

Research Article

Some Further Generalizations of Hölder's Inequality and Related Results on Fractal Space

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We establish some new generalizations and refinements of the local fractional integral Hölder's inequality and some related results on fractal space. We also show that many existing inequalities related to the local fractional integral Hölder's inequality are special cases of the main inequalities which are presented here.

1. Introduction

Let p_j ($j = 1, 2, \dots$) be constrained by

$$\sum_{j=1}^m \frac{1}{p_j} = 1. \quad (1)$$

Suppose also that $f_j(x) > 0$ and f_j ($j = 1, 2, \dots, m$) are continuous real-valued functions on $[a, b]$. Then each of the following assertions holds true.

(1) For $p_j > 0$ ($j = 1, 2, \dots, m$), we have the following inequality known as the Hölder inequality (see [1]):

$$\int_a^b \prod_{j=1}^m f_j(x) dx \leq \prod_{j=1}^m \left(\int_a^b f_j^{p_j}(x) dx \right)^{1/p_j}. \quad (2)$$

(2) For $0 < p_m < 1$ and $p_j < 0$ ($j = 1, 2, \dots, m-1$), we have the following reverse Hölder inequality (see [2]):

$$\int_a^b \prod_{j=1}^m f_j(x) dx \geq \prod_{j=1}^m \left(\int_a^b f_j^{p_j}(x) dx \right)^{1/p_j}. \quad (3)$$

In the special case when $m = 2$ and $p_1 = p_2$, inequality (2) reduces to the celebrated Cauchy inequality (see [3]). Both the

Cauchy inequality and the Hölder inequality play significant roles in many different branches of modern pure and applied mathematics. A great number of generalizations, refinements, variations, and applications of each of these inequalities have been studied in the literature (see [3–13] and the references cited therein). Recently, Yang [14] established the following local fractional integral Hölder's inequality on fractal space.

Let $f(x), g(x) \in C_\alpha(a, b)$, $p > 1$, and $1/p + 1/q = 1$. Then

$$\begin{aligned} & \frac{1}{\Gamma(1+\alpha)} \int_a^b |f(x)g(x)|(dx)^\alpha \\ & \leq \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b |f(x)|^p(dx)^\alpha \right)^{1/p} \\ & \quad \times \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b |g(x)|^q(dx)^\alpha \right)^{1/q}. \end{aligned} \quad (4)$$

More recently, Chen [15] gave a generalization of inequality (4) and its corresponding reverse form as follows.

Let $f_j(x) \in C_\alpha(a, b)$, $p_j \in \mathbb{R}$ ($j = 1, 2, \dots, m$), and

$$\sum_{j=1}^m \frac{1}{p_j} = 1. \quad (5)$$

Then each of the following assertions holds true. (1) For $p_j > 1$ ($j = 1, 2, \dots, m$), we have

$$\begin{aligned} & \frac{1}{\Gamma(1 + \alpha)} \int_a^b \prod_{j=1}^m |f_j(x)| (dx)^\alpha \\ & \leq \prod_{j=1}^m \left(\int_a^b \frac{1}{\Gamma(1 + \alpha)} |f_j(x)|^{p_j} (dx)^\alpha \right)^{1/p_j}. \end{aligned} \tag{6}$$

(2) For $0 < p_1 < 1$ and $p_j < 0$ ($j = 2, \dots, m$), we have

$$\begin{aligned} & \frac{1}{\Gamma(1 + \alpha)} \int_a^b \prod_{j=1}^m |f_j(x)| (dx)^\alpha \\ & \geq \prod_{j=1}^m \left(\int_a^b \frac{1}{\Gamma(1 + \alpha)} |f_j(x)|^{p_j} (dx)^\alpha \right)^{1/p_j}. \end{aligned} \tag{7}$$

The study of local fractional calculus has been an interesting topic (see [14–25]). In fact, local fractional calculus [14, 16, 17] has turned out to be a very useful tool to deal with the continuously nondifferentiable functions and fractals. This formalism has had a great variety of applications in describing physical phenomena, for example, elasticity [17, 26, 27], continuum mechanics [26], quantum mechanics [28, 29], wave phenomena and heat-diffusion analysis [30–34], and other branches of pure and applied mathematics [15, 35–37] and nonlinear dynamics [38, 39]. For more details and other applications of local fractional calculus, the interested reader may refer to the recent works [14–42] (see also the monograph [43] dealing extensively with fractional differential equations).

The purpose of this paper is to give some new generalizations and refinements of inequalities (6) and (7). Some related inequalities are also considered. This paper is structured as follows. In Section 2, we introduce some basic facts about local fractional calculus. In Section 3, we establish some new generalizations and refinements of the local fractional integral Hölder inequality and their corresponding reverse forms. Finally, we give our concluding remarks and observations in Section 4.

2. Preliminaries

In this section, we recall some known results of local fractional calculus (see [14, 16, 17]). Throughout this section we will always assume that F is a subset of the real line and is a fractal.

Lemma 1 (see [17]). *Assume that $f : (F, d) \rightarrow (\Omega', d')$ is a bi-Lipschitz mapping; then there are two positive constants ρ, τ , and $F \subset R$,*

$$\rho^s H^s(F) \leq H^s(f(F)) \leq \tau^s H^s(F), \tag{8}$$

such that

$$\rho^\alpha |x_1 - x_2|^\alpha \leq |f(x_1) - f(x_2)| \leq \tau^\alpha |x_1 - x_2|^\alpha \tag{9}$$

holds true for all $x_1, x_2 \in F$.

Based on Lemma 1, it is easy to show that [14]

$$|f(x_1) - f(x_2)| \leq \tau^\alpha |x_1 - x_2|^\alpha, \tag{10}$$

such that the following inequality holds true [14]:

$$|f(x_1) - f(x_2)| \leq \varepsilon^\alpha, \tag{11}$$

where α is fractal dimension of F .

Definition 2 (see [14, 17]). Assume that $\varepsilon, \delta > 0, |x - x_0|^\alpha \leq \delta$, and $\varepsilon, \delta \in R$; if

$$|f(x) - f(x_0)| \leq \varepsilon^\alpha, \tag{12}$$

then $f(x)$ is called local fractional continuous at $x = x_0$, denoted by $\lim_{x \rightarrow x_0} f(x) = f(x_0)$. If $f(x)$ is local fractional continuous on the interval (a, b) , then we write (see, e.g., [14])

$$f(x) \in C_\alpha(a, b), \tag{13}$$

where $C_\alpha(a, b)$ denotes the space of local fractional continuous functions on (a, b) .

Definition 3 (see [16, 17]). Suppose that $f(x)$ is a nondifferentiable function of exponent α ($0 < \alpha \leq 1$). If the following inequality holds true

$$|f(x) - f(y)| \leq C|x - y|^\alpha, \tag{14}$$

then $f(x)$ is a Hölder function of exponent α for $x, y \in F$.

Definition 4 (see [16, 17]). If $f(x)$ satisfies the following inequality

$$|f(x) - f(x_0)| \leq o((x - x_0)^\alpha), \tag{15}$$

then $f(x)$ is continuous of order α ($0 < \alpha \leq 1$) or, briefly, α -continuous.

Definition 5 (see [14, 16–18]). Suppose that $f(x)$ is local fractional continuous on the interval (a, b) ; then the local fractional derivative of $f(x)$ of order α at $x = x_0$ is given by

$$\begin{aligned} f^{(\alpha)}(x_0) &= \left. \frac{d^\alpha f(x)}{dx^\alpha} \right|_{x=x_0} \\ &= \lim_{x \rightarrow x_0} \frac{\Gamma(1 + \alpha) \Delta(f(x) - f(x_0))}{(x - x_0)^\alpha}, \end{aligned} \tag{16}$$

provided this limit exists.

From Definition 5, we have the following conclusion (see [14]):

$$f^{(\alpha)}(x) = D_x^{(\alpha)} f(x), \tag{17}$$

which is denoted by (see [14])

$$f(x) \in D_x^{(\alpha)}(a, b), \tag{18}$$

where $D_x^{(\alpha)}(a, b)$ denotes the space of local fractional derivable functions on (a, b) .

Definition 6 (see [14, 16–18]). Suppose that $f(x)$ is local fractional continuous on the interval (a, b) ; then the local fractional integral of the function $f(x)$ in the interval $[a, b]$ is defined by

$$\begin{aligned} {}_a I_b^{(\alpha)} f(x) &= \frac{1}{\Gamma(1+\alpha)} \int_a^b f(t) (dt)^\alpha \\ &= \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t \rightarrow 0} \sum_{j=0}^{N-1} f(t_j) (\Delta t_j)^\alpha, \end{aligned} \tag{19}$$

where $\Delta t_j = t_{j+1} - t_j$, $\Delta t = \max\{\Delta t_1, \Delta t_2, \dots, \Delta t_j, \dots\}$, and $[t_j, t_{j+1}]$ ($j = 1, 2, \dots, N - 1$; $t_0 = a$; $t_N = b$) are a partition of the interval $[a, b]$.

Let ${}_a I_x^{(\alpha)}(a, b)$ denote the space of local fractional integrable functions on (a, b) ; from Definition 6, we can obtain the following result (see, for details, [14]):

$$f(x) \in {}_a I_x^{(\alpha)}(a, b), \tag{20}$$

if there exists (see [14])

$${}_a I_x^{(\alpha)} f(x). \tag{21}$$

Remark 7 (see [14]). If we suppose that $f(x) \in D_x^{(\alpha)}(a, b)$ or $C_\alpha(a, b)$, then we have

$$f(x) \in {}_\alpha I_x^{(\alpha)}(a, b). \tag{22}$$

3. Main Results

In this section, we state and prove our main results.

Theorem 8. Assume that $\alpha_{kj} \in \mathbb{R}$ ($j = 1, 2, \dots, m$; $k = 1, 2, \dots, s$),

$$\sum_k \frac{1}{p_k} = 1, \quad \sum_{k=1}^s \alpha_{kj} = 0. \tag{23}$$

If $f_j(x) > 0$ and $f_j \in C_\alpha(a, b)$ ($j = 1, 2, \dots, m$), then each of the following assertions holds true.

(1) For $p_k > 0$ ($k = 1, 2, \dots, s$), one has

$$\begin{aligned} &\frac{1}{\Gamma(1+\alpha)} \int_a^b \prod_{j=1}^m f_j(x) (dx)^\alpha \\ &\leq \prod_{k=1}^s \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b \prod_{j=1}^m f_j^{1+p_k \alpha_{kj}}(x) (dx)^\alpha \right)^{1/p_k}. \end{aligned} \tag{24}$$

(2) For $0 < p_s < 1$ and $p_k < 0$ ($k = 1, 2, \dots, s - 1$), one has

$$\begin{aligned} &\frac{1}{\Gamma(1+\alpha)} \int_a^b \prod_{j=1}^m f_j(x) (dx)^\alpha \\ &\geq \prod_{k=1}^s \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b \prod_{j=1}^m f_j^{1+p_k \alpha_{kj}}(x) (dx)^\alpha \right)^{1/p_k}. \end{aligned} \tag{25}$$

Proof. (1) Let

$$g_k(x) = \left(\prod_{j=1}^m f_j^{1+p_k \alpha_{kj}}(x) \right)^{1/p_k}. \tag{26}$$

Applying the assumptions $\sum_k (1/p_k) = 1$ and $\sum_{k=1}^s \alpha_{kj} = 0$, a direct computation shows that

$$\begin{aligned} \prod_{k=1}^s g_k(x) &= g_1 g_2 \cdots g_s \\ &= \left(\prod_{j=1}^m f_j^{1+\alpha_{1j}}(x) \right)^{1/a_1} \left(\prod_{j=1}^m f_j^{1+\alpha_{2j}}(x) \right)^{1/a_2} \\ &\quad \cdots \left(\prod_{j=1}^m f_j^{1+\alpha_{sj}}(x) \right)^{1/a_s} \\ &= \prod_{j=1}^m f_j^{1/a_1 + \alpha_{1j}}(x) \prod_{j=1}^m f_j^{1/a_2 + \alpha_{2j}}(x) \cdots \prod_{j=1}^m f_j^{1/a_s + \alpha_{sj}}(x) \\ &= \prod_{j=1}^m f_j^{1/a_1 + 1/a_2 + \cdots + 1/a_s + \alpha_{1j} + \alpha_{2j} + \cdots + \alpha_{sj}}(x) \\ &= \prod_{j=1}^m f_j(x); \end{aligned} \tag{27}$$

that is,

$$\prod_{k=1}^s g_k(x) = \prod_{j=1}^m f_j(x). \tag{28}$$

It is easy to see that

$$\frac{1}{\Gamma(1+\alpha)} \int_a^b \prod_{j=1}^m f_j(x) (dx)^\alpha = \frac{1}{\Gamma(1+\alpha)} \int_a^b \prod_{k=1}^s g_k(x) (dx)^\alpha. \tag{29}$$

It follows from the Hölder inequality (6) that

$$\begin{aligned} &\frac{1}{\Gamma(1+\alpha)} \int_a^b \prod_{k=1}^s g_k(x) (dx)^\alpha \\ &\leq \prod_{k=1}^s \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b g_k^{p_k}(x) (dx)^\alpha \right)^{1/p_k}. \end{aligned} \tag{30}$$

Substitution of $g_k(x)$ into (30) leads us immediately to inequality (24). This proves inequality (24).

(2) The proof of inequality (25) is similar to the proof of inequality (24). Indeed, by using (26), (29), and (7), we have

$$\begin{aligned} &\frac{1}{\Gamma(1+\alpha)} \int_a^b \prod_{k=1}^s g_k(x) (dx)^\alpha \\ &\geq \prod_{k=1}^s \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b g_k^{p_k}(x) (dx)^\alpha \right)^{1/p_k}. \end{aligned} \tag{31}$$

Substitution of $g_k(x)$ into (31) leads to inequality (25) immediately. \square

Remark 9. Upon setting $s = m$, $\alpha_{kj} = -1/p_k$, for $j \neq k$, and $\alpha_{kk} = 1 - 1/p_k$, inequalities (24) and (25) are reduced to inequalities (6) and (7), respectively.

As we remarked earlier, many existing inequalities related to the local fractional integral Hölder's inequality are special cases of inequalities (24) and (25). For example, we have the following corollary.

Corollary 10. Under the assumptions of Theorem 8 with $s = m$, $\alpha_{kj} = -t/p_k$, for $j \neq k$, and $\alpha_{kk} = t(1 - 1/p_k)$ ($t \in \mathbb{R}$), each of the following assertions holds true.

(1) For $p_k > 0$ ($k = 1, 2, \dots, s$), one has

$$\begin{aligned} & \frac{1}{\Gamma(1 + \alpha)} \int_a^b \prod_{j=1}^m f_j(x) (dx)^\alpha \\ & \leq \prod_{k=1}^m \left(\frac{1}{\Gamma(1 + \alpha)} \int_a^b \left(\prod_{j=1}^m f_j(x) \right)^{1-t} (f_k^{p_k})^t(x) (dx)^\alpha \right)^{1/p_k}. \end{aligned} \tag{32}$$

(2) For $0 < p_m < 1$ and $p_k < 0$ ($k = 1, 2, \dots, m - 1$), one has

$$\begin{aligned} & \frac{1}{\Gamma(1 + \alpha)} \int_a^b \prod_{j=1}^m f_j(x) (dx)^\alpha \\ & \geq \prod_{k=1}^m \left(\frac{1}{\Gamma(1 + \alpha)} \int_a^b \left(\prod_{j=1}^m f_j(x) \right)^{1-t} (f_k^{p_k})^t(x) (dx)^\alpha \right)^{1/p_k}. \end{aligned} \tag{33}$$

Theorem 11. Assume that $r \in \mathbb{R}$, $\alpha_{kj} \in \mathbb{R}$ ($j = 1, 2, \dots, m$; $k = 1, 2, \dots, s$),

$$\sum_k \frac{1}{p_k} = r, \quad \sum_k \alpha_{kj} = 0. \tag{34}$$

If $f_j(x) > 0$ and $f_j \in C_\alpha(a, b)$ ($j = 1, 2, \dots, m$), then each of the following assertions holds true.

(1) For $rp_k > 0$ ($k = 1, 2, \dots, s$), one has

$$\begin{aligned} & \frac{1}{\Gamma(1 + \alpha)} \int_a^b \prod_{j=1}^m f_j(x) (dx)^\alpha \\ & \leq \prod_{k=1}^s \left(\frac{1}{\Gamma(1 + \alpha)} \int_a^b \prod_{j=1}^m f_j^{1+rp_k\alpha_{kj}}(x) (dx)^\alpha \right)^{1/rp_k}. \end{aligned} \tag{35}$$

(2) For $0 < rp_s < 1$ and $rp_k < 0$ ($k = 1, 2, \dots, s - 1$), one has

$$\begin{aligned} & \frac{1}{\Gamma(1 + \alpha)} \int_a^b \prod_{j=1}^m f_j(x) (dx)^\alpha \\ & \geq \prod_{k=1}^s \left(\frac{1}{\Gamma(1 + \alpha)} \int_a^b \prod_{j=1}^m f_j^{1+rp_k\alpha_{kj}}(x) (dx)^\alpha \right)^{1/rp_k}. \end{aligned} \tag{36}$$

Proof. (1) Since $rp_k > 0$ and $\sum_k (1/p_k) = r$, we get $\sum_k (1/rp_k) = 1$. Then, by applying (24), we immediately obtain inequality (35).

(2) Since $0 < rp_s < 1$, $rp_k < 0$, and $\sum_k (1/p_k) = r$, we have $\sum_k (1/rp_k) = 1$. Thus, by applying (25), we immediately have inequality (36). This completes the proof of Theorem 11. \square

From Theorem 11, we obtain Corollary 12, which is a generalization of Theorem 11.

Corollary 12. Under the assumptions of Theorem 11, let $s = 2$, $p_1 = p$, $p_2 = q$, and $\alpha_{1j} = -\alpha_{2j} = \alpha_j$. Then each of the following assertions holds true.

(1) For $rp > 0$, one has

$$\begin{aligned} & \frac{1}{\Gamma(1 + \alpha)} \int_a^b \prod_{j=1}^m f_j(x) (dx)^\alpha \\ & \leq \left(\frac{1}{\Gamma(1 + \alpha)} \int_a^b \prod_{j=1}^m f_j^{1+rp\alpha_j}(x) (dx)^\alpha \right)^{1/rp} \\ & \quad \cdot \left(\frac{1}{\Gamma(1 + \alpha)} \int_a^b \prod_{j=1}^m f_j^{1-rq\alpha_j}(x) (dx)^\alpha \right)^{1/rq}. \end{aligned} \tag{37}$$

(2) For $0 < rp < 1$, one has

$$\begin{aligned} & \frac{1}{\Gamma(1 + \alpha)} \int_a^b \prod_{j=1}^m f_j(x) (dx)^\alpha \\ & \geq \left(\frac{1}{\Gamma(1 + \alpha)} \int_a^b \prod_{j=1}^m f_j^{1+rp\alpha_j}(x) (dx)^\alpha \right)^{1/rp} \\ & \quad \cdot \left(\frac{1}{\Gamma(1 + \alpha)} \int_a^b \prod_{j=1}^m f_j^{1-rq\alpha_j}(x) (dx)^\alpha \right)^{1/rq}. \end{aligned} \tag{38}$$

Next we present a refinement of each of inequalities (35) and (36).

Theorem 13. Under the assumptions of Theorem 11, each of the following assertions holds true.

(1) For $rp_k > 0$ ($k = 1, 2, \dots, s$), one has

$$\begin{aligned} & \frac{1}{\Gamma(1 + \alpha)} \int_a^b \prod_{j=1}^m f_j(x) (dx)^\alpha \\ & \leq \varphi(c) \leq \prod_{k=1}^s \left(\frac{1}{\Gamma(1 + \alpha)} \int_a^b \prod_{j=1}^m f_j^{1+rp_k\alpha_{kj}}(x) (dx)^\alpha \right)^{1/rp_k}, \end{aligned} \tag{39}$$

where

$$\begin{aligned} \varphi(c) & \equiv \frac{1}{\Gamma(1 + \alpha)} \int_a^c \prod_{j=1}^m f_j(x) (dx)^\alpha \\ & \quad + \prod_{k=1}^s \left(\frac{1}{\Gamma(1 + \alpha)} \int_c^b \prod_{j=1}^m f_j^{1+rp_k\alpha_{kj}}(x) (dx)^\alpha \right)^{1/rp_k} \end{aligned} \tag{40}$$

is a nonincreasing function with $a \leq c \leq b$.

(2) For $0 < rp_s < 1$ and $rp_k < 0 (k = 1, 2, \dots, s - 1)$, one has

$$\begin{aligned} & \frac{1}{\Gamma(1 + \alpha)} \int_a^b \prod_{j=1}^m f_j(x) (dx)^\alpha \\ & \geq \phi(c) \geq \prod_{k=1}^s \left(\frac{1}{\Gamma(1 + \alpha)} \int_a^b \prod_{j=1}^m f_j^{1+rp_k\alpha_{kj}}(x) (dx)^\alpha \right)^{1/rp_k}, \end{aligned} \tag{41}$$

where

$$\begin{aligned} \phi(c) \equiv & \frac{1}{\Gamma(1 + \alpha)} \int_a^c \prod_{j=1}^m f_j(x) (dx)^\alpha \\ & + \prod_{k=1}^s \left(\frac{1}{\Gamma(1 + \alpha)} \int_c^b \prod_{j=1}^m f_j^{1+rp_k\alpha_{kj}}(x) (dx)^\alpha \right)^{1/rp_k} \end{aligned} \tag{42}$$

is a nondecreasing function with $a \leq c \leq b$.

Proof. (1) Let

$$g_k(x) = \left(\prod_{j=1}^m f_j^{1+rp_k\alpha_{kj}}(x) \right)^{1/rp_k}. \tag{43}$$

By rearrangement, it follows from the assumptions of Theorem 11 that

$$\prod_{j=1}^m f_j(x) = \prod_{k=1}^s g_k(x). \tag{44}$$

Then, by Hölder’s inequality (6), we obtain

$$\begin{aligned} & \frac{1}{\Gamma(1 + \alpha)} \int_a^b \prod_{j=1}^m f_j(x) (dx)^\alpha \\ & = \frac{1}{\Gamma(1 + \alpha)} \int_a^b \prod_{k=1}^s g_k(x) (dx)^\alpha \\ & = \frac{1}{\Gamma(1 + \alpha)} \int_a^c \prod_{k=1}^s g_k(x) (dx)^\alpha + \frac{1}{\Gamma(1 + \alpha)} \int_c^b \prod_{k=1}^s g_k(x) (dx)^\alpha \\ & \leq \frac{1}{\Gamma(1 + \alpha)} \int_a^c \prod_{k=1}^s g_k(x) (dx)^\alpha \\ & \quad + \prod_{k=1}^s \left(\frac{1}{\Gamma(1 + \alpha)} \int_c^b g_k^{rp_k}(x) (dx)^\alpha \right)^{1/rp_k} \\ & \leq \prod_{k=1}^s \left(\frac{1}{\Gamma(1 + \alpha)} \int_a^c g_k^{rp_k}(x) (dx)^\alpha \right. \\ & \quad \left. + \frac{1}{\Gamma(1 + \alpha)} \int_c^b g_k^{rp_k}(x) (dx)^\alpha \right)^{1/rp_k} \end{aligned}$$

$$\begin{aligned} & = \prod_{k=1}^s \left(\frac{1}{\Gamma(1 + \alpha)} \int_a^b g_k^{rp_k}(x) (dx)^\alpha \right)^{1/rp_k} \\ & = \prod_{k=1}^s \left(\frac{1}{\Gamma(1 + \alpha)} \int_a^b \prod_{j=1}^m f_j^{1+rp_k\alpha_{kj}}(x) (dx)^\alpha \right)^{1/rp_k}. \end{aligned} \tag{45}$$

Hence, the desired result is obtained.

(2) The proof of inequality (41) is similar to the proof of inequality (39), so we omit the details involved. \square

4. Concluding Remarks and Observations

Integral inequalities play a major role in the development of local fractional calculus. In this work, we considered some new generalizations and refinements of the local fractional integral Hölder’s inequality and some related results on fractal space. Hölder’s inequality was obtained by Yang [14] using local fractional integral. Moreover, the reverse local fractional integral Hölder’s inequality was established by Chen [15]. In our present investigation, we have offered further generalizations and refinements of these inequalities by using the local fractional integral which was introduced and investigated by Yang [14, 16, 17]. Special cases of the various results derived in this paper are shown to be related to a number of known results.

For the relevant details about the mathematical, physical, and engineering applications and interpretations of the operators of fractional calculus and local fractional calculus in dealing with the intermediate processes and the intermediate phenomena, the interested reader may be referred to the monographs by Yang [17] and Kilbas et al. [43] (and indeed also to some of the other recent investigations which are cited in this paper).

Conflict of Interests

The authors declare that they have no conflict of interests.

Authors’ Contribution

This paper is the result of joint work of all authors who contributed equally to the final version of this paper. All authors read and approved the final paper.

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