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Research Article

Controllability for a Wave Equation with Moving Boundary

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We investigate the controllability for a one-dimensional wave equation in domains with moving boundary. This model characterizes small vibrations of a stretched elastic string when one of the two endpoints varies. When the speed of the moving endpoint is less than $1 - 1/\sqrt{e}$, by Hilbert uniqueness method, sidewise energy estimates method, and multiplier method, we get partial Dirichlet boundary controllability. Moreover, we will give a sharper estimate on controllability time that only depends on the speed of the moving endpoint.

1. Introduction and Main Results

This paper concerns a finite vibrating string described by a wave equation. The left boundary endpoint of the string is fixed, while the right boundary endpoint is moving. Given that $T > 0$, write $\widehat{Q}_T^k$ for the following noncylindrical domain:

$$\left\{(y,t) \in \mathbb{R}^2; 0 < y < \alpha_k(t),\ \forall t \in (0,T)\right\}, \tag{1}$$

where

$$\alpha_k(t) = 1 + kt, \quad 0 < k < 1. \tag{2}$$

Consider the following wave equation in the noncylindrical domain $\widehat{Q}_T^k$:

$$\begin{gathered} u_{tt} - u_{yy} = 0 \quad \text{in } \widehat{Q}_T^k, \\ u(0,t) = v(t), \quad u(\alpha_k(t), t) = 0 \quad \text{on } (0,T), \\ u(0) = u^0, \quad u_t(0) = u^1 \quad \text{in } (0,1), \end{gathered} \tag{3}$$

where v is the control variable and is put on fixed endpoint. The constant k is called to be the speed of the moving endpoint. By [1], for $0 < k < 1$, any $(u^0, u^1) \in L^2(0,1) \times H^{-1}(0,1)$, and $v \in L^2(0,T)$, (3) admits a unique transposition solution.

The exact controllability problem for (3) is formulated as follows: given $T > 0$ large enough. For each $(u^0, u^1) \in L^2(0,1) \times H^{-1}(0,1)$ and for each $(u_d^0, u_d^1) \in L^2(0, \alpha_k(T)) \times H^{-1}(0, \alpha_k(T))$, is it possible to find a control $v \in L^2(0,T)$ such that the corresponding solution of (3) satisfies

$$u(T) = u_d^0, \qquad u_t(T) = u_d^1? \tag{4}$$

The main purpose of this paper is to study the exact controllability of (3). As we all know, there exist pieces of literature on the controllability problems of wave equations in a cylindrical domain. However, as far as we know, there are only a few works on the exact controllability for wave equations defined in noncylindrical domains. We refer to [1–3] for some known results in this respect. In [1], the boundary controllability problem for a multidimensional wave equation with constant coefficients in a noncylindrical domain was discussed. However, in [1] in the one-dimensional case, the following condition seems necessary:

$$\int_0^\infty \left|\alpha_k'(t)\right| dt < \infty. \tag{5}$$

It is easy to check that this condition is not satisfied for the moving boundary in (3). In [2], the exact controllability of a multidimensional wave equation with constant coefficients in a noncylindrical domain was established, while a control entered the system through the whole noncylindrical domain. In [3], the exact controllability of the following system is studied:

$$\begin{gathered} u_{tt} - u_{yy} = 0 \quad \text{in } \widehat{Q}_T^k, \\ u(0,t) = 0, \quad u(\alpha_k(t), t) = v(t) \quad \text{on } (0,T), \\ u(0) = u^0, \quad u_t(0) = u^1 \quad \text{in } (0,1). \end{gathered} \tag{6}$$

But the control is put on moving endpoint. In order to overcome these difficulties and drop the additional conditions for the moving boundary, we use sidewise energy estimates method to obtain observability inequality.

The main result of this paper is stated as follows.

Theorem 1. *Suppose that* $0 < k < 1 - 1/\sqrt{e}$. *For any given* $T > T_k^*$, (3) *is exactly controllable at time* T.

Remark 2. T_k^* will be defined during the course of the proof.

Remark 3. It seems natural to expect that the exact controllability of (3) holds when $k \in (0,1)$. However, we do not succeed in extending the approach developed in Theorem 1 to this case.

In order to prove Theorem 1, we first transform (3) into a wave equation with variable coefficients in a cylindrical domain. For this aim, set

$$x = \frac{y}{\alpha_k(t)}, \qquad w(x,t) = u(y,t) = u(\alpha_k(t)x, t) \qquad \text{for } (y,t) \in \widehat{Q}_T^k. \tag{7}$$

Then, it is easy to check that (x,t) varies in $Q := (0,1)\times(0,T)$. Also, (3) is transformed into the following equivalent wave equation in the cylindrical domain Q:

$$\begin{gathered} w_{tt} - \left[\frac{\beta_k(x,t)}{\alpha_k(t)} w_x\right]_x + \frac{\gamma_k(x)}{\alpha_k(t)} w_{tx} = 0 \quad \text{in } Q, \\ w(0,t) = v(t), \quad w(1,t) = 0 \quad \text{on } (0,T), \\ w(0) = w^0, \quad w_t(0) = w^1 \quad \text{in } (0,1), \end{gathered} \tag{8}$$

where

$$\begin{gathered} \beta_k(x,t) = \frac{1 - k^2x^2}{\alpha_k(t)}, \qquad \gamma_k(x) = -2kx, \\ w^0 = u^0, \qquad w^1 = u^1 + kxu_x^0. \end{gathered} \tag{9}$$

Equation (8) admits a unique solution in the sense of transposition

$$w \in C\left([0,T]; L^2(0,1)\right) \cap C^1\left([0,T]; H^{-1}(0,1)\right) \tag{10}$$

(see [4]).

Therefore, the exact controllability of (3) (Theorem 1) is reduced to the following controllability result for the wave equation (8).

Theorem 4. *Suppose that* $0 < k < 1 - 1/\sqrt{e}$. *Let* $T > T_k^*$. *Then, for any initial value* $(w^0, w^1) \in L^2(0,1) \times H^{-1}(0,1)$ *and target* $(w_d^0, w_d^1) \in L^2(0,1) \times H^{-1}(0,1)$, *there exists a control* $v \in L^2(0,T)$ *such that the corresponding solution* w *of* (8) *in the sense of transposition satisfies*

$$w(T) = w_d^0, \qquad w_t(T) = w_d^1. \tag{11}$$

In order to obtain Theorem 4, we will use Hilbert uniqueness method. The main idea is to define a weighted energy function for the following wave equation with variable coefficients in cylindrical domains:

$$\begin{gathered} \alpha_k(t) z_{tt} - \left[\beta_k(x,t) z_x\right]_x + \gamma_k(x) z_{tx} = 0 \quad \text{in } Q, \\ z(0,t) = 0, \quad z(1,t) = 0 \quad \text{on } (0,T), \\ z(0) = z^0, \quad z_t(0) = z^1 \quad \text{in } (0,1), \end{gathered} \tag{12}$$

where $k \in (0,1)$, $(z^0, z^1) \in H_0^1(0,1) \times L^2(0,1)$ is any given initial value, and α_k, β_k, and γ_k are the functions given in (8). Similar to Theorem 3.2 in [4], we have that (12) has a unique weak solution

$$z \in C\left([0,T]; H_0^1(0,1)\right) \cap C^1\left([0,T]; L^2(0,1)\right). \tag{13}$$

In the sequel, we denote by C a positive constant depending only on T and k, which may be different from one place to another.

The energy function of system (12) is defined as

$$E(t) = \frac{1}{2}\int_0^1 \left[\alpha_k(t)\left|z'(x,t)\right|^2 + \beta_k(x,t)\left|z_x(x,t)\right|^2\right] dx \qquad (t \geq 0). \tag{14}$$

In particular

$$E_0 = E(0) = \frac{1}{2}\int_0^1 \left[\left|z^1\right|^2 + \beta_k(x,0)\left|z_x^0\right|^2\right] dx. \tag{15}$$

Note that this weighted energy is different from the usual one, but they are equivalent. We will obtain explicit energy equality. Using this energy equality, we will first prove the following observability result.

Theorem 5. *Let* $T > T_k^*$. *For any* $(z^0, z^1) \in H_0^1(0,1)\times L^2(0,1)$, *there exists a constant* $C > 0$ *such that the corresponding solution* z *of* (12) *satisfies*

$$\begin{aligned} C\left(\left|z^0\right|_{H_0^1(0,1)}^2 + \left|z^1\right|_{L^2(0,1)}^2\right) &\leq \int_0^T \beta_k(0,t)\left|z_x(0,t)\right|^2 dt \\ &\leq C\left(\left|z^0\right|_{H_0^1(0,1)}^2 + \left|z^1\right|_{L^2(0,1)}^2\right). \end{aligned} \tag{16}$$

Then, applying Hilbert uniqueness method, we will deduce Theorem 4.

The rest of this paper is organized as follows. In Section 2, we derive Theorem 5. Section 3 is devoted to a proof of Theorem 4.

2. Observability: Proof of Theorem 5

In this section, first we give the following two lemmas (see the detailed proof in [3]).

Lemma 6. *For any* $(z^0, z^1) \in H_0^1(0,1) \times L^2(0,1)$ *and* $t \in [0,T]$, *any solution* z *of* (12) *satisfies the following estimate:*

$$E(t) = \frac{1}{\alpha_k(t)} E_0. \tag{17}$$

Lemma 7. *Suppose that* $q \in C^1([0,1])$ *is any given function. Then any solution* z *of* (12) *satisfies the following estimate:*

$$\begin{aligned}
&\left[\frac{1}{2}\int_0^T \beta_k(x,t)q(x)|z_x(x,t)|^2 dt\right]\Big|_0^1 \\
&= \frac{1}{2}\int_0^T\int_0^1 \left[q_x(x)\alpha_k(t)|z_t(x,t)|^2\right. \\
&\qquad + (q_x(x)\beta_k(x,t) - \beta_{k,x}(x,t)q(x)) \\
&\qquad \left.\times |z_x(x,t)|^2\right] dx\,dt \\
&\quad - \int_0^T\int_0^1 \alpha_{k,t}(t)q(x)z_t(x,t)z_x(x,t)\,dx\,dt \\
&\quad + \int_0^1 \left[\alpha_k(t)q(x)z_t(x,t)z_x(x,t)\right. \\
&\qquad \left. + \frac{1}{2}\gamma_k(x)q(x)|z_x(x,t)|^2\right] dx \Big|_0^T.
\end{aligned} \tag{18}$$

Now, we give a proof of Theorem 5.

Proof. We first give the proof of the second inequality in (16).

Define

$$G(x) = \frac{1}{2}\int_{A_1(x)}^{T-A_1(x)} \left[\alpha_k(t)|z_t(x,t)|^2 + \beta_k(x,t)|z_x(x,t)|^2\right] dt. \tag{19}$$

Note that

$$G(0) = \frac{1}{2}\int_{A_1(x)}^{T-A_1(x)} \beta_k(0,t)|z_x(0,t)|^2 dt. \tag{20}$$

The derivative of the functional of G is

$$\begin{aligned}
&G'(x) \\
&= \int_{A_1(x)}^{T-A_1(x)} \left[\alpha_k(t)z_t(x,t)z_{t,x}(x,t)\right. \\
&\qquad + \beta_k(x,t)z_x(x,t)z_{xx}(x,t) \\
&\qquad \left. + \frac{1}{2}\beta_{k,x}(x,t)|z_x(x,t)|^2\right] dt \\
&\quad + \left(-\frac{1}{2}A_1'(x)\right) \\
&\quad \times \sum_{t=T-A_1(x),A_1(x)} \left[\alpha_k(t)|z_t(x,t)|^2 + \beta_k(x,t)|z_x(x,t)|^2\right],
\end{aligned} \tag{21}$$

where

$$\begin{aligned}
&\int_{A_1(x)}^{T-A_1(x)} \alpha_k(t)z_t(x,t)z_{t,x}(x,t)\,dt \\
&= \left[\alpha_k(t)z_t(x,t)z_x(x,t)\right]\Big|_{A_1(x)}^{T-A_1(x)} \\
&\quad - \int_{A_1(x)}^{T-A_1(x)} \left[\alpha_{k,t}(t)z_t(x,t)z_x(x,t)\right. \\
&\qquad \left. + \alpha_k(t)z_{tt}(x,t)z_x(x,t)\right] dt.
\end{aligned} \tag{22}$$

By (12), it follows that

$$\begin{aligned}
&\alpha_k(t)z_{tt}(x,t) \\
&= [\beta_k(x,t)z_x]_x(x,t) - \gamma_k(x)z_{tx}(x,t) \\
&= \beta_{k,x}(x,t)z_x(x,t) + \beta_k(x,t)z_{xx}(x,t) \\
&\quad - \gamma_k(x)z_{tx}(x,t),
\end{aligned} \tag{23}$$

from which and using integrating by parts, we have that

$$\begin{aligned}
&\int_{A_1(x)}^{T-A_1(x)} \alpha_k(t)z_t(x,t)z_{t,x}(x,t)\,dt \\
&= \left[\alpha_k(t)z_t(x,t)z_x(x,t)\right]\Big|_{A_1(x)}^{T-A_1(x)} \\
&\quad - \int_{A_1(x)}^{T-A_1(x)} \left[\alpha_{k,t}(t)z_t(x,t)z_x(x,t)\right. \\
&\qquad + \beta_{k,x}(x,t)|z_x(x,t)|^2 \\
&\qquad + \beta_k(x,t)z_{xx}(x,t)z_x(x,t) \\
&\qquad \left. - \gamma_k(x)z_{tx}(x,t)z_x(x,t)\right] dt \\
&= \left[\alpha_k(t)z_t(x,t)z_x(x,t)\right]\Big|_{A_1(x)}^{T-A_1(x)} \\
&\quad - \int_{A_1(x)}^{T-A_1(x)} \left[\alpha_{k,t}(t)z_t(x,t)z_x(x,t)\right. \\
&\qquad + \beta_{k,x}(x,t)|z_x(x,t)|^2 \\
&\qquad \left. + \beta_k(x,t)z_{xx}(x,t)z_x(x,t)\right] dt \\
&\quad + \left[\frac{1}{2}\gamma_k(x)|z_x(x,t)|^2\right]\Big|_{A_1(x)}^{T-A_1(x)} \\
&= \left[\alpha_k(t)z_t(x,t)z_x(x,t)\right. \\
&\qquad \left. + \frac{1}{2}\gamma_k(x)|z_x(x,t)|^2\right]\Big|_{A_1(x)}^{T-A_1(x)} \\
&\quad - \int_{A_1(x)}^{T-A_1(x)} \left[\alpha_{k,t}(t)z_t(x,t)z_x(x,t)\right. \\
&\qquad + \beta_{k,x}(x,t)|z_x(x,t)|^2 \\
&\qquad \left. + \beta_k(x,t)z_{xx}(x,t)z_x(x,t)\right] dt.
\end{aligned} \tag{24}$$

We conclude, using (24), that

$$\begin{aligned}
G'(x) &= -\int_{A_1(x)}^{T-A_1(x)} \Big[\alpha_{k,t}(t)\, z_t(x,t)\, z_x(x,t) \\
&\qquad + \frac{1}{2}\beta_{k,x}(x,t)\left|z_x(x,t)\right|^2\Big]\, dt \\
&\quad + \Big[\alpha_k(t)\, z_t(x,t)\, z_x(x,t) \\
&\qquad + \frac{1}{2}\gamma_k(x)\left|z_x(x,t)\right|^2\Big]\Big|_{A_1(x)}^{T-A_1(x)} \\
&\quad + \left(-\frac{1}{2}A_1'(x)\right) \\
&\quad \times \sum_{t=T-A_1(x),A_1(x)} \left[\alpha_k(t)\left|z_t(x,t)\right|^2 + \beta_k(x,t)\left|z_x(x,t)\right|^2\right].
\end{aligned} \tag{25}$$

We will choose $A_1(x)$ later which satisfies

$$\begin{aligned}
&\left[\alpha_k(t) z_t(x,t) z_x(x,t) + \frac{1}{2}\gamma_k(x)\left|z_x(x,t)\right|^2\right]\Big|_{A_1(x)}^{T-A_1(x)} \\
&\quad + \left(-\frac{1}{2}A_1'(x)\right) \\
&\quad \times \sum_{t=T-A_1(x),A_1(x)} \left[\alpha_k(t)\left|z_t(x,t)\right|^2 + \beta_k(x,t)\left|z_x(x,t)\right|^2\right] \\
&\le 0.
\end{aligned} \tag{26}$$

From (25) and (26), for $\varepsilon > 0$, it concludes that

$$\begin{aligned}
G'(x) &\le -\int_{A_1(x)}^{T-A_1(x)} \Big[\alpha_{k,t}(t)\, z_t(x,t)\, z_x(x,t) \\
&\qquad + \frac{1}{2}\beta_{k,x}(x,t)\left|z_x(x,t)\right|^2\Big]\, dt \\
&= \int_{A_1(x)}^{T-A_1(x)} \Big[-k z_t(x,t)\, z_x(x,t) \\
&\qquad + \frac{k^2 x}{\alpha_k(t)}\left|z_x(x,t)\right|^2\Big]\, dt \\
&\le \left|\int_{A_1(x)}^{T-A_1(x)} \sqrt{\alpha_k(t)} z_t(x,t)\, \frac{k}{\sqrt{\alpha_k(t)}} z_x(x,t)\, dt\right| \\
&\quad + \left|\int_{A_1(x)}^{T-A_1(x)} \frac{k^2 x}{1-k^2x^2}\beta_k(x,t)\left|z_x(x,t)\right|^2 dt\right| \\
&\le \frac{1}{2\varepsilon}\int_{A_1(x)}^{T-A_1(x)} \alpha_k(t)\left|z_t(x,t)\right|^2 dt \\
&\quad + \frac{\varepsilon}{2}\int_{A_1(x)}^{T-A_1(x)} \frac{k^2}{1-k^2x^2}\beta_k(x,t)\left|z_x(x,t)\right|^2 dt \\
&\quad + \int_{A_1(x)}^{T-A_1(x)} \frac{k^2 x}{1-k^2x^2}\beta_k(x,t)\left|z_x(x,t)\right|^2 dt \\
&\le \frac{1}{2}\frac{1}{\varepsilon}\int_{A_1(x)}^{T-A_1(x)} \alpha_k(t)\left|z_t(x,t)\right|^2 dt \\
&\quad + \frac{1}{2}\frac{k^2(\varepsilon+2)}{1-k^2}\int_{A_1(x)}^{T-A_1(x)} \beta_k(x,t)\left|z_x(x,t)\right|^2 dt.
\end{aligned} \tag{27}$$

Take $\varepsilon = (1-k)/k$; then it is easy to check that

$$\frac{1}{\varepsilon} = \frac{k^2(\varepsilon+2)}{1-k^2}. \tag{28}$$

Hence

$$G'(x) \le \frac{k}{1-k}G(x) \qquad x \in (0,1). \tag{29}$$

By Gronwall inequality, there exists $C > 0$ such that

$$G(x) \le CG(0). \tag{30}$$

Integrating (30) in $[0,1]$, we have

$$\int_0^1 G(x)\, dx \le CG(0). \tag{31}$$

By (17), we deduce that

$$\begin{aligned}
&[T - 2A_1(1)]\, E_0 \\
&\quad = \int_{A_1(1)}^{T-A_1(1)} E_0 dt \\
&\quad \le (1+kT)\int_{A_1(1)}^{T-A_1(1)} E(t)\, dt \\
&\quad = (1+kT)\frac{1}{2}\int_{A_1(1)}^{T-A_1(1)}\int_0^1 \Big[\alpha_k(t)\left|z_t(x,t)\right|^2 dt \\
&\qquad + \beta_k(x,t)\left|z_x(x,t)\right|^2\Big]\, dt\, dx.
\end{aligned} \tag{32}$$

Now choose that $A_1(x)$ also satisfies

$$A_1'(x) > 0, \qquad A_1(1) \ge A_1(x) \ge A_1(0) \ge 0. \tag{33}$$

Then from (19), (31), and (32), it follows that

$$\begin{aligned}
&[T - 2A_1(1)]\, E_0 \\
&\leq (1+kT)\frac{1}{2}\int_{A_1(x)}^{T-A_1(x)}\int_0^1 \left[\alpha_k(t)\left|z_t(x,t)\right|^2 dt \right.\\
&\qquad\qquad \left. + \beta_k(x,t)\left|z_x(x,t)\right|^2\right] dt\, dx \\
&\leq (1+kT)\int_0^1 G(x)\, dx \\
&\leq C(1+kT)\, G(0),
\end{aligned} \tag{34}$$

from which and from (20), we have that

$$\begin{aligned}
G(0) &= \frac{1}{2}\int_{A_1(x)}^{T-A_1(x)} \beta_k(0,t)\left|z_x(0,t)\right|^2 dt \\
&\geq \frac{1}{C(1+kT)}\,[T - 2A_1(1)]\, E_0.
\end{aligned} \tag{35}$$

Let

$$T_k^* \triangleq 2A_1(1). \tag{36}$$

When $T > T_k^*$, by (35), (16) follows.

In the following, when $T > T_k^*$, we choose $A_1(x)$ which satisfies (33) and (26). In (26), for $0 < \varepsilon \leq 1$, we have that

$$\begin{aligned}
&\left|\alpha_k(t)\, z_t(x,t)\, z_x(x,t)\right| \\
&= \left|\sqrt{\alpha_k(t)} z_t(x,t)\, \sqrt{\alpha_k(t)} z_x(x,t)\right| \\
&= \left|\sqrt{\alpha_k(t)}\sqrt{A_1'(x)} z_t(x,t)\, \frac{\sqrt{\alpha_k(t)}}{\sqrt{A_1'(x)}} z_x(x,t)\right| \\
&\leq \frac{\varepsilon}{2}\alpha_k(t)\, A_1'(x)\left|z_t(x,t)\right|^2 + \frac{1}{2\varepsilon}\frac{\alpha_k(t)}{A_1'(x)}\left|z_x(x,t)\right|^2 \\
&= \frac{A_1'(x)}{2}\left[\varepsilon\alpha_k(t)\left|z_t(x,t)\right|^2\right] \\
&\quad + \frac{A_1'(x)}{2}\left[\frac{1}{\varepsilon}\frac{\alpha_k(t)}{\left(A_1'(x)\right)^2}\left|z_x(x,t)\right|^2\right], \\
&\left|\frac{1}{2}\gamma_k(x)\left|z_x(x,t)\right|^2\right| \leq \frac{1}{2}A_1'(x)\,\frac{\left|\gamma_k(x)\right|}{A_1'(x)}\left|z_x(x,t)\right|^2.
\end{aligned} \tag{37}$$

Assume that

$$\begin{aligned}
&\left[\alpha_k(t) z_t(x,t) z_x(x,t) + \frac{1}{2}\gamma_k(x)\left|z_x(x,t)\right|^2\right]\Bigg|_{A_1(x)}^{T-A_1(x)} \\
&\quad + \left(-\frac{1}{2}A_1'(x)\right) \\
&\quad \times \sum_{t=T-A_1(x),A_1(x)} \left[\alpha_k(t)\left|z_t(x,t)\right|^2 + \beta_k(x,t)\left|z_x(x,t)\right|^2\right] \\
&\leq \frac{1}{2}A_1'(x) \\
&\quad \times \sum_{t=T-A_1(x),A_1(x)} \left\{(\varepsilon-1)\,\alpha_k(t)\left|z_t(x,t)\right|^2\right. \\
&\qquad + \left[\frac{1}{\varepsilon}\frac{\alpha_k(t)}{\left(A_1'(x)\right)^2} + \frac{\left|\gamma_k(x)\right|}{A_1'(x)} - \beta_k(x,t)\right] \\
&\qquad \left. \times \left|z_x(x,t)\right|^2\right\} \leq 0.
\end{aligned} \tag{38}$$

We must take $0 < \varepsilon \leq 1$ and $A_1(x)$ satisfies

$$\frac{1}{\varepsilon}\frac{\alpha_k(t)}{\left(A_1'(x)\right)^2} + \frac{\left|\gamma_k(x)\right|}{A_1'(x)} - \beta_k(x,t) \leq 0. \tag{39}$$

From (39), we derive

$$\frac{1}{\varepsilon\left(A_1'(x)\right)^2} + \frac{2kx}{A_1'(x)\,\alpha_k(t)} + \frac{k^2x^2}{\alpha_k^2(t)} \leq \frac{1}{\alpha_k^2(t)}. \tag{40}$$

Let $\varepsilon = 1$. Then it follows that

$$\left[\frac{1}{A_1'(x)} + \frac{kx}{\alpha_k(t)}\right]^2 \leq \frac{1}{\alpha_k^2(t)}; \tag{41}$$

that is,

$$\frac{1}{A_1'(x)} + \frac{x}{\alpha_k(t)} \leq \frac{1}{\alpha_k(t)}. \tag{42}$$

By (42), we deduce that $T > T_k^*$,

$$A_1'(x) \geq \frac{\alpha_k(t)}{1-kx} \geq \frac{1+kT_k^*}{1-kx}. \tag{43}$$

Integrating into $(0,x)$, we have

$$A_1(x) \geq \frac{1+kT_k^*}{k}\left[-\ln(1-kx)\right]. \tag{44}$$

Hence, we choose

$$A_1(x) = \frac{1+kT_k^*}{k}\left[-\ln(1-kx)\right], \tag{45}$$

which satisfies (33) and (26). It follows that

$$A_1(1) = \frac{1+kT_k^*}{k}\left[-\ln(1-k)\right]. \tag{46}$$

From the definition of T_k^* (see (36)) and (46), we deduce that $k \in (0, 1 - 1/\sqrt{e})$,

$$T_k^* = \frac{-2\ln(1-k)}{k[1+2\ln(1-k)]} = 2A_1(1). \tag{47}$$

In the following, we give the proof of the second inequality in (16).

We choose $q(x) = x - 1$ for $x \in [0,1]$ in (18). Noting that $\alpha_k'(t) = k$, $\beta_{k,x}(x,t) = -2k^2x/(1+kt)$, and $\gamma_k(x) = -2kx$, it follows that

$$\begin{aligned} \frac{1}{2}\int_0^T \beta_k(0,t)\left|z_x(0,t)\right|^2 dt &= \int_0^T E(t)\,dt \\ &\quad - \int_0^T \int_0^1 k(x-1) z_t(x,t) z_x(x,t)\,dx\,dt \\ &\quad + \int_0^T \int_0^1 \frac{k^2 x(x-1)}{1+kt}\left|z_x(x,t)\right|^2 dx\,dt \\ &\quad + \int_0^1 \Big[\alpha_k(t)(x-1) z_t(x,t) z_x(x,t) \\ &\qquad - kx(x-1)|z_x(x,t)|^2\Big]\,dx\Big|_0^T. \end{aligned} \tag{48}$$

Next, we estimate every term in the right side of (48). Notice that $1 \le \alpha_k(t) \le 1 + kT$ and $0 < (1-k^2)/(1+kT) \le \beta_k(x,t) \le 1$ for any $(x,t) \in Q$. By (17), we have

$$\begin{aligned} &\int_0^T E(t)\,dt - \int_0^T \int_0^1 k(x-1) z_t(x,t) z_x(x,t)\,dx\,dt \\ &\quad + \int_0^T \int_0^1 \frac{k^2 x(x-1)}{1+kt}\left|z_x(x,t)\right|^2 dx\,dt \\ &\le \int_0^T E(t)\,dt + C\int_0^T \int_0^1 \left[\left|z_t(x,t)\right|^2 + \left|z_x(x,t)\right|^2\right] dx\,dt \\ &\le \int_0^T E(t)\,dt \\ &\quad + C\int_0^T \int_0^1 \left[\alpha_t(t)\left|z_t(x,t)\right|^2 + \beta_k(x,t)\left|z_x(x,t)\right|^2\right] dx\,dt \\ &\le C\int_0^T E(t)\,dt \le CE_0. \end{aligned} \tag{49}$$

On the other hand, for each $t \in [0,T]$, it holds that

$$\begin{aligned} &\left|\int_0^1 \Big[\alpha_k(t)(x-1) z_t(x,t) z_x(x,t) \right. \\ &\qquad \left. - kx(x-1)\left|z_x(x,t)\right|^2\Big]\,dx\right| \\ &\le C\int_0^T E(t)\,dt \le CE_0. \end{aligned} \tag{50}$$

Therefore, by (48)–(50), we have

$$\begin{aligned} \frac{1}{2}\int_0^T \beta_k(0,t)\left|z_x(0,t)\right|^2 dt &\le CE_0 \\ &\le C\left(\left|z^0\right|^2_{H_0^1(0,1)} + \left|z^1\right|^2_{L^2(0,1)}\right). \end{aligned} \tag{51}$$

□

Remark 8. Theorem 5 implies that, for any $(z_0, z_1) \in H_0^1(0,1) \times L^2(0,1)$, the corresponding solution z of (12) satisfies $z_x(0,\cdot) \in L^2(0,T)$.

3. Controllability: Proof of Theorem 4

In this section, we prove the exact controllability for the wave equation (8) in the cylindrical domain Q (Theorem 4) by Hilbert uniqueness method. The main idea is to seek a control in the form $v = z_x(0,t)$, where z is the solution of (12) for some suitable initial data. The other proof is similar to the proof of Theorem 2.1 in [3].

Remark 9. It is easy to check that

$$T_0^* \triangleq \lim_{k\to 0} T_k^* = \lim_{k\to 0}\frac{-2\ln(1-k)}{k[1+2\ln(1-k)]} = 2. \tag{52}$$

It is well known that the wave equation (3) in the cylindrical domain is null controllable at any time $T > T_0^*$. However, we do not know whether the controllability time T_k^* is sharp.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] M. M. Miranda, "Exact controllability for the wave equation in domains with variable boundary," *Revista Matemática de la Universidad Complutense de Madrid*, vol. 9, no. 2, pp. 435–457, 1996.

[2] C. Bardos and G. Chen, "Control and stabilization for the wave equation—part III: domain with moving boundary," *SIAM Journal on Control and Optimization*, vol. 19, no. 1, pp. 123–138, 1981.

[3] L. Cui, X. Liu, and H. Gao, "Exact controllability for a one-dimensional wave equation in non-cylindrical domains," *Journal of Mathematical Analysis and Applications*, vol. 402, no. 2, pp. 612–625, 2013.

[4] M. M. Miranda, "HUM and the wave equation with variable coefficients," *Asymptotic Analysis*, vol. 11, no. 4, pp. 317–341, 1995.