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Research Article

Iterates of Bernstein Type Operators on a Triangle with All Curved Sides

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We consider some Bernstein-type operators as well as their product and Boolean sum for a function defined on a triangle with all curved sides. Using the weakly Picard operators technique and the contraction principle, we study the convergence of the iterates of these operators.

1. Bernstein Type Operators

In this paper, using the weakly Picard operators technique and the contraction principle, we study the convergence of the iterates of some operators introduced in [1]. Similar operators with the ones from [1] were studied in [2, 3] and [4], where the authors construct interpolation and Bernstein-type operators on triangles and squares with one and all curved sides. They studied the operators, their product and Boolean sum, as well as their interpolation properties, the order of accuracy, and the remainder of the corresponding approximation formulas.

We recall some results regarding Bernstein-type operators on a triangle with all curved sides from [1].

We denote by $\widetilde{T}_h$ the triangle with all curved sides, which has the vertices $V_1 = (0,h)$, $V_2 = (h,0)$, $V_3 = (0,0)$, and the three curved sides γ_1, γ_2 (along the coordinate axis), and γ_3 (opposite to the vertex V_3); $h \in \mathbb{R}_+$. We have that γ_1 is defined by $(x, f_1(x))$, with $f_1(0) = f_1(h) = 0$, $f_1(x) \le 0$, for $x \in [0,h]$; γ_2 is defined by $(g_2(y), y)$, with $g_2(0) = g_2(h) = 0$, $g_2(y) \le 0$, for $y \in [0,h]$ and γ_3 is defined by the one-to-one functions f_3 and g_3, where g_3 is the inverse of the function f_3; that is, $y = f_3(x)$ and $x = g_3(y)$, with $x, y \in [0,h]$ and $f_3(0) = g_3(0) = h$ (see Figure 1). In the sequel we denote by $e_{ij}(x,y) = x^i y^j$, for $i, j \in \mathbb{N}$.

Let F be a real-valued function defined on $\widetilde{T}_h$ and $(g_2(y), y), (g_3(y), y)$, respectively, and let $(x, f_1(x)), (x, f_3(x))$ be the points in which the parallel lines to the coordinate axes, passing through the point $(x,y) \in \widetilde{T}_h$, intersecting the sides γ_1, γ_2, and γ_3. We consider the uniform partitions of the intervals $[g_2(y), g_3(y)]$ and $[f_1(x), f_3(x)]$, $x, y \in [0,h]$:

$$\Delta_m^x = \left\{ g_2(y) + i\frac{g_3(y) - g_2(y)}{m} \,\middle|\, i = \overline{0,m} \right\}, \tag{1}$$

respectively,

$$\Delta_n^y = \left\{ f_1(x) + j\frac{f_3(x) - f_1(x)}{n} \,\middle|\, j = \overline{0,n} \right\}, \tag{2}$$

and the Bernstein-type operators B_m^x and B_n^y defined by

$$\begin{aligned}(B_m^x F)(x,y) &= \sum_{i=0}^{m} p_{m,i}(x,y)\, F\left(g_2(y) + i\frac{g_3(y) - g_2(y)}{m}, y\right),\end{aligned} \tag{3}$$

with

$$p_{m,i}(x,y) = \binom{m}{i}\left[\frac{x - g_2(y)}{g_3(y) - g_2(y)}\right]^i \left[1 - \frac{x - g_2(y)}{g_3(y) - g_2(y)}\right]^{m-i}, \tag{4}$$

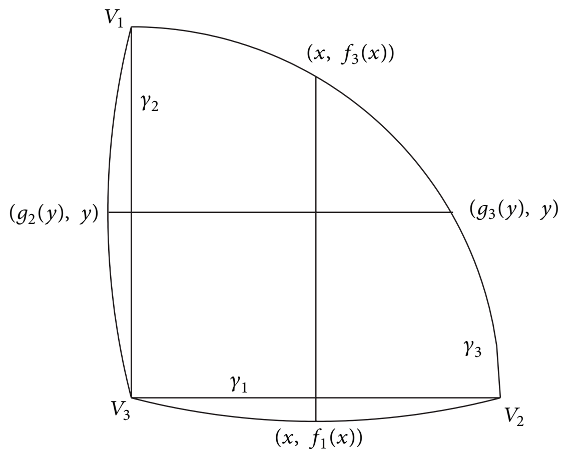

Figure 1: Triangle $\widetilde{T}_h$.

respectively,

$$(B_n^y F)(x,y) = \sum_{j=0}^{n} q_{n,j}(x,y) F\left(x, f_1(x) + j\frac{f_3(x) - f_1(x)}{n}\right), \tag{5}$$

with

$$q_{n,j}(x,y) = \binom{n}{j}\left[\frac{y - f_1(x)}{f_3(x) - f_1(x)}\right]^j \left[1 - \frac{y - f_1(x)}{f_3(x) - f_1(x)}\right]^{n-j}. \tag{6}$$

Remark 1. In Figures 2(a) and 2(b) we plot the points $(g_2(y) + i((g_3(y) - g_2(y))/m), y)$, $i = \overline{0,m}$ and, respectively, $(x, f_1(x) + j((f_3(x) - f_1(x))/n)$, $j = \overline{0,n}$, for $x, y \in [0,h]$.

Theorem 2. *If F is a real-valued function defined on $\widetilde{T}_h$, then*

(i) $B_m^x F = F$ *on* $\gamma_2 \cup \gamma_3$,

(ii) $B_n^y F = F$ *on* $\gamma_1 \cup \gamma_3$,

(iii) $(B_m^x e_{i0})(x,y) = x^i$, $i = 0,1$,

$(B_m^x e_{20})(x,y) = x^2 + [x - g_2(y)][g_3(y) - x]/m$,

$(B_m^x e_{ij})(x,y) = y^j (B_m^x e_{i0})(x,y)$, $i = 0,1,2$; $j \in \mathbb{N}$,

(iv) $(B_n^y e_{0j})(x,y) = y^j$, $j = 0,1$,

$(B_n^y e_{02})(x,y) = y^2 + [y - f_1(x)][f_3(x) - y]/n$,

$(B_n^y e_{ij})(x,y) = x^i (B_n^y e_{0j})(x,y)$, $j = 0,1,2$; $i \in \mathbb{N}$.

Let $P_{mn} = B_m^x B_n^y$, respectively, and let $Q_{nm} = B_n^y B_m^x$ be the products of the operators B_m^x and B_n^y.

We have

$$(P_{mn}F)(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}(x,y)\, q_{n,j}(x_i,y) \times F\left(x_i, f_1(x_i) + j\frac{f_3(x_i) - f_1(x_i)}{n}\right) \tag{7}$$

with $x_i = g_2(y) + i((g_3(y) - g_2(y))/m)$, respectively,

$$(Q_{nm}F)(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}(x,y_j)\, q_{n,j}(x,y) \times F\left(g_2(y_j) + i\frac{g_3(y_j) - g_2(y_j)}{m}, y_j\right), \tag{8}$$

with $y_j = f_1(x) + j((f_3(x) - f_1(x))/n)$.

Theorem 3. *If F is a real-valued function defined on $\widetilde{T}_h$, then*

(i) $(P_{mn}F)(V_3) = F(V_3)$, $P_{mn}F = F$, *on* γ_3 *and*

(ii) $(Q_{nm}F)(V_3) = F(V_3)$, $Q_{nm}F = F$, *on* γ_3.

We consider the Boolean sums of the operators B_m^x and B_n^y; that is,

$$S_{mn} := B_m^x \oplus B_n^y = B_m^x + B_n^y - B_m^x B_n^y, \tag{9}$$

respectively,

$$T_{nm} := B_n^y \oplus B_m^x = B_n^y + B_m^x - B_n^y B_m^x. \tag{10}$$

Theorem 4. *If F is a real-valued function defined on $\widetilde{T}_h$, then*

$$S_{mn}F\big|_{\partial\widetilde{T}_h} = F\big|_{\partial\widetilde{T}_h}, \qquad T_{nm}F\big|_{\partial\widetilde{T}_h} = F\big|_{\partial\widetilde{T}_h}. \tag{11}$$

2. Weakly Picard Operators

We recall some results regarding weakly Picard operators that will be used in the sequel (see, e.g., [5]).

Let (X,d) be a metric space and $A : X \to X$ an operator. We denote by $F_A := \{x \in X \mid A(x) = x\}$, the fixed points set of A; $I(A) := \{Y \subset X \mid A(Y) \subset Y, Y \neq \emptyset\}$, the family of the nonempty invariant subsets of A; $A^0 := 1_X, A^1 := A, \ldots, A^{n+1} := A \circ A^n$, $n \in \mathbb{N}$.

Definition 5. The operator $A : X \to X$ is a Picard operator if there exists $x^* \in X$ such that

(i) $F_A = \{x^*\}$;

(ii) the sequence $(A^n(x_0))_{n\in\mathbb{N}}$ converges to x^* for all $x_0 \in X$.

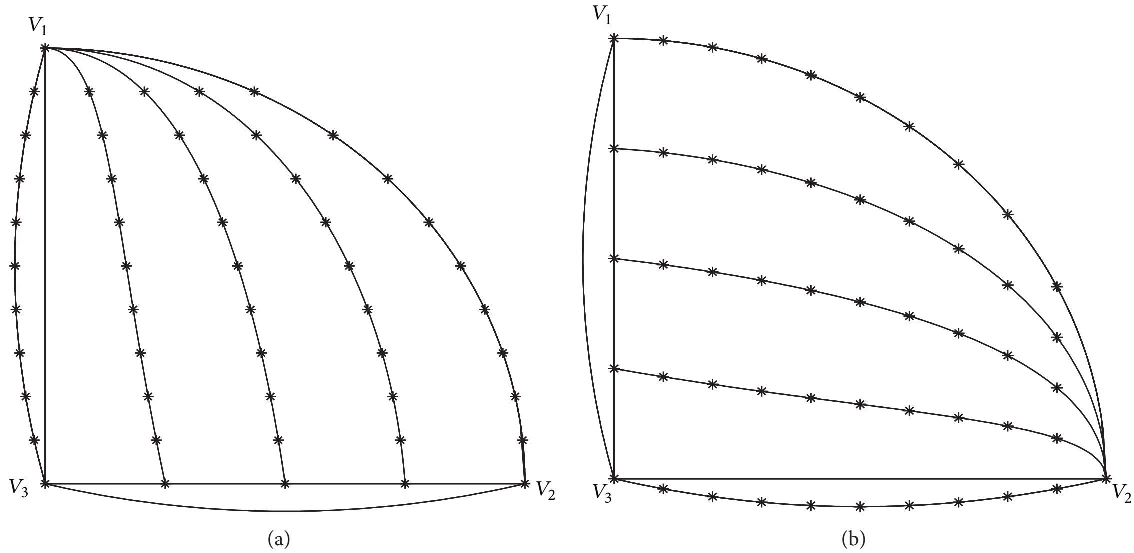

FIGURE 2: (a) Points of Δ_m^x, for $m = 4$. (b) Points of Δ_y^n, for $n = 4$.

Definition 6. The operator A is a weakly Picard operator if the sequence $(A^n(x))_{n\in\mathbb{N}}$ converges, for all $x \in X$, and the limit (which may depend on x) are a fixed point of A.

Definition 7. If A is a weakly Picard operator, then we consider the operator A^∞, $A^\infty : X \to X$, defined by

$$A^\infty(x) := \lim_{n\to\infty} A^n(x). \tag{12}$$

Theorem 8. *An operator A is a weakly Picard operator if and only if there exists a partition of X, $X = \bigcup_{\lambda\in\Lambda} X_\lambda$, such that*

(a) $X_\lambda \in I(A), \forall\lambda \in \Lambda$;

(b) $A|_{X_\lambda} : X_\lambda \to X_\lambda$ *is a Picard operator,* $\forall\lambda \in \Lambda$.

3. Iterates of Bernstein Type Operators

Let F be a real-valued function defined on $\widetilde{T}_h$; $h \in \mathbb{R}_+$.

Using the weakly Picard operators technique and the contraction principle, we study the convergence of the iterates of the Bernstein-type operators (3) and (5) and of their product and Boolean sum operators (7), (8), (9) and (10). The same approach for some other linear and positive operators leads to similar results in [6–12].

The limit behavior for the iterates of some classes of positive linear operators was also studied, for example, in [13–23]. In the papers [19–21] new methods were introduced (e.g., Korovkin type technique) for the study of the asymptotic behavior of the iterates of positive linear operators, positive linear operators preserving the affine functions and defined on the space of bounded real-valued functions on [0, 1]. This techniques enlarge the class of operators for which the limit of the iterates can be computed. In [13, 14] some methods were proposed to determine the degree of convergence for the iterates of certain positive linear operators towards the first Bernstein operator. Using the spectrum of the operators involved in [15], convergence results were proved for overiterates of certain (generalized) Bernstein-Stancu operators (see, e.g., [24–26]). In [16, 17] new techniques were introduced (infinite products, rates of convergence), based on the results from [18], in order to prove that infinite products of certain positive linear operators weakly converge to the first Bernstein operator.

Now we study the convergence of the iterates of the Bernstein-type operators (3) and (5).

Theorem 9. *The operators B_m^x and B_n^y are weakly Picard operators and*

$$\begin{aligned}(B_m^{x,\infty}F)(x,y) &= \frac{F(g_3(y),y) - F(g_2(y),y)}{g_3(y)-g_2(y)}x \\ &\quad + \frac{g_3(y)F(g_2(y),y) - g_2(y)F(g_3(y),y)}{g_3(y)-g_2(y)},\end{aligned} \tag{13}$$

$$\begin{aligned}(B_n^{y,\infty}F)(x,y) &= \frac{F(x,f_3(x)) - F(x,f_1(x))}{f_3(x)-f_1(x)}y \\ &\quad + \frac{f_3(x)F(x,f_1(x)) - f_1(x)F(x,f_3(x))}{f_3(x)-f_1(x)}.\end{aligned} \tag{14}$$

Proof. Taking into account the interpolation properties of B_m^x and B_n^y (from Theorem 2), let us consider

$$\begin{aligned}X^{(1)}_{\varphi|_{\gamma_2},\varphi|_{\gamma_3}} &= \left\{F \in C(\widetilde{T}_h) \mid F(g_2(y),y) = \varphi|_{\gamma_2},\right. \\ &\qquad \left. F(g_3(y),y) = \varphi|_{\gamma_3}\right\}, \quad \text{for } y \in [0,h],\end{aligned}$$

$$X^{(2)}_{\psi|_{\gamma_1},\psi|_{\gamma_3}} = \left\{F \in C\left(\widetilde{T}_h\right) \mid F\left(x, f_1\left(x\right)\right) = \psi|_{\gamma_1}, \right.$$
$$\left. F\left(x, f_3\left(x\right)\right) = \psi|_{\gamma_3}\right\}, \quad \text{for } x \in [0,h] \tag{15}$$

and denote

$$\begin{aligned} F^{(1)}_{\varphi|_{\gamma_2},\varphi|_{\gamma_3}}\left(x,y\right) &:= \frac{\varphi|_{\gamma_3} - \varphi|_{\gamma_2}}{g_3\left(y\right) - g_2\left(y\right)} x \\ &\quad + \frac{g_3\left(y\right)\varphi|_{\gamma_2} - g_2\left(y\right)\varphi|_{\gamma_3}}{g_3\left(y\right) - g_2\left(y\right)}, \\ F^{(2)}_{\psi|_{\gamma_1},\psi|_{\gamma_3}}\left(x,y\right) &:= \frac{\psi|_{\gamma_3} - \psi|_{\gamma_1}}{f_3\left(x\right) - f_1\left(x\right)} y \\ &\quad + \frac{f_3\left(x\right)\psi|_{\gamma_1} - f_1\left(x\right)\psi|_{\gamma_3}}{f_3\left(x\right) - f_1\left(x\right)}, \end{aligned} \tag{16}$$

with $\varphi, \psi \in C(\widetilde{T}_h)$.

We have the following properties:

(i) $X^{(1)}_{\varphi|_{\gamma_2},\varphi|_{\gamma_3}}$ and $X^{(2)}_{\psi|_{\gamma_1},\psi|_{\gamma_3}}$ are closed subsets of $C(\widetilde{T}_h)$;

(ii) $X^{(1)}_{\varphi|_{\gamma_2},\varphi|_{\gamma_3}}$ is an invariant subset of B^x_m and $X^{(2)}_{\psi|_{\gamma_1},\psi|_{\gamma_3}}$ is an invariant subset of B^y_n, for $\varphi, \psi \in C(\widetilde{T}_h)$ and $n, m \in \mathbb{N}^*$;

(iii) $C(\widetilde{T}_h) = \bigcup_{\varphi \in C(\widetilde{T}_h)} X^{(1)}_{\varphi|_{\gamma_2},\varphi|_{\gamma_3}}$ and $C(\widetilde{T}_h) = \bigcup_{\psi \in C(\widetilde{T}_h)} X^{(2)}_{\psi|_{\gamma_1},\psi|_{\gamma_3}}$ are partitions of $C(\widetilde{T}_h)$;

(iv) $F^{(1)}_{\varphi|_{\gamma_2},\varphi|_{\gamma_3}} \in X^{(1)}_{\varphi|_{\gamma_2},\varphi|_{\gamma_3}} \cap F_{B^x_m}$ and $F^{(2)}_{\psi|_{\gamma_1},\psi|_{\gamma_3}} \in X^{(2)}_{\psi|_{\gamma_1},\psi|_{\gamma_3}} \cap F_{B^y_n}$, where $F_{B^x_m}$ and $F_{B^y_n}$ denote the fixed points sets of B^x_m and B^y_n.

The statements (i) and (iii) are obvious.

(ii), by linearity of Bernstein operators and Theorem 2, it follows that $\forall F^{(1)}_{\varphi|_{\gamma_2},\varphi|_{\gamma_3}} \in X^{(1)}_{\varphi|_{\gamma_2},\varphi|_{\gamma_3}}$ and $\forall F^{(2)}_{\psi|_{\gamma_1},\psi|_{\gamma_3}} \in X^{(2)}_{\psi|_{\gamma_1},\psi|_{\gamma_3}}$; we have

$$\begin{aligned} B^x_m F^{(1)}_{\varphi|_{\gamma_2},\varphi|_{\gamma_3}}\left(x,y\right) &= F^{(1)}_{\varphi|_{\gamma_2},\varphi|_{\gamma_3}}\left(x,y\right), \\ B^y_n F^{(2)}_{\psi|_{\gamma_1},\psi|_{\gamma_3}}\left(x,y\right) &= F^{(2)}_{\psi|_{\gamma_1},\psi|_{\gamma_3}}\left(x,y\right). \end{aligned} \tag{17}$$

So, $X^{(1)}_{\varphi|_{\gamma_2},\varphi|_{\gamma_3}}$ and $X^{(2)}_{\psi|_{\gamma_1},\psi|_{\gamma_3}}$ are invariant subsets of B^x_m and, respectively, of B^y_n, for $\varphi, \psi \in C(\widetilde{T}_h)$ and $n, m \in \mathbb{N}^*$.

(iv), we prove that

$$\begin{aligned} B^x_m|_{X^{(1)}_{\varphi|_{\gamma_2},\varphi|_{\gamma_3}}} &: X^{(1)}_{\varphi|_{\gamma_2},\varphi|_{\gamma_3}} \longrightarrow X^{(1)}_{\varphi|_{\gamma_2},\varphi|_{\gamma_3}}, \\ B^y_n|_{X^{(2)}_{\psi|_{\gamma_1},\psi|_{\gamma_3}}} &: X^{(2)}_{\psi|_{\gamma_1},\psi|_{\gamma_3}} \longrightarrow X^{(2)}_{\psi|_{\gamma_1},\psi|_{\gamma_3}}, \end{aligned} \tag{18}$$

are contractions for $\varphi, \psi \in C(\widetilde{T}_h)$ and $n, m \in \mathbb{N}^*$.

Let $F, G \in X^{(1)}_{\varphi|_{\gamma_2},\varphi|_{\gamma_3}}$. From (3) we have

$$\begin{aligned} &\left|B^x_m\left(F\right)\left(x,y\right) - B^x_m\left(G\right)\left(x,y\right)\right| \\ &\quad = \left|B^x_m\left(F-G\right)\left(x,y\right)\right| \\ &\quad \le \left|1 - \left(1 - \frac{x - g_2(y)}{g_3(y) - g_2(y)}\right)^m - \left(\frac{x - g_2(y)}{g_3(y) - g_2(y)}\right)^m\right| \\ &\qquad \|F-G\|_\infty \le \left(1 - \frac{1}{2^{m-1}}\right)\|F-G\|_\infty, \end{aligned} \tag{19}$$

where $\|\cdot\|_\infty$ denotes the Chebyshev norm. So,

$$\begin{aligned} &\left\|B^x_m(F)(x,y) - B^x_m(G)(x,y)\right\|_\infty \\ &\quad \le \left(1 - \frac{1}{2^{m-1}}\right)\|F-G\|_\infty, \quad \forall F, G \in X^{(1)}_{\varphi|_{\gamma_2},\varphi|_{\gamma_3}}, \end{aligned} \tag{20}$$

that is, $B^x_m|_{X^{(1)}_{\varphi|_{\gamma_2},\varphi|_{\gamma_3}}}$ is a contraction for $\varphi \in C(\widetilde{T}_h)$.

Analogously we have

$$\begin{aligned} &\left|B^y_n\left(F\right)\left(x,y\right) - B^y_n\left(G\right)\left(x,y\right)\right| \\ &\quad = \left|B^y_n\left(F-G\right)\left(x,y\right)\right| \\ &\quad \le \left|1 - \left(1 - \frac{y - f_1(x)}{f_3(x) - f_1(x)}\right)^n - \left(\frac{y - f_1(x)}{f_3(x) - f_1(x)}\right)^n\right| \\ &\qquad \|F-G\|_\infty \le \left(1 - \frac{1}{2^{n-1}}\right)\|F-G\|_\infty, \end{aligned} \tag{21}$$

whence

$$\begin{aligned} &\left\|B^y_n(F)(x,y) - B^y_n(G)(x,y)\right\|_\infty \\ &\quad \le \left(1 - \frac{1}{2^{n-1}}\right)\|F-G\|_\infty, \quad \forall F, G \in X^{(2)}_{\psi|_{\gamma_1},\psi|_{\gamma_3}}, \end{aligned} \tag{22}$$

that is, $B^y_n|_{X^{(2)}_{\psi|_{\gamma_1},\psi|_{\gamma_3}}}$ is a contraction for $\psi \in C(\widetilde{T}_h)$.

On the other hand, $((\varphi|_{\gamma_3} - \varphi|_{\gamma_2})/(g_3(y) - g_2(y)))(\cdot) + (g_3(y)\varphi|_{\gamma_2} - g_2(y)\varphi|_{\gamma_3})/(g_3(y) - g_2(y)) \in X^{(1)}_{\varphi|_{\gamma_2},\varphi|_{\gamma_3}}$, $((\psi|_{\gamma_3} - \psi|_{\gamma_1})/(f_3(x) - f_1(x)))(\cdot) + (f_3(x)\psi|_{\gamma_1} - f_1(x)\psi|_{\gamma_3})/(f_3(x) - f_1(x)) \in X^{(2)}_{\psi|_{\gamma_1},\psi|_{\gamma_3}}$ are fixed points of B^x_m and B^y_n; that is,

$$\begin{aligned} &B^x_m\left(\frac{\varphi|_{\gamma_3} - \varphi|_{\gamma_2}}{g_3\left(y\right) - g_2\left(y\right)}\left(\cdot\right) + \frac{g_3\left(y\right)\varphi|_{\gamma_2} - g_2\left(y\right)\varphi|_{\gamma_3}}{g_3\left(y\right) - g_2\left(y\right)}\right) \\ &\quad = \frac{\varphi|_{\gamma_3} - \varphi|_{\gamma_2}}{g_3\left(y\right) - g_2\left(y\right)}\left(\cdot\right) + \frac{g_3\left(y\right)\varphi|_{\gamma_2} - g_2\left(y\right)\varphi|_{\gamma_3}}{g_3\left(y\right) - g_2\left(y\right)}, \\ &B^y_n\left(\frac{\psi|_{\gamma_3} - \psi|_{\gamma_1}}{f_3\left(x\right) - f_1\left(x\right)}\left(\cdot\right) + \frac{f_3\left(x\right)\psi|_{\gamma_1} - f_1\left(x\right)\psi|_{\gamma_3}}{f_3\left(x\right) - f_1\left(x\right)}\right) \\ &\quad = \frac{\psi|_{\gamma_3} - \psi|_{\gamma_1}}{f_3\left(x\right) - f_1\left(x\right)}\left(\cdot\right) + \frac{f_3\left(x\right)\psi|_{\gamma_1} - f_1\left(x\right)\psi|_{\gamma_3}}{f_3\left(x\right) - f_1\left(x\right)}. \end{aligned} \tag{23}$$

From the contraction principle, $F^{(1)}_{\varphi|_{\gamma_2},\varphi|_{\gamma_3}}(x,y) := ((\varphi|_{\gamma_3} - \varphi|_{\gamma_2})/(g_3(y) - g_2(y)))x + (g_3(y)\varphi|_{\gamma_2} - g_2(y)\varphi|_{\gamma_3})/(g_3(y) - g_2(y))$ is the unique fixed point of B^x_m in $X^{(1)}_{\varphi|_{\gamma_2},\varphi|_{\gamma_3}}$ and $B^x_m|_{X^{(1)}_{\varphi|_{\gamma_2},\varphi|_{\gamma_3}}}$ is a Picard operator, with

$$\begin{aligned}(B^{x,\infty}_m F)(x,y) &= \frac{F(g_3(y),y) - F(g_2(y),y)}{g_3(y) - g_2(y)}x \\ &+ \frac{g_3(y)F(g_2(y),y) - g_2(y)F(g_3(y),y)}{g_3(y) - g_2(y)},\end{aligned} \tag{24}$$

and, similarly, $F^{(2)}_{\psi|_{\gamma_1},\psi|_{\gamma_3}}(x,y) := ((\psi|_{\gamma_3} - \psi|_{\gamma_1})/(f_3(x) - f_1(x)))y + (f_3(x)\psi|_{\gamma_1} - f_1(x)\psi|_{\gamma_3})/(f_3(x) - f_1(x))$ is the unique fixed point of B^y_n in $X^{(2)}_{\psi|_{\gamma_1},\psi|_{\gamma_3}}$ and $B^y_n|_{X^{(2)}_{\psi|_{\gamma_1},\psi|_{\gamma_3}}}$ is a Picard operator, with

$$\begin{aligned}(B^{y,\infty}_n F)(x,y) = {} & \frac{F(x,f_3(x)) - F(x,f_1(x))}{f_3(x) - f_1(x)}y \\ &+ \frac{f_3(x)F(x,f_1(x)) - f_1(x)F(x,f_3(x))}{f_3(x) - f_1(x)}.\end{aligned} \tag{25}$$

Consequently, taking into account (ii), by Theorem 8 it follows that the operators B^x_m and B^y_n are weakly Picard operators. □

Now we study the convergence of the product and Boolean sum operators (7) and (9).

Theorem 10. *The operator P_{mn} is a weakly Picard operator and*

$$\begin{aligned}(P^{\infty}_{mn}F)(x,y) &= \frac{1}{[g_3(y) - g_2(y)][f_3(x) - f_1(x)]} \\ &\times [g_3(y)f_3(x_0)F(x_0,f_1(x_0)) + g_2(y)f_1(x_1) \\ &\quad\times F(x_1,f_3(x_1)) - g_3(y)f_1(x_1) \\ &\quad\times F(x_0,f_3(x_0)) - g_2(y)f_3(x_0)F(x_1,f_1(x_1))] \\ &+ \frac{x}{[g_3(y) - g_2(y)][f_3(x) - f_1(x)]} \\ &\times [f_1(x_1)F(x_0,f_3(x_0)) + f_3(x_0)F(x_1,f_1(x_1)) \\ &\quad - f_3(x_0)F(x_0,f_1(x_0)) \\ &\quad - f_1(x_0)F(x_1,f_3(x_1))] \\ &+ \frac{y}{[g_3(y) - g_2(y)][f_3(x) - f_1(x)]} \\ &\times [g_3(y)F(x_0,f_3(x_0)) \\ &\quad + g_2(y)F(x_1,f_1(x_1)) - g_3(y) \\ &\quad\times F(x_0,f_1(x_0)) - g_2(y)F(x_1,f_3(x_1))] \\ &+ \frac{xy}{[g_3(y) - g_2(y)][f_3(x) - f_1(x)]} \\ &\times [F(x_0,f_1(x_0)) + F(x_1,f_3(x_1)) \\ &\quad - F(x_0,f_3(x_0)) - F(x_1,f_1(x_1))], \\ &\textit{with } x_0 = g_2(y),\ x_1 = g_3(y).\end{aligned} \tag{26}$$

Proof. Let $X_{\alpha,\beta,\gamma,\delta} = \{F \in C(\widetilde{T}_h) \mid F(x_0,f_1(x_0)) = \alpha, F(x_1,f_1(x_1)) = \beta, F(x_1,f_3(x_1)) = \gamma, F(x_0,f_3(x_0)) = \delta\}$ and denote

$$\begin{aligned}&F_{\alpha,\beta,\gamma,\delta}(x,y) \\ &:= (g_3(y)f_3(x_0)\alpha + g_2(y)f_1(x_1)\gamma \\ &\quad - g_3(y)f_1(x_1)\delta - g_2(y)f_3(x_0)\beta) \\ &\quad\times ([g_3(y) - g_2(y)][f_3(x) - f_1(x)])^{-1} \\ &\quad + \frac{f_1(x_1)\delta + f_3(x_0)\beta - f_3(x_0)\alpha - f_1(x_0)\gamma}{[g_3(y) - g_2(y)][f_3(x) - f_1(x)]}x \\ &\quad + \frac{g_3(y)\delta + g_2(y)\beta - g_3(y)\alpha - g_2(y)\gamma}{[g_3(y) - g_2(y)][f_3(x) - f_1(x)]}y \\ &\quad + \frac{\alpha + \gamma - \beta - \delta}{[g_3(y) - g_2(y)][f_3(x) - f_1(x)]}xy\end{aligned} \tag{27}$$

with $\alpha,\beta,\gamma,\delta \in \mathbb{R}$.

We remark that

(i) $X_{\alpha,\beta,\gamma,\delta}$ is a closed subset of $C(\widetilde{T}_h)$;

(ii) $X_{\alpha,\beta,\gamma,\delta}$ is an invariant subset of P_{mn}, for $\alpha,\beta,\gamma,\delta \in \mathbb{R}$ and $n,m \in \mathbb{N}^*$;

(iii) $C(\widetilde{T}_h) = \bigcup_{\alpha,\beta,\gamma,\delta} X_{\alpha,\beta,\gamma,\delta}$ is a partition of $C(\widetilde{T}_h)$;

(iv) $F_{\alpha,\beta,\gamma,\delta} \in X_{\alpha,\beta,\gamma,\delta} \cap F_{P_{mn}}$, where $F_{P_{mn}}$ denote the fixed points sets of P_{mn}.

The statements (i) and (iii) are obvious.

(ii), similarly with the proof of Theorem 9, by linearity of Bernstein operators and Theorem 3, it follows that $X_{\alpha,\beta,\gamma,\delta}$ is an invariant subset of P_{mn}, for $\alpha,\beta,\gamma,\delta \in \mathbb{R}$ and $n,m \in \mathbb{N}^*$.

(iv), we prove that

$$P_{mn}|_{X_{\alpha,\beta,\gamma,\delta}} : X_{\alpha,\beta,\gamma,\delta} \longrightarrow X_{\alpha,\beta,\gamma,\delta} \tag{28}$$

is a contraction for $\alpha,\beta,\gamma,\delta \in \mathbb{R}$ and $n,m \in \mathbb{N}^*$. Let $F,G \in X_{\alpha,\beta,\gamma,\delta}$. From [7, Lemma 8] it follows that

$$\begin{aligned}|P_{mn}(F)(x,y) - P_{mn}(G)(x,y)| &= |P_{mn}(F-G)(x,y)| \\ &\le \left(1 - \frac{1}{2^{m+n-2}}\right)\|F-G\|_{\infty}.\end{aligned} \tag{29}$$

So,

$$\begin{aligned}&\left\|P_{mn}(F)(x,y)-P_{mn}(G)(x,y)\right\|_\infty\\&\quad\le\left(1-\frac{1}{2^{m+n-2}}\right)\|F-G\|_\infty,\quad\forall F,G\in X_{\alpha,\beta,\gamma,\delta},\end{aligned}\tag{30}$$

that is, $P_{mn}|_{X_{\alpha,\beta,\gamma,\delta}}$ is a contraction for $\alpha,\beta,\gamma,\delta\in\mathbb{R}$.

From the contraction principle we have that $F_{\alpha,\beta,\gamma,\delta}$ is the unique fixed point of P_{mn} in $X_{\alpha,\beta,\gamma,\delta}$ and $P_{mn}|_{X_{\alpha,\beta,\gamma,\delta}}$ is a Picard operator, so (26) holds. Consequently, taking into account (ii), by Theorem 8 it follows that the operator P_{mn} is a weakly Picard operator. □

Remark 11. We have a similar result for the operator Q_{nm}.

Theorem 12. *The operator S_{mn} is a weakly Picard operator and*

$$\begin{aligned}&(S_{mn}^\infty F)(x,y)\\&=\frac{F(g_3(y),y)-F(g_2(y),y)}{g_3(y)-g_2(y)}x\\&+\frac{g_3(y)F(g_2(y),y)-g_2(y)F(g_3(y),y)}{g_3(y)-g_2(y)}\\&+\frac{F(x,f_3(x))-F(x,f_1(x))}{f_3(x)-f_1(x)}y\\&+\frac{f_3(x)F(x,f_1(x))-f_1(x)F(x,f_3(x))}{f_3(x)-f_1(x)}\\&-\frac{1}{[g_3(y)-g_2(y)][f_3(x)-f_1(x)]}\\&\times[g_3(y)f_3(x_0)F(x_0,f_1(x_0))+g_2(y)f_1(x_1)\\&\quad\times F(x_1,f_3(x_1))-g_3(y)f_1(x_1)F(x_0,f_3(x_0))\\&\quad-g_2(y)f_3(x_0)F(x_1,f_1(x_1))]\\&-\frac{x}{[g_3(y)-g_2(y)][f_3(x)-f_1(x)]}\\&\times[f_1(x_1)F(x_0,f_3(x_0))+f_3(x_0)F(x_1,f_1(x_1))\\&\quad-f_3(x_0)F(x_0,f_1(x_0))-f_1(x_0)F(x_1,f_3(x_1))]\\&-\frac{y}{[g_3(y)-g_2(y)][f_3(x)-f_1(x)]}\\&\times[g_3(y)F(x_0,f_3(x_0))+g_2(y)F(x_1,f_1(x_1))\\&\quad-g_3(y)F(x_0,f_1(x_0))-g_2(y)F(x_1,f_3(x_1))]\\&-\frac{xy}{[g_3(y)-g_2(y)][f_3(x)-f_1(x)]}\\&\times[F(x_0,f_1(x_0))+F(x_1,f_3(x_1))\\&\quad-F(x_0,f_3(x_0))-F(x_1,f_1(x_1))],\\&\quad\textit{with } x_0=g_2(y),x_1=g_3(y).\end{aligned}\tag{31}$$

Proof. The proof follows the same steps as in the previous theorems but using the following inequality:

$$\begin{aligned}&\left\|S_{mn}(F)(x,y)-S_{mn}(G)(x,y)\right\|_\infty\\&\quad\le\left[1-\left(\frac{1}{2^{m-1}}+\frac{1}{2^{n-1}}-\frac{1}{2^{m+n-2}}\right)\right]\|F-G\|_\infty,\end{aligned}\tag{32}$$

in order to prove that S_{mn} is a contraction. □

Remark 13. We have a similar result for the operator T_{nm}.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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