$ of a and b are defined by

$$T(a,b) = \frac{a-b}{2\arctan((a-b)/(a+b))}, \qquad (1)$$

$$M_r(a,b) = \left(\frac{a^r+b^r}{2}\right)^{1/r} \quad (r \neq 0), \qquad M_0(a,b) = \sqrt{ab}, \qquad (2)$$

respectively.

It is well-known that the power mean $M_r(a,b)$ is strictly increasing with respect to $r \in \mathbb{R}$ for fixed $a, b > 0$ with $a \neq b$. In the recent past, both mean values have been the subject of intensive research. In particular, many remarkable inequalities for T and M_r can be found in the literature [1–5].

Seiffert [6] proved that the double inequality

$$M_1(a,b) < T(a,b) < M_2(a,b) \qquad (3)$$

holds for all $a, b > 0$ with $a \neq b$.

In [7], Hästö proved that the function $T(1,x)/M_p(1,x)$ is strictly increasing on $[1,\infty)$ if $p \leq 1$ and presented an improvement for the first inequality in (3).

Costin and Toader [8] proved that the inequality

$$T(a,b) > M_{3/2}(a,b) \qquad (4)$$

holds for all $a, b > 0$ with $a \neq b$.

In [9], Witkowski proved that the double inequality

$$\frac{2\sqrt{2}}{\pi}M_2(a,b) < T(a,b) < \frac{4}{\pi}M_1(a,b) \qquad (5)$$

holds for all $a, b > 0$ with $a \neq b$.

Recently, the following optimal estimations for the second Seiffert mean by power means were obtained independently in [10, 11]:

$$M_{\log 2/(\log\pi - \log 2)}(a,b) < T(a,b) < M_{5/3}(a,b) \qquad (6)$$

for all $a, b > 0$ with $a \neq b$.

The main purpose of this paper is to give the necessary and sufficient condition for the monotonicity of the function $T(1,x)/M_p(1,x)$ on $(0,1)$ and present the best possible parameters α and β such that the double inequality

$$\alpha M_{5/3}(a,b) < T(a,b) \leq \beta M_{\log 2/(\log\pi - \log 2)}(a,b) \qquad (7)$$

holds for all $a, b > 0$ with $a \neq b$.

2. Main Results

In order to prove our main results we first establish a lemma.

Lemma 1. *Let $f(p,x)$ be defined on $\mathbb{R} \times (0,1)$ by*

$$f(p,x) = \frac{(1-x)(1+x^p)}{(1+x^2)(1+x^{p-1})} - \arctan\frac{1-x}{1+x}. \qquad (8)$$

Then there exists $\lambda \in (0,1)$ such that $f(p,x)$ is strictly decreasing with respect to x on $(0,\lambda]$ and strictly increasing with respect to x on $[\lambda,1)$ if $p \in (1,5/3)$.

Proof. Let

$$\begin{aligned} f_1(p,x) &= (1-p)x^p + (1+p)x^{p-1} \\ &\quad - (1+p)x^{p-2} + (p-1)x^{p-3} \\ &\quad - 2x^{2p-3} + 2. \end{aligned} \tag{9}$$

Then,

$$f_1(p,1) = 0, \qquad f_1(p,0^+) = \infty, \tag{10}$$

$$\frac{\partial f(p,x)}{\partial x} = -\frac{x(1-x)}{(1+x^2)^2(1+x^{p-1})^2} f_1(p,x), \tag{11}$$

$$\begin{aligned} x^{4-p}\frac{\partial f_1(p,x)}{\partial x} &= -2(2p-3)x^p - p(p-1)x^3 \\ &\quad + (p-1)(p+1)x^2 \\ &\quad - (p+1)(p-2)x + (p-1)(p-3) \\ &:= f_2(p,x), \end{aligned} \tag{12}$$

$$\begin{aligned} f_2(p,0) &= (p-1)(p-3) < 0, \\ f_2(p,1) &= 2(5-3p) > 0, \end{aligned} \tag{13}$$

$$\begin{aligned} \frac{\partial f_2(p,x)}{\partial x} &= -2p(2p-3)x^{p-1} - 3p(p-1)x^2 \\ &\quad + 2(p-1)(p+1)x - (p+1)(p-2). \end{aligned} \tag{14}$$

We divide two cases to prove that $\partial f_2(p,x)/\partial x > 0$ for all $x \in (0,1)$ and $p \in (1,5/3)$. □

Case 1. Consider that $p \in (1,3/2]$. From (14) we clearly see that

$$\frac{\partial^3 f_2(p,x)}{\partial x^3} = -2p(p-1)\left[3 + (2-p)(3-2p)x^{p-3}\right] < 0, \tag{15}$$

$$\begin{aligned} \frac{\partial f_2}{\partial x}(p,0) &= (p+1)(2-p) > 0, \\ \frac{\partial f_2}{\partial x}(p,1) &= 2p(5-3p) > 0. \end{aligned} \tag{16}$$

Equation (15) implies that $\partial f_2(p,x)/\partial x$ is strictly concave with respect to x on the interval $(0,1)$. Then (16) and the basic properties of concave function lead to the conclusion that

$$\frac{\partial f_2}{\partial x}(p,x) > (1-x)\frac{\partial f_2}{\partial x}(p,0) + x\frac{\partial f_2}{\partial x}(p,1) > 0. \tag{17}$$

Case 2. Consider that $p \in (3/2,5/3)$. Making use of the weighted arithmetic-geometric inequality $\lambda a + (1-\lambda)b \geq a^\lambda b^{1-\lambda}$ $(0 \leq \lambda \leq 1)$ we get

$$x^{p-1} \leq (p-1)x + (2-p). \tag{18}$$

Equations (14) and (18) lead to

$$\begin{aligned} &\frac{\partial f_2(p,x)}{\partial x} \\ &\geq -2p(2p-3)\left[(p-1)x + (2-p)\right] \\ &\quad - 3p(p-1)x^2 + 2(p-1)(p+1)x - (p+1)(p-2) \\ &= -3p(p-1)x^2 - 2(p-1)(2p^2-4p-1)x \\ &\quad + (p-2)(4p^2-7p-1) := f_3(p,x). \end{aligned} \tag{19}$$

Note that

$$\begin{aligned} \frac{\partial^2 f_3(p,x)}{\partial x^2} &= -6p(p-1) < 0, \\ f_3(p,1) &= 2p(5-3p) > 0, \\ f_3(p,0) &= 4(p-2)\left(p - \frac{\sqrt{65}+7}{8}\right)\left(p + \frac{\sqrt{65}+7}{8}\right) > 0. \end{aligned} \tag{20}$$

It follows from (20) and the concavity of the function $f_3(p,x)$ with respect to x on the interval $(0,1)$ that

$$f_3(p,x) > (1-x)f_3(p,0^+) + xf_3(p,1) > 0. \tag{21}$$

Therefore, $\partial f_2(p,x)/\partial x > 0$ follows from (19) and (21).

Next we prove the desired result. From (12) and (13) together with the fact that $\partial f_2(p,x)/\partial x > 0$ we clearly see that there exists $\lambda_1 \in (0,1)$ such that $f_1(p,x)$ is strictly decreasing with respect to x on $(0,\lambda_1]$ and strictly increasing with respect to x on $[\lambda_1,1)$. Therefore, Lemma 1 follows easily from (10) and (11) together with the piecewise monotonicity of $f_1(p,x)$ with respect to x on the interval $(0,1)$.

Theorem 2. *Let $F(p,x)$ be defined on $\mathbb{R} \times (0,1)$ by*

$$\begin{aligned} F(p,x) &= \log\frac{T(1,x)}{M_p(1,x)} = \log\frac{1-x}{2\arctan((1-x)/(1+x))} \\ &\quad - \frac{1}{p}\log\frac{1+x^p}{2} \quad (p \neq 0), \end{aligned} \tag{22}$$

$$\begin{aligned} F(0,x) &= \lim_{p\to 0} F(p,x) \\ &= \log\frac{1-x}{2\arctan((1-x)/(1+x))} - \frac{1}{2}\log x. \end{aligned} \tag{23}$$

Then the following statements are true.

(1) *$F(p,x)$ is strictly increasing with respect to x on $(0,1)$ if and only if $p \geq 5/3$.*

(2) $F(p,x)$ *is strictly decreasing with respect to* x *on* $(0,1)$ *if and only if* $p \le 1$.

(3) *If* $p \in (1, 5/3)$, *then there exists* $\mu \in (0,1)$ *such that* $F(p,x)$ *is strictly increasing with respect to* x *on* $(0,\mu]$ *and strictly decreasing with respect to* x *on* $[\mu, 1)$.

Proof. It follows from (22) and (23) that

$$\frac{\partial F(p,x)}{\partial x} = \frac{1+x^{p-1}}{x(1-x)(1+x^p)\arctan((1-x)/(1+x))} f(p,x), \tag{24}$$

where $f(p,x)$ is defined by (8). And

$$\frac{\partial f(p,x)}{\partial x} = -\frac{x(1-x)}{(1+x^2)^2(1+x^{p-1})} g(p,x), \tag{25}$$

where

$$\begin{aligned} g(p,x) = {} & (1-p)x^p + (1+p)x^{p-1} \\ & - 2x^{2p-3} - (1+p)x^{p-2} \\ & + (p-1)x^{p-3} + 2. \end{aligned} \tag{26}$$

(1) If $F(p,x)$ is strictly increasing with respect to x on $(0,1)$, then (24) leads to $f(p,x) > 0$ for all $x \in (0,1)$. Making use of L'Höspital's rule and (8) we get

$$\lim_{x \to 1^-} \frac{f(p,x)}{(1-x)^3} = \frac{1}{24}(3p-5) \ge 0, \tag{27}$$

which implies that $p \ge 5/3$.

If $p \ge 5/3$, then from (8) and (26) together with the fact that the function $p \to (1+x^p)/(1+x^{p-1})$ is strictly increasing on $\mathbb{R}$ we get

$$f(p,x) \ge f\left(\frac{5}{3}, x\right), \tag{28}$$

$$\begin{aligned} g\left(\frac{5}{3}, x\right) = {} & \frac{2}{3} x^{-4/3} (1 - x^{1/3})^3 (1 + x^{2/3}) \\ & \times (1 + 3x^{1/3} + 5x^{2/3} + 3x + x^{4/3}) > 0 \end{aligned} \tag{29}$$

for all $x \in (0,1)$.

Equations (8) and (25) together with inequality (29) lead to the conclusion that

$$f\left(\frac{5}{3}, x\right) > f\left(\frac{5}{3}, 1\right) = 0 \tag{30}$$

for all $x \in (0,1)$.

Therefore, $F(p,x)$ is strictly increasing with respect to x on $(0,1)$ which follows easily from (24), (28), and (30).

(2) If $F(p,x)$ is strictly decreasing with respect to x on $(0,1)$, then (24) implies that $f(p,x) < 0$ for all $x \in (0,1)$. In particular, we have $f(p,0^+) \le 0$ and $p \le 1$. Indeed, if $p > 1$, then (8) leads to the conclusion that $f(p,0^+) = 1 - \pi/4 > 0$.

If $p \le 1$, then from (8) and (26) together with the fact that the function $p \to (1+x^p)/(1+x^{p-1})$ is strictly increasing on $\mathbb{R}$ we get

$$f(p,x) \le f(1,x), \tag{31}$$

$$g(1,x) = 4\left(1 - \frac{1}{x}\right) < 0 \tag{32}$$

for all $x \in (0,1)$.

Equations (8) and (25) together with inequality (32) lead to the conclusion that

$$f(1,x) < f(1,1) = 0 \tag{33}$$

for all $x \in (0,1)$.

Therefore, $F(p,x)$ is strictly decreasing with respect to x on $(0,1)$ which follows easily from (24), (31), and (33).

(3) If $p \in (1, 5/3)$, then (8) leads to

$$f(p,0) = 1 - \frac{\pi}{4} > 0, \qquad f(p,1) = 0. \tag{34}$$

It follows from Lemma 1 and (34) that we clearly see that there exists $\mu \in (0,1)$ such that $f(p,x) > 0$ for $x \in (0,\mu)$ and $f(p,x) < 0$ for $x \in (\mu, 1)$. Then from (24) we get Theorem 2(3) immediately. □

Theorem 3. *For all* $a, b > 0$ *with* $a \ne b$, *the double inequality*

$$\alpha M_{5/3}(a,b) < T(a,b) \le \beta M_{\log 2/(\log \pi - \log 2)}(a,b) \tag{35}$$

holds with the best possible constants $\beta = e^{F(\log 2/(\log \pi - \log 2), \mu)} = 1.0136\ldots$ *and* $\alpha = 2^{8/5}/\pi = 0.9649\ldots$, *where* μ *is the solution of the equation* $f(\log 2/(\log \pi - \log 2), x) = 0$ *on* $(0,1)$ *and* $f(p,x)$ *and* $F(p,x)$ *are defined by* (8) *and* (22), *respectively.*

Proof. Without loss of generality, we assume that $a > b > 0$. Let $x = b/a \in (0,1)$; then from (1) and (2) we get

$$\log T(a,b) - \log M_p(a,b) = F(p,x). \tag{36}$$

If $p = 5/3$, then from (22) and Theorem 2(1) we get

$$F\left(\frac{5}{3}, x\right) > F\left(\frac{5}{3}, 0\right) = \log \frac{2^{8/5}}{\pi}. \tag{37}$$

Therefore, the first inequality in (35) with the best possible constant $\alpha = 2^{8/5}/\pi$ follows from (36) and (37) together with the monotonicity of $F(5/3, x)$ given in Theorem 2(1).

If $p = \log 2/(\log \pi - \log 2) \in (1, 5/3)$, then Lemma 1 and (24) together with (34) imply that there exists $\mu \in (0,1)$ such that $f(\log 2/(\log \pi - \log 2), x) = 0$, and $F(\log 2/(\log \pi - \log 2), x)$ is strictly increasing on $(0,\mu]$ and strictly decreasing on $[\mu, 1)$. Therefore, we have

$$F\left(\frac{\log 2}{\log \pi - \log 2}, x\right) \le F\left(\frac{\log 2}{\log \pi - \log 2}, \mu\right). \tag{38}$$

Making use of MATHEMATICA software, numerical computations show that

$$0.186930110570624 < \mu < 0.186930110570625, \quad e^{F(\log 2/(\log \pi - \log 2),\mu)} = 1.0136\ldots. \tag{39}$$

Therefore, the second inequality in (35) with the best possible constant $\beta = e^{F(\log 2/(\log \pi - \log 2),\mu)} = 1.0136\ldots$ follows from (36) and (38) together with the piecewise monotonicity of $F(\log 2/(\log \pi - \log 2), x)$. □

Corollary 4. *The double inequality*

$$\frac{Q^2(a,b)}{L_{p-1}(a,b)} < T(a,b) < \frac{Q^2(a,b)}{L_{q-1}(a,b)} \tag{40}$$

holds for all $a, b > 0$ *with* $a \neq b$ *if and only if* $p \geq 5/3$ *and* $q \leq 1$, *where* $Q(a,b) = \sqrt{(a^2+b^2)/2}$ *and* $L_p(a,b) = (a^{p+1} + b^{p+1})/(a^p + b^p)$ *are, respectively, the quadratic and pth Lehmer means of a and b.*

Proof. Without loss of generality, we assume that $a > b > 0$. Let $x = b/a \in (0,1)$. Then from Theorem 2 and (24) we clearly see that the $f(p,x) > 0$ if and only if $p \geq 5/3$ and $f(p,x) < 0$ if and only if $p \leq 1$. Then (8) leads to the conclusion that the inequalities

$$\frac{(1-x)(1+x^p)}{(1+x^2)(1+x^{p-1})} > \arctan\frac{1-x}{1+x}, \tag{41}$$

$$\frac{(1-x)(1+x^q)}{(1+x^2)(1+x^{q-1})} < \arctan\frac{1-x}{1+x} \tag{42}$$

hold for all $x \in (0,1)$ if and only $p \geq 5/3$ and $q \leq 1$.

Therefore, Corollary 4 follows easily from inequalities (41) and (42) together with (1). □

Corollary 5. *Let* $a_1, b_1, a_2, b_2 > 0$ *with* $a_1/b_1 < a_2/b_2$. *Then Theorem 2 leads to the following Ky Fan type inequality:*

$$\frac{T(a_1,b_1)}{T(a_2,b_2)} < (>) \frac{M_p(a_1,b_1)}{M_p(a_2,b_2)} \tag{43}$$

if $p \geq 5/3$ $(p \leq 1)$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] E. Neuman and J. Sándor, "On the Schwab-Borchardt mean," *Mathematica Pannonica*, vol. 14, no. 2, pp. 253–266, 2003.

[2] E. Neuman and J. Sándor, "On the Schwab-Borchardt mean. II," *Mathematica Pannonica*, vol. 17, no. 1, pp. 49–59, 2006.

[3] W.-D. Jiang and F. Qi, "Some sharp inequalities involving Seiffert and other means and their concise proofs," *Mathematical Inequalities & Applications*, vol. 15, no. 4, pp. 1007–1017, 2012.

[4] W.-D. Jiang, "Some sharp inequalities involving reciprocals of the Seiffert and other means," *Journal of Mathematical Inequalities*, vol. 6, no. 4, pp. 593–599, 2012.

[5] A. Witkowski, "Optimal weighted harmonic interpolations between Seiffert means," *Colloquium Mathematicum*, vol. 130, no. 2, pp. 265–279, 2013.

[6] H.-J. Seiffert, "Aufgabe β16," *Die Wurzel*, vol. 29, pp. 221–222, 1995.

[7] P. A. Hästö, "A monotonicity property of ratios of symmetric homogeneous means," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 3, no. 5, article 71, 23 pages, 2002.

[8] I. Costin and G. Toader, "A nice separation of some Seiffert-type means by power means," *International Journal of Mathematics and Mathematical Sciences*, vol. 2012, Article ID 430692, 6 pages, 2012.

[9] A. Witkowski, "Interpolations of Schwab-Borchardt mean," *Mathematical Inequalities & Applications*, vol. 16, no. 1, pp. 193–206, 2013.

[10] I. Costin and G. Toader, "Optimal evaluations of some Seiffert-type means by power means," *Applied Mathematics and Computation*, vol. 219, no. 9, pp. 4745–4754, 2013.

[11] Z.-H. Yang, "Sharp bounds for the second Seiffert mean in terms of power means," http://arxiv.org/pdf/1206.5494v1.pdf.