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Research Article

Lightlike Hypersurfaces and Canal Hypersurfaces of Lorentzian Surfaces

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The lightlike hypersurfaces in semi-Euclidean space are of special interest in Relativity Theory. In particular, the singularities of these lightlike hypersurfaces provide good models for the study of different horizon types. And we obtain some geometrical propositions of the canal hypersurfaces of Lorentzian surfaces. We introduce the notions of flatness for these hypersurfaces and study their singularities.

1. Introduction

The extrinsic differential geometry of submanifolds in 4-dimensional semi-Euclidean space is of special interest in Relativity Theory. In particular the lightlike hypersurfaces, which can be constructed as lightlike ruled hypersurfaces over Lorentzian surfaces in anti-de Sitter space, provide good models for the study of different horizon types of black holes, such as Kerr black hole, Cauchy black hole, and Schwarzschild black hole [1–8]. Hiscock described that the horizon was constituted by lightlike hypersurfaces and lightlike wave front was lightlike hypersurface [6]; Dąbrowski et al. have studied the null (lightlike) strings form the photon sphere, moving in the single spacetime of general relativity, including lightlike hypersurfaces [1, 3, 4]. The authors gave the null string evolution in Schwarzschild spacetime by the solutions of null string equations, which are also the null geodesic equations of general relativity appended by an additional stringy constant [3, 4]. In the view of geometry, the null string (null curve) in lightlike surfaces is null geodesic [9]. In this sense, the singularities of lightlike hypersurfaces are deeply related to the shapes of horizons.

M. Kossowski introduced a Gauss map on its associated spacelike surface, obtaining in this way interesting conclusions on the lightlike hypersurfaces which parallel the known results for surfaces in Euclidean 3-space concerning their contacts with the model surfaces [10]. When working in semi-Euclidean space, we observe that the properties associated with the contacts of a given submanifold with null cone and lightlike hyperplanes have a special relevance from the geometrical viewpoint. In [11–13], the current authors and so forth pursued with this line by describing the invariant geometric properties of Lorentzian surfaces of codimension two in semi-Euclidean space that arise from their contacts with null cone. For this purpose, the task of this paper is to study some local properties of these Lorentzian surfaces in semi-Euclidean $(n+1)$-space.

Canal hypersurfaces, which are generated by surfaces with codimension 2 along fixed direction, are envelopes of families of hyperspheres. In three-dimensional space, canal surfaces were considered in many classical texts on differential geometry [14]. Since the property of a hypersurface to be a canal hypersurface is conformally invariant, canal hypersurfaces in a multidimensional Euclidean space were investigated in many papers, such as [15, 16]. However, in all these works the authors did not note the singularities of canal hypersurface in semi-Euclidean space. In this paper, we analyze the geometric meaning of the canal hypersurfaces from the view point of singularity. And we obtain the conclusion that the canal hypersurfaces have the similar singularities as Lorentzian surfaces.

The remainder of this paper is organized as follows. In Section 2, we give some basic notions about Lorentzian surfaces and lightlike hypersurfaces. Meanwhile, the Lorentzian Gauss-Kronecker curvatures of Lorentzian surfaces are also introduced. In Section 3, we describe Lorentzian distance-squared functions, whose discriminant sets and wave front sets are just right of the given lightlike hypersurfaces. In Section 4, we discuss the contact between lightlike hypersurfaces and null cone by Montald's theorem. We give an example about the classification of singularities to lightlike hypersurfaces generated by Lorentzian surfaces in anti-de Sitter space in Section 5. In the last section, we consider some geometric properties of canal hypersurfaces, which are generated by Lorentzian surfaces in anti-de Sitter 3-space and the conclusion that the types of singularity of canal hypersurfaces are the same as the Lorentzian surfaces.

We will assume throughout the whole paper that all manifolds and maps are C^∞ unless the contrary is explicitly stated.

2. Preliminaries

Einstein formulated general relativity as a theory of space, time, and gravitation in semi-Euclidean space in 1915. However, this subject has remained dormant for much of its history because its understanding requires advanced mathematics knowledge. Since the end of the twentieth century, semi-Euclidean geometry has been an active area of mathematical research, and it has been applied to a variety of subjects related to differential geometry and general relativity. In this section, we illustrated some basic knowledge of semi-Euclidean space.

Let $\mathbb{R}^{n+1} = \{(x_0, x_1, \ldots, x_n) \mid x_i \in \mathbb{R},\ i = 0, 1, \ldots, n\}$ be an $(n+1)$-dimensional vector space. For any vectors $\mathbf{x} = (x_0, x_1, \ldots, x_n)$ and $\mathbf{y} = (y_0, y_1, \ldots, y_n)$ in $\mathbb{R}^{n+1}$, the pseudoscalar product of $\mathbf{x}$ and $\mathbf{y}$ is defined by

$$\langle \mathbf{x}, \mathbf{y} \rangle = -x_0 y_0 - x_1 y_1 + \sum_{i=2}^{n} x_i y_i. \tag{1}$$

The space $(\mathbb{R}^{n+1}, \langle , \rangle)$ is called semi-Euclidean $(n+1)$-dimensional space with index two, denoted by $\mathbb{R}_2^{n+1}$. A vector $\mathbf{x} \in \mathbb{R}_2^{n+1} \setminus \{\mathbf{0}\}$ is called *spacelike, lightlike,* or *timelike,* if $\langle \mathbf{x}, \mathbf{x} \rangle$ is positive, zero, or negative, respectively. There exist some special submanifolds in $\mathbb{R}_2^{n+1}$, such as unit pseudo-n-sphere $\mathbb{S}_2^n$, anti de Sitter n-space $\mathbb{H}_1^n$, null cone Λ_1^n, and Lorentz torus $S_t^1 \times S_s^{n-1}$, which have the same definitions as in [17].

Definition 1. Let $\mathbf{X} : U \to \mathbb{H}_1^n$ be an embedding, where $U \subset \mathbb{R}^{n-1}$ is an open subset; if there exists i such that $\mathbf{X}(u), \mathbf{X}_{u_i}(u)$ is timelike vector and $\mathbf{X}_{u_j}(u)\ (j \neq i)$ is spacelike vector, we call $M = \mathbf{X}(U)$ Lorentzian surface in anti-de Sitter space.

Without loss of generality, we only consider $i = 1$; the other cases are the same. We construct a unit spacelike normal vector

$$\mathbf{N}(u) = \frac{\mathbf{X}(u) \wedge \mathbf{X}_{u_1}(u) \wedge \cdots \wedge \mathbf{X}_{u_{n-1}}(u)}{\left\| \mathbf{X}(u) \wedge \mathbf{X}_{u_1}(u) \wedge \cdots \wedge \mathbf{X}_{u_{n-1}}(u) \right\|}, \tag{2}$$

and the vectors $\mathbf{X}(u) \pm \mathbf{N}(u)$ are lightlike. Since $\{\mathbf{X}_{u_1}(u), \ldots, \mathbf{X}_{u_{n-1}}(u)\}$ is a basis of T_pM, so the system $\{\mathbf{X}(u), \mathbf{X}_{u_1}(u), \ldots, \mathbf{X}_{u_{n-1}}(u), \mathbf{N}(u)\}$ provides a basis for $\mathbb{R}_2^{n+1}$.

We define a map $\mathbb{L}^\pm : U \to S_t^1 \times S_s^{n-1}$ by

$$\mathbb{L}^\pm(u) = \widetilde{\mathbf{X}(u) \pm \mathbf{N}}(u), \tag{3}$$

which is called *the lightcone Gauss indicatrix* of $\mathbf{X}(u)$. We have shown that $\pi^t : T_pM \oplus N_pM \to T_pM$ and $\pi^n : T_pM \oplus N_pM \to N_pM$ [11]. Under the identification of U and M, the derivative $d\mathbf{X}(u_0)$ can be identified to the identity mapping id_{T_pM} on the tangent space T_pM, where $p = \mathbf{X}(u_0)$. This means that $d\mathbb{L}^\pm(u_0) = \mathrm{id}_{T_pM} \pm d\mathbf{N}(u_0)$. Thus, $d\mathbb{L}^\pm(u_0)$ can be regarded as a linear transformation on the tangent space T_pM. We call the linear transformation $S_p^\pm(u) = -d_p(\mathbb{L}^\pm(u)) : T_pM \to T_pM$ the *Lorentzian shape operator* of M at $p = \mathbf{X}(u_0)$. We denote the eigenvalue of $S_p^\pm$ by $k^\pm$, which is called a Lorentzian principal curvature of M at point p. The *Lorentzian Gauss-Kronecker curvature* of M is defined as $K_l^\pm(u)(p) = \det S_p^\pm(u)$.

Definition 2. A point $p = \mathbf{X}(u)$ is an umbilic point if all the principal curvatures coincide at p. M is called totally umbilic surface if all points on M are umbilics.

Supposing $M = \mathbf{X}(U)$ is totally umbilic, we have the following propositions by simple computation.

Proposition 3. *If $\mathbf{X}(U)$ is totally umbilic, we have the following classification.*

(1) *Suppose that $k^\pm \neq 0$.*

  (a) *If $0 \leq |k^\pm + 1| < 1$, then M is a part of an anti-de Sitter space. In particular, if $k^\pm = -1$, then M is a part of a small anti-de Sitter space.*
  (b) *If $|k^\pm + 1| > 1$, then M is a part of unit pseudo-n-sphere.*

(2) *Suppose $k^\pm = 0$. Then M is a part of hyperhorosphere.*

Proposition 4. *Let $\mathbf{X}(U)$ be a Lorentzian surface in anti-de Sitter space, the lightcone Gauss indicatrix is constant if and only if there exists a unique lightlike hyperplane $HP(\mathbf{n}, -1)$ in $\mathbb{R}_2^{n+1}$, such that the $M = \mathbf{X}(U)$ is a part of $\mathbb{H}_1^n \cap HP(\mathbf{n}, -1)$, where $\mathbf{n} = \mathbb{L}^\pm(u)$.*

Since $\mathbf{X}_{u_1}$ is timelike vector, $\mathbf{X}_{u_i}$ $(i = 2, \ldots, n-1)$ is spacelike vector, and semi-Riemannian metric on $M = \mathbf{X}(U)$ defined by $ds^2 = \sum_{i=1}^{n-1} \delta_i g_{ij} du_i du_j$ [18, 19], where $g_{ij} = \langle \mathbf{X}_{u_i}(u), \mathbf{X}_{u_j}(u) \rangle$, $\delta_1 = -1$, and $\delta_i = 1$, for any $i = 2, \ldots, n-1$, we have a Lorentzian second fundamental invariant with respect to the vectors $\mathbf{X}(u)$, $\mathbf{N}(u)$ defined by $h_{ij}(\mathbf{X}, \mathbf{N})(u) = \langle -\mathbb{L}_{u_i}^\pm(u), \mathbf{X}_{u_j}(u) \rangle$, for any $u \in U$.

Proposition 5. *The Lorentzian Weingarten formulas with respect to $\mathbf{X}(u)$, $\mathbf{N}(u)$ are as follows.*

(1) $\mathbb{L}_{u_i}^\pm = -\sum_{j=1}^{n} \delta_i h_i^{j\pm} \mathbf{X}_{u_j}$,

(2) $\pi^t \circ \mathbb{L}^{\pm}_{u_i} = -\sum_{j=1}^{n} \delta_i h_i^{j\pm} \mathbf{X}_{u_j}$, *where* $h_i^{j\pm} = h_{ik}^{\pm} g^{kj}$, $g^{kj} = g_{kj}^{-1}$.

Proof. There exist real numbers λ, μ, $\Gamma_i^{j\pm}$ such that $\mathbb{L}^{\pm}_{u_i} = \lambda \mathbf{X}(u) + \mu \mathbf{N}(u) + \sum_{j=1}^{n-1} \Gamma_i^{j\pm} \mathbf{X}_{u_j}$. Since $\langle \pi^n \circ \mathbb{L}^{\pm}_{u_i}, \mathbf{X}_{u_j} \rangle = 0$, we have

$$-h_{ik} = \left\langle \mathbb{L}^{\pm}_{u_i}, \mathbf{X}_{u_k} \right\rangle = \sum_{j=1}^{n-1} \Gamma_i^{j\pm} \left\langle \mathbf{X}_{u_j}, \mathbf{X}_{u_k} \right\rangle = \sum_{j=1}^{n-1} \Gamma_i^{j} \delta_j g_{jk}. \quad (4)$$

Hence, we have $h_i^{j\pm} = \sum_{k=1}^{n-1} h_{ik}^{\pm} g^{kj} = -\sum_{k=1}^{n-1} \sum_{m=1}^{n-1} \Gamma_i^{m\pm} \delta_i g_{mk} g^{kj} = -\delta_i \Gamma_i^{j\pm}$; the formula (2) follows the conclusion of item (1). □

As a corollary of the Proposition 5, we have an explicit expression of the Lorentzian Gauss-Kronecker curvature by Riemannian metric and the second fundamental invariant.

Corollary 6. *Under the same notations as in the above proposition, the Lorentzian Gauss-Kronecker curvature is given by* $K_l^{\pm}(u) = \det(h_{ij}^{\pm}) / \det(g_{km})$.

Proof. By the above proposition, the representation matrix of the Lorentzian shape operator with respect to the basis $\{\mathbf{X}_{u_1}, \mathbf{X}_{u_2}, \ldots, \mathbf{X}_{u_{n-1}}\}$ is $h_i^{j\pm}(\mathbf{N}) = (h_{i\beta}^{\pm})(g^{\beta j})$. It follows from this fact that

$$\begin{aligned} K_l^{\pm}(u) &= \det S_p^{\pm}(\mathbf{X}(u), \mathbf{N}(u)) = \det(-d\mathbb{L}^{\pm}) \\ &= -\det \Gamma_i^{j\pm} = \det h_i^{j\pm} = \det h_{ij}^{\pm} g^{jk} = \frac{\det\left(h_{ij}^{\pm}\right)}{\det\left(g_{km}\right)}. \end{aligned} \quad (5)$$

So we complete the proof. □

Since $\langle -\mathbb{L}^{\pm}(u), \mathbf{X}_{u_j}(u) \rangle = 0$, we have $h_{ij}^{\pm} = \langle \mathbb{L}^{\pm}(u), \mathbf{X}_{u_i u_j}(u) \rangle$. Therefore, the Lorentzian second fundamental invariant depends on the values $\mathbb{L}^{\pm}(u_0)$, $\mathbf{X}_{u_i u_j}(u_0)$. By the above corollary, the Lorentzian Gauss-Kronecker curvature depends only on $\mathbb{L}^{\pm}(u_0)$, $\mathbf{X}_{u_i}(u_0)$, $\mathbf{X}_{u_i u_j}(u_0)$. It is independent on the choice of the normal vector field $\mathbf{N}(u)$.

Definition 7. Let $M = \mathbf{X}(u)$ be a Lorentzian surface in anti-de Sitter space and let $\mathbf{N}(u)$ be its spacelike normal vector; a hypersurface $LH_M^{\pm}$ defined by $LH_M^{\pm}(u, \mu) = \mathbf{X}(u) + \mu(\widetilde{\mathbf{X}(u) \pm \mathbf{N}}(u))$ is called $LH_M^{\pm}$ the lightlike hypersurface along M.

3. Lorentzian Distance-Squared Function

To describe the existence of singularities of lightlike hypersurfaces, we should construct contact functions, whose wave front set is the singularity set of lightlike hypersurfaces. In this section, we introduce some notions of Lorentzian distance-squared functions on Lorentzian surfaces in anti-de Sitter space, which can supply the contact relationship between Lorentzian surfaces and standard spherical surfaces. Meanwhile, we obtain the Lorentzian distance-squared functions as Morse family.

A function $G : M \times \mathbb{R}_2^{n+1} \to \mathbb{R}$ on the Lorentzian surface is given by

$$G(u, \boldsymbol{\lambda}) = \langle \mathbf{X}(u) - \boldsymbol{\lambda}, \mathbf{X}(u) - \boldsymbol{\lambda} \rangle, \quad (6)$$

which is called *Lorentzian distance-squared function* on M. For any fixed $\boldsymbol{\lambda}_0 \in \mathbb{R}_2^{n+1}$, we write $g_{\lambda_0}(u) = G(u, \boldsymbol{\lambda}_0)$ and have the following propositions by simple computing.

Proposition 8. *Let M be a Lorentzian surface in anti-de Sitter space and let $G : M \times \mathbb{R}_2^{n+1} \to \mathbb{R}$ be Lorentzian distance-squared function on M. Suppose that $\boldsymbol{\lambda}_0 \neq p_0 = \mathbf{X}(u_0)$. Then we have*

(1) $g_{\lambda_0}(p_0) = \partial g_{\lambda_0}/\partial u_0 = \cdots = \partial g_{\lambda_0}/\partial u_{n-1} = 0$ *if and only if* $p_0 - \boldsymbol{\lambda}_0 = \mu(\widetilde{\mathbf{X}(u_0) \pm \mathbf{N}}(u_0))$ *for* $\mu \in \mathbb{R} \setminus \{0\}$.

(2) $g_{\lambda_0}(p_0) = \partial g_{\lambda_0}/\partial u_0 = \cdots = \partial g_{\lambda_0}/\partial u_{n-1} = \det$ *Hess*$(g_{\lambda_0})(u_0) = 0$ *if and only if* $p_0 - \boldsymbol{\lambda}_0 = \mu(\widetilde{\mathbf{X}(u_0) \pm \mathbf{N}}(u_0))$ *for* $\mu \in \mathbb{R} \setminus \{0\}$ *and* $K_l^{\pm}(u_0) = 0$.

Proposition 9. *Let $\boldsymbol{\lambda}_0 \in \mathbb{R}_2^{n+1}$ and let M be a Lorentzian surface without any umbilic point satisfying $K_l^{\pm}(u) \neq 0$. Then $M \subset \Lambda_{1\lambda_0}^n$ if and only if $\boldsymbol{\lambda}_0$ is an isolated singular value of the lightlike hypersurface LH_M and $LH_M \subset \Lambda_{1\lambda_0}^n$.*

Proof. By definition, $M \subset \Lambda_{1\lambda_0}^n$ if and only if $g_{\lambda_0}(u) = 0$, for any $u \in U$, where $g_{\lambda_0}(u) = G(u, \boldsymbol{\lambda}_0)$ is the Lorentzian distance-squared function on M. It follows from Proposition 8 that there exists a smooth function $\rho : U \to \mathbb{R}$ such that $\mathbf{X}(u) = \boldsymbol{\lambda}_0 + \rho(u)(\widetilde{\mathbf{X}(u) \pm \mathbf{N}}(u))$. Therefore,

$$LH_M(\mu, \rho) = \boldsymbol{\lambda}_0 + (\mu + \rho(u)) \left(\widetilde{\mathbf{X}(u) \pm \mathbf{N}}(u)\right). \quad (7)$$

Hence, we have $LH_M(\mu, \rho) \subset \Lambda_{1\lambda_0}^n$. Moreover, we get that

$$\begin{aligned} \frac{\partial LH_M}{\partial \mu} &= \widetilde{\mathbf{X}(u) \pm \mathbf{N}}(u), \\ \frac{\partial LH_M}{\partial u_i} &= \rho_{u_i} \left(\widetilde{\mathbf{X}(u) \pm \mathbf{N}}(u)\right) + (\rho + \mu) \left(\widetilde{\mathbf{X}(u) + \mathbf{N}}(u)\right)_{u_i} \end{aligned} \quad (8)$$

for any $i = 1, 2 \ldots, n-1$, and from above formulas, we can obtain

$$\begin{aligned} &\frac{\partial LH_M}{\partial \rho} \wedge \frac{\partial LH_M}{\partial u_1} \wedge \cdots \wedge \frac{\partial LH_M}{\partial u_{n-1}} \\ &\quad = (\rho + \mu)^{n-1} \left(\widetilde{\mathbf{X}(u) \pm \mathbf{N}}(u)\right) \wedge \left(\widetilde{\mathbf{X}(u) \pm \mathbf{N}}(u)\right)_{u_1} \\ &\quad \wedge \cdots \wedge \left(\widetilde{\mathbf{X}(u) \pm \mathbf{N}}(u)\right)_{u_{n-1}}. \end{aligned} \quad (9)$$

Therefore, we have $\mathbf{X}(u) - \boldsymbol{\lambda}_0 = \rho(u)(\mathbf{X}(\widetilde{u) \pm \mathbf{N}}(u))$, since $\mathbf{X}(u) - \boldsymbol{\lambda}_0$ is lightlike, $\mathbf{X}_{u_1}$ is a timelike vector, and $\mathbf{X}_{u_i}$ ($i = 2, \ldots, n-1$) is spacelike vector. $\mathbf{X}(u) - \boldsymbol{\lambda}_0, \mathbf{X}_{u_1}, \ldots, \mathbf{X}_{u_{n-1}}$ are linearly independent. Therefore,

$$\begin{aligned}(\mathbf{X}(u) - \boldsymbol{\lambda}_0) \wedge \mathbf{X}_{u_1} \wedge \cdots \wedge \mathbf{X}_{u_{n-1}} &\\ = \mu^n(u)\left(\mathbf{X}(\widetilde{u) \pm \mathbf{N}}(u)\right) \wedge \left(\mathbf{X}(\widetilde{u) \pm \mathbf{N}}(u)\right)_{u_1} &\\ \wedge \cdots \wedge \left(\mathbf{X}(\widetilde{u) \pm \mathbf{N}}(u)\right)_{u_{n-1}} \neq \mathbf{0}, &\end{aligned} \tag{10}$$

so $\partial LH_M/\partial\rho \wedge \partial LH_M/\partial u_1 \wedge \cdots \wedge \partial LH_M/\partial u_{n-1} = \mathbf{0}$ if and only if $\mu + \rho(u) = 0$ under the assumption that $K_l(u) \neq 0$. This means that $\boldsymbol{\lambda}_0$ is an isolated singularity of $LH_M^{\pm}$. $\square$

Since we only consider local properties, we may assume that $M = \mathbb{R}^n$. As the definitions in [11], it follows that

$$\begin{aligned}\Sigma_*(G) = \Big\{&(u,\lambda) \in (U \times \mathbb{R}^{n+1}, \mathbf{0}) \mid \\ &G(u,\lambda) = \frac{\partial G}{\partial u_1}(u,\lambda) = \cdots = \frac{\partial G}{\partial u_k}(u,\lambda) = 0\Big\}.\end{aligned} \tag{11}$$

The set $\Sigma_*(G)$ is defined as the wave front set of G. Also, we can write $\Sigma_*(G)$ as

$$\mathscr{D}_G = \{\mathbf{X}(u) + \mu\mathbb{L}^{\pm}(u),\ \mu \in \mathbb{R}\}. \tag{12}$$

Thus, a singular point of the lightlike hypersurface satisfied $\boldsymbol{\lambda}_0 = \mathbf{X}(u_0) + \mu_0(\mathbf{X}(\widetilde{u_0) \pm \mathbf{N}}(u_0))$.

Definition 10. Let G be a Morse family, a map germ $\mathscr{L}_G : (\Sigma_*(G), \mathbf{0}) \to PT^*\mathbb{R}_2^{n+1}$, which satisfied $\mathscr{L}_G(q,x) = (x, [(\partial G/\partial q_1)(q,x) : \cdots : (\partial G/\partial q_k)(q,x)])$, is called *Legendrian immersion germ*.

Let $\pi : PT^*\mathbb{R}_2^{n+1} \to \mathbb{R}_2^{n+1}$ be the projective cotangent bundle over an n-dimensional manifold in $\mathbb{R}_2^{n+1}$. This fibration can be considered as a Legendrian fibration with the canonical contact structure K on $PT^*\mathbb{R}_2^{n+1}$. Let us consider the tangent bundle $\tau : TPT^*\mathbb{R}_2^{n+1} \to PT^*\mathbb{R}_2^{n+1}$ and $d\pi : TPT^*\mathbb{R}_2^{n+1} \to T\mathbb{R}_2^{n+1}$. The property $\alpha(V) = 0$ does not depend on the choice of representative of the class $[\alpha]$. Thus we can define the canonical contact structure on $PT^*\mathbb{R}_2^{n+1}$ by $K = \{\mathbf{X} \in TPT^*\mathbb{R}_2^{n+1} \mid \tau(X)(d\pi(X)) = 0\}$. For a local coordinate neighbourhood in $\mathbb{R}_2^{n+1}$, we have a trivialization $PT^*U \cong U \times P(\mathbb{R}^n)^*$ and we call $((x_1, \ldots, x_n), [\xi_1 : \cdots : \xi_n])$ *homogeneous coordinates*, where $[\xi_1 : \cdots : \xi_n]$ are homogeneous coordinates of the dual projective space $P(\mathbb{R}^n)^*$. It is known that any Legendrian fibration is locally equivalent to $\pi : PT^*\mathbb{R}_2^{n+1} \to \mathbb{R}_2^{n+1}$ [20].

Proposition 11. *All Legendrian submanifold germs in $PT^*\mathbb{R}_2^{n+1}$ are constructed by the above method.*

Proposition 12. *Let G be the Lorentzian distance-squared function on M. For any point $(u, \boldsymbol{\lambda}) \in G^{-1}(0)$, G is a Morse family around $(u, \boldsymbol{\lambda})$.*

Proof. Denote $\mathbf{X}(u) = (X_0(u), X_1(u), \ldots, X_n(u))$ and $\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \ldots, \lambda_n)$. By definition, we have

$$\begin{aligned}G(u,\boldsymbol{\lambda}) = &-(X_0(u) - \lambda_0)^2 - (X_1(u) - \lambda_1)^2 \\ &+ \sum_{i=2}^{n}(X_i(u) - \lambda_i)^2.\end{aligned} \tag{13}$$

We now prove that the mapping $\Delta^*G = \{G, \partial G/\partial u_0, \ldots, \partial G/\partial u_{n-1}\}$ is nonsingular at $(u, \boldsymbol{\lambda}) \in G^{-1}(0)$. Indeed, the Jacobian matrix of Δ^*G is given by

$$\begin{pmatrix} & 2(X_0(u) - \lambda_0) & 2(X_1(u) - \lambda_1) & \cdots & -2(X_n(u) - \lambda_n) \\ & 2X_{0u_0}(u) & 2X_{1u_0}(u) & \cdots & -2X_{nu_0}(u) \\ \mathbf{A} & & & & \\ & \vdots & \vdots & \vdots & \vdots \\ & 2X_{0u_{n-1}}(u) & 2X_{1u_{n-1}}(u) & \cdots & -2X_{nu_{n-1}}(u) \end{pmatrix}, \tag{14}$$

where the matrix $\mathbf{A}$ is given by

$$\begin{pmatrix} 2\langle \mathbf{X} - \boldsymbol{\lambda}, \mathbf{X}_{u_0}\rangle & \cdots & 2\langle \mathbf{X} - \boldsymbol{\lambda}, \mathbf{X}_{u_n}\rangle \\ 2(\langle \mathbf{X}_{u_0}, \mathbf{X}_{u_0}\rangle + \langle \mathbf{X} - \boldsymbol{\lambda}, \mathbf{X}_{u_0u_0}\rangle) & \cdots & 2(\langle \mathbf{X}_{u_0}, \mathbf{X}_{u_n}\rangle + \langle \mathbf{X} - \boldsymbol{\lambda}, \mathbf{X}_{u_nu_0}\rangle) \\ \vdots & \vdots & \vdots \\ 2(\langle \mathbf{X}_{u_n}, \mathbf{X}_{u_0}\rangle + \langle \mathbf{X} - \boldsymbol{\lambda}, \mathbf{X}_{u_0u_n}\rangle) & \cdots & 2(\langle \mathbf{X}_{u_n}, \mathbf{X}_{u_n}\rangle + \langle \mathbf{X} - \boldsymbol{\lambda}, \mathbf{X}_{u_nu_n}\rangle) \end{pmatrix}, \tag{15}$$

and $\mathbf{X}_{u_i u_j}(u) = \partial^2 \mathbf{X}(u)/\partial_{u_i}\partial_{u_j}(u)$. Since $\mathbf{X}(u)$ is an immersion, the rank of the matrix

$$\begin{pmatrix} 2\mathbf{X}_{0u_0}(u) & 2\mathbf{X}_{1u_0}(u) & -2\mathbf{X}_{2u_0}(u) & \cdots & -2\mathbf{X}_{nu_0}(u) \\ 2\mathbf{X}_{0u_1}(u) & 2\mathbf{X}_{1u_1}(u) & -2\mathbf{X}_{2u_1}(u) & \cdots & -2\mathbf{X}_{nu_1}(u) \\ \vdots & \vdots & \vdots & \vdots & \\ 2\mathbf{X}_{0u_{n-1}}(u) & 2\mathbf{X}_{1u_{n-1}}(u) & -2\mathbf{X}_{2u_{n-1}}(u) & \cdots & -2\mathbf{X}_{nu_{n-1}}(u) \end{pmatrix} \tag{16}$$

is equal to $n-1$ and $\mathbf{X}(u) - \boldsymbol{\lambda}$ is lightlike, so that it is linearly independent of tangent vector $\mathbf{X}_{u_0}, \ldots, \mathbf{X}_{u_{n-1}}$. This means that the rank of $\mathbf{B}$ is equal to n, where

$$\mathbf{B} = \begin{pmatrix} 2(X_0(u) - \lambda_0) & 2(X_1(u) - \lambda_1) & \cdots & -2(X_n(u) - \lambda_n) \\ 2X_{0u_0}(u) & 2X_{1u_0}(u) & \cdots & -2X_{nu_0}(u) \\ \vdots & \vdots & \vdots & \vdots \\ 2X_{0u_{n-1}}(u) & 2X_{1u_{n-1}}(u) & \cdots & -2X_{nu_{n-1}}(u) \end{pmatrix}. \tag{17}$$

Therefore, the Jacobi matrix of $\Delta^* G$ is nonsingularity at $(u, \boldsymbol{\lambda}) \in G^{-1}(0)$. □

4. Contact with Null Cone

In this section, we gave the singularities of lightlike hypersurfaces are stable, whose types are not changed with small disturbance under the view of $\mathscr{K}$-equivalent and $\mathscr{P}$-$\mathscr{K}$-equivalent. Before we start to consider the contact between lightlike hypersurfaces and null cone, we briefly review the theory of contact due to Montaldi [21, 22]. Let X_i and Y_i $(i = 1, 2)$ be submanifolds in $\mathbb{R}^n$ with $\dim X_1 = \dim X_2$ and $\dim Y_1 = \dim Y_2$. We say that the *contact* of X_1 and Y_1 at y_1 is of the same type as the contact of X_2 and Y_2 at y_2 if there is a diffeomorphism germ $\phi : (\mathbb{R}^n, y_1) \to (\mathbb{R}^n, y_2)$ such that $\phi(X_1) = X_2$ and $\phi(Y_1) = Y_2$. In this case, we write $K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$. In his paper [21], Montaldi gives a characterization of the notion of contact by using the terminology of singularity theory.

Theorem 13 (see [21]). *Let X_i and Y_i $(i = 1, 2)$ be submanifolds of $\mathbb{R}^n$ with $\dim X_1 = \dim X_2$ and $\dim Y_1 = \dim Y_2$. Let $g_i : (X_i, x_i) \to (\mathbb{R}^n, y_i)$ be immersion germs and let $f_i : (\mathbb{R}^n, y_i) \to (\mathbb{R}^p, \mathbf{0})$ be submersion germs with $(Y_i, y_i) = (f_i^{-1}(\mathbf{0}), y_i)$. Then $K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$ if and only if $f_1 \circ g_1$ and $f_2 \circ g_2$ are $\mathscr{K}$-equivalent.*

For the $\mathscr{K}$-equivalent among smooth map germs, considering the function $\mathscr{G} : \mathbb{R}_2^{n+1} \times \mathbb{R}_2^{n+1} \to \mathbb{R}$ by $\mathbf{G}(\mathbf{X}, \boldsymbol{\lambda}) = \langle \mathbf{X} - \boldsymbol{\lambda}, \mathbf{X} - \boldsymbol{\lambda} \rangle$ and denoting $\mathfrak{g}_{\lambda_0}(\mathbf{X}) = \mathscr{G}(\mathbf{X}, \boldsymbol{\lambda}_0)$, we have $\mathfrak{g}_{\lambda_0}^{-1}(0) = \Lambda_1^n$. For $p_0 = \mathbf{X}(u_0)$, we can take the vector $\boldsymbol{\lambda}_0 = \mathbf{X}(u_0) + \mu_0(\widetilde{\mathbf{X}(u_0) \pm \mathbf{N}}(u_0))$. Then $\mathfrak{g}_{\lambda_0} \circ \mathbf{X}(u_0) = \mathscr{G} \circ (\mathbf{X} \times \mathrm{id}_{\mathbb{R}_2^{n+1}}) = G(u_0, \boldsymbol{\lambda}_0) = 0$ and the relations are $((\partial \mathfrak{g}_{\lambda_0} \circ \mathbf{X})/\partial u_i)(p_0) = (\partial G/\partial u_i)(p_0, \boldsymbol{\lambda}_0) = 0$ $(i = 1, 2, \ldots, n-1)$. This means that the lightcone $\mathfrak{g}_{\lambda_0}^{-1}(0) = \Lambda_{1\lambda_0}^n$ is tangent to M at p_0. In this case, we call each $\Lambda_{1\lambda_0}^n$ a tangent null cone of M at p_0. We denote by $\mathscr{E}_n$ the local ring of function germs $(\mathbb{R}^n, \mathbf{0}) \to (\mathbb{R}, 0)$ with the unique maximal ideal

$$\mathfrak{M}_n = \{h \in \mathscr{E}_n \mid h(\mathbf{0}) = 0\}. \tag{18}$$

Let $F, G : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \to (\mathbb{R}, 0)$ be function germs. We say that F, G are $\mathscr{P}$-$\mathscr{K}$-equivalent if there exists a diffeomorphism germ $\psi : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \to (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$ of the form $\psi(q, x) = (\psi_1(q, x), \psi_2(x))$ for $(q, x) \in (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$ such that $\psi^*(\langle F \rangle_{\mathscr{E}_{k+n}}) = \langle G \rangle_{\mathscr{E}_{k+n}}$, where $\psi^* : \mathscr{E}_{k+n} \to \mathscr{E}_{k+n}$ is the pullback $\mathscr{R}$-algebra isomorphism defined by $\psi^*(h) = h \circ \psi$.

We apply the tools for the study of the contact theory. Let $\mathbb{L}_i^{\pm} : U \to \Lambda_1^n$ be two null cone Legendrian Gauss map germs of Lorentzian surface germs $\mathbf{X}_i : U \to \mathbb{H}_1^n$ $(i = 1, 2)$. We say that $\mathbb{L}_1^{\pm}$ and $\mathbb{L}_2^{\pm}$ are $\mathscr{A}$*-equivalent* if there exist diffeomorphism germs $\varphi : (U, (u_{11}, \ldots, u_{1n})) \to (U, (u_{21}, \ldots, u_{2n}))$ and $\phi : (\mathbb{R}_2^{n+1}, \boldsymbol{\lambda}_1) \to (\mathbb{R}_2^{n+1}, \boldsymbol{\lambda}_2)$ such that $\phi \circ \mathbb{L}_1^{\pm} = \mathbb{L}_2^{\pm} \circ \varphi$.

Let $F : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \to (\mathbb{R}, 0)$ be a function germ, F is $\mathscr{K}$-versal deformation of $f = F \mid \mathbb{R}^k \times \{\mathbf{0}\}$ if $\varepsilon_k = T_e(K)(f) + \langle (\partial F/\partial x_1)|_{\mathbb{R}^k \times \{\mathbf{0}\}}, \ldots, (\partial F/\partial x_n)|_{\mathbb{R}^k \times \{\mathbf{0}\}} \rangle$, where $T_e(K)(f) = \langle \partial f/\partial q_1, \ldots, \partial f/\partial q_k \rangle_{\varepsilon_k}$. The main result in the theory [22] is as follows.

Theorem 14 (see [22]). *Let $F, G : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \to (\mathbb{R}, 0)$ be Morse families. Then*

(1) *$\mathscr{L}_F$ and $\mathscr{L}_G$ are Legendrian equivalent if and only if F and G are $\mathscr{P}$-$\mathscr{K}$-equivalent.*

(2) *$\mathscr{L}_F$ is Legendrian stable if and only if F is a $\mathscr{K}$-versal deformation of $F \mid \mathbb{R}^k \times \{\mathbf{0}\}$.*

Since F and G are function germs on the common space, by the uniqueness result of the versal deformation of a function germ, we have the following classification results of Legendrian stable germs. For a map germ $f : (\mathbb{R}^n, \mathbf{0}) \to (\mathbb{R}^p, \mathbf{0})$, we give the local ring of f by $Q(f) = \varepsilon_n / f(\mathfrak{M}_p)$.

Proposition 15. *Let F and $G : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \to (\mathbb{R}, 0)$ be Morse families. Suppose that $\mathscr{L}_F$ and $\mathscr{L}_G$ are Legendrian stable. Then the following conditions are equivalent.*

(1) *$(W(\mathscr{L}_F), \mathbf{0})$ and $(W(\mathscr{L}_G), \mathbf{0})$ are diffeomorphic as germs.*

(2) *$\mathscr{L}_F$ and $\mathscr{L}_G$ are Legendrian equivalent.*

(3) *$Q(f)$ and $Q(g)$ are isomorphic as $\mathscr{R}$-algebras, where $f = F \mid \mathbb{R}^k \times \{\mathbf{0}\}$ and $g = G \mid \mathbb{R}^k \times \{\mathbf{0}\}$.*

Let $G_i : U \times \mathbb{R}^{n+1}_2 \to \mathbb{R}$ be the Lorentzian distance-squared function germs of $\mathbf{X}_i$ $(i = 1, 2)$. We denote $g_{i\lambda_i}(u) = G_i(u, \boldsymbol{\lambda}_i)$, then $g_{i\lambda_i} = \mathfrak{g}_{\lambda_i} \circ \mathbf{X}_i$. By Theorem 13, we know $K(X_1, \Lambda^n_{1\lambda_1}; \boldsymbol{\lambda}_1) = K(X_2, \Lambda^n_{1\lambda_2}; \boldsymbol{\lambda}_2)$ if and only if $g_{1\lambda_1}$ and $g_{2\lambda_2}$ are $\mathscr{K}$-equivalent. Therefore, we can denote the local ring of the function $\tilde{g}_{\lambda_0} : U \to \mathbb{R}$, we remark that we can explicitly write the local ring as follows:

$$Q(\mathbf{X}, (u_1, \ldots, u_n)) = \frac{C^\infty_{u_0}(U)}{\left(\left\langle \mathbf{X}(u), \mathbf{X}\widetilde{(u_0) + \mathbf{N}}(u_0) \right\rangle\right)^{-1}_{C^\infty_{u_0}(U)}}, \tag{19}$$

where $C^\infty_{u_0}(U)$ is the local ring of function germs with the maximal ideal $\mathfrak{M}(u)$ in [12].

Theorem 16 (see [12]). *Let $\mathbf{X}_i : U_i \to \mathbb{H}^n_1$ $(i = 1, 2)$ be Lorentzian surface germs such that the corresponding Legendrian lift germs are Legendrian stable. Then the following conditions are equivalent.*

(1) *The lightlike hypersurface germs $LH^\pm_{M_1}$ and $LH^\pm_{M_2}$ are $\mathscr{A}$-equivalent.*

(2) *G_1 and G_2 are $\mathscr{P}$-$\mathscr{K}$-equivalent.*

(3) *$g_{1\lambda_1}$ and $g_{2\lambda_2}$ are $\mathscr{K}$-equivalent.*

(4) *$K(X_1, \Lambda^n_{1\lambda_1}; \boldsymbol{\lambda}_1) = K(X_2, \Lambda^n_{1\lambda_2}; \boldsymbol{\lambda}_2)$.*

(5) *$Q(\mathbf{X}_1)$ and $Q(\mathbf{X}_2)$ are isomorphic as $\mathscr{R}$-algebras.*

Proof. Since the Lorentzian distance-squared function is a Morse family of functions, conditions (1) and (2) are equivalent. Moreover, $LH^\pm_{M_i}$ is Lagrangian stable, G_i is the $\mathscr{R}$-versal deformation of $g_{i\lambda_i}$; by the uniqueness result of the $\mathscr{R}$-versal deformation, condition (2) implies condition (3). By definition, we know condition (3) implies condition (2). It follows from Theorem 13 that conditions (3) and (4) are equivalent. As the same way, we can obtain conditions (5) and (1) as equivalent by Proposition 15, so we complete the proof. □

Given a Lorentzian surface $\mathbf{X} : U \to \mathbb{H}^n_1$, we call $(\mathbf{X}^{-1}(\Lambda^n_{1\lambda}), u_0)$ the tangent indicatrix germ of $\mathbf{X}$, where $\boldsymbol{\lambda} = \mathbf{X}(u_0) + \mu_0(\mathbf{X}\widetilde{(u_0) \pm \mathbf{N}}(u_0))$ and $\mu_0 = \mp(1/k_i)$ $(i = 1, 2, \ldots, n-1)$.

Corollary 17. *The lightlike hypersurface germs LH_{M_1} and LH_{M_2} are $\mathscr{A}$-equivalent, then tangent indicatrix germs $(\mathbf{X}_1^{-1}, u_1)$ and $(\mathbf{X}_2^{-1}, u_2)$ are diffeomorphic as set germs.*

Proof. The tangent indicatrix germ of $\mathbf{X}_i$ is the zero level set of g_{i,λ_i} Since $\mathscr{K}$-equivalent among function germs preserves the zero-level sets of function germs, the assertion follows Theorem 16. □

5. Singularities of Lightlike Hypersurfaces in $\mathbb{R}^4_2$

In this section, we study the classification of singularities of 3-dimensional lightlike hypersurfaces, which are generated by Lorentzian surface in anti-de Sitter 3-space, also, we consider the space of Lorentzian embeddings $\mathrm{Emb}_L(U, \mathbb{H}^3_1)$ with Whitney C^∞-topology, where $U \subset \mathbb{R}^2$ is an open subset. As the choose of the standard arguments in [11], we consider a function $\mathscr{G} : \mathbb{R}^4_2 \times \mathbb{R}^4_2 \to \mathbb{R}$ by $\mathscr{G}(\mathbf{v}, \boldsymbol{\lambda}) = \langle \mathbf{v} - \boldsymbol{\lambda}, \mathbf{v} - \boldsymbol{\lambda} \rangle$ and claim that $\mathscr{G}_\lambda(\mathbf{v})$ is a submersion at $\mathbf{v} \neq \boldsymbol{\lambda}$ for any fixed $\boldsymbol{\lambda} \in \mathbb{R}^4_2$. Given $\mathbf{X} \in \mathrm{Emb}_L(U, \mathbb{H}^3_1)$, we have $G = \mathscr{G} \circ (\mathbf{X} \times \mathrm{id}_{\mathbb{R}^4_2})$. We have the l-jet extension $j^l_1 G : U \times \mathbb{R}^4_2 \to J^l(U, \mathbb{R})$ defined by $j^l_1 G(u, \boldsymbol{\lambda}) = j^l g_\lambda(u)$. Consider the trivialization $J^l(U, \mathbb{R}) = U \times \mathbb{R} \times J^l(2, 1)$. For any submanifold $Q \subset J^l(2, 1)$, we denote $\widetilde{Q} = U \times \mathbb{R} \times Q$. Then we have the following proposition [12, 17].

Proposition 18. *Let Q be a submanifold of $J^l(2, 1)$. Consider*

$$T_Q = \left\{\mathbf{X} \in Emb_L\left(U, \mathbb{H}^3_1\right) \mid j^l_1 G \text{ is transversal to } \widetilde{Q}\right\} \tag{20}$$

is a residual subset of $Emb_L(U, \mathbb{H}^3_1)$. If Q is a closed subset, then T_Q is open.

On the other hand, we have a stratification given by the set of $\mathscr{K}$-orbits in $J^l(2, 1) \setminus W^l(2, 1)$ (for the definition of $W^l(2, 1)$ and additional properties refer to [12]).

Theorem 19. *There exists an open dense subset $\mathscr{O} \subset Emb_L(U, \mathbb{H}^3_1)$ such that for any $\mathbf{X} \subset \mathscr{O}$, the germ of the Legendrian lift of the corresponding lightlike hypersurface $LH^\pm_M$ at each point is Legendrian stable.*

Proposition 20. *There exists an open dense subset $\mathscr{O} \subset Emb_L(U, \mathbb{H}^3_1)$ such that for any $\mathbf{X} \subset \mathscr{O}$, the germ of the corresponding lightlike hypersurface $LH^\pm_M$ at any point $(x, y, u) \in U \times \mathbb{R}$ is $\mathscr{A}$-equivalent to one of the map germs A_k $(1 \le k \le 4)$ or $D^\pm_4$, where*

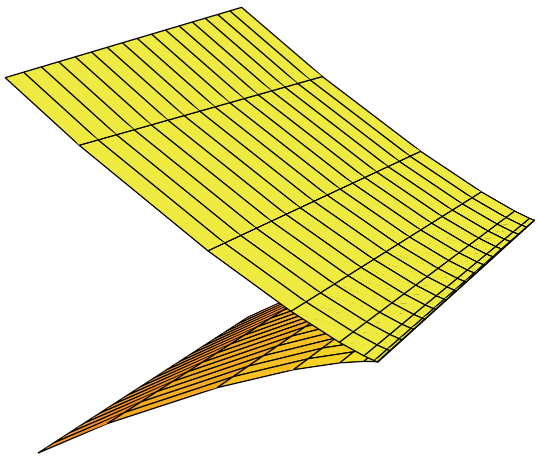

Figure 1: Cuspidal edge.

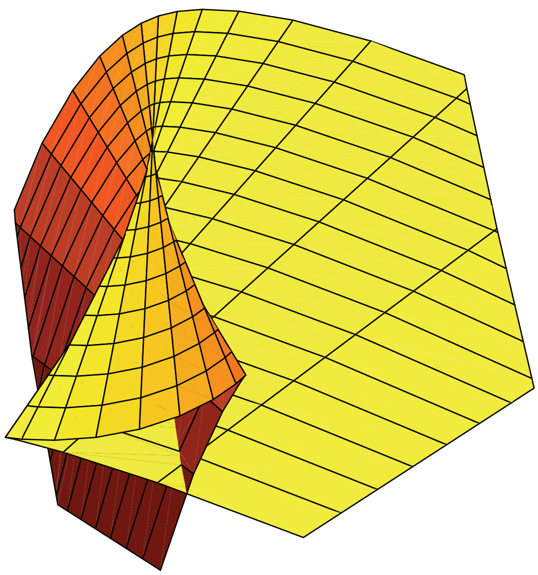

Figure 2: Swallowtail.

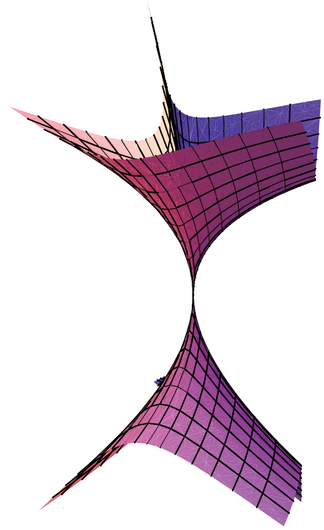

Figure 3: Purse.

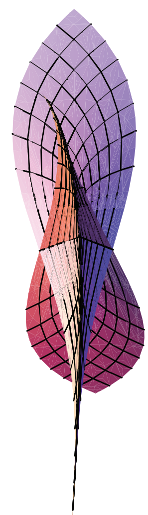

Figure 4: Pyramid.

(A_1) $f(u_1, u_2, u_3) = (u_1, u_2, u_3, 0)$ *(embedding),*

(A_2) $f(u_1, u_2, u_3) = (3u_1^2, 2u_1^3, u_2, u_3)$ *(cuspidal edge) (Figure 1),*

(A_3) $f(u_1, u_2, u_3) = (4u_1^3 + 2u_1u_2, 3u_1^4 + u_1^2u_2, u_2, u_3)$ *(swallowtail) (Figure 2),*

(A_4) $f(u_1, u_2, u_3) = (5u_1^4 + 3u_2u_1^2 + 2u_1u_3, 4u_1^5 + 2u_1^3u_2 + u_1^2u_3, u_1, u_2)$ *(butterfly),*

(D_4^+) $f(u_1, u_2, u_3) = (2u_1^2 + 2u_2^2 + u_1u_2u_3, 3u_1^2 + u_2u_3, 3u_2^2 + u_1u_3, u_3)$ *(purse) (Figure 3),*

(D_4^-) $f(u_1, u_2, u_3) = (2u_1^3 - 2u_1u_2^2 + u_1^2u_3 + u_2^2u_3, u_2^2 - 3u_1^2 - 2u_1u_3, u_1u_3 - u_2u_3, u_3)$ *(pyramid) (Figure 4).*

By using the generic normal forms of generating families and Corollary 17, we have the following corollary.

Corollary 21. *There exists an open dense subset $\mathcal{O} \subset Emb_L(U, \mathbb{H}_1^3)$ such that for any $\mathbf{X} \subset \mathcal{O}$, the germ of the corresponding tangent indicatrix at any point $(x_0, y_0) \in U$ is diffeomorphic to one of the germs in the following lists.*

(1) $\{(x, y) \in (\mathbb{R}^2, \mathbf{0}) \mid x^3 + y^2 = 0\}$ *(ordinary cusp); (Figure 5),*

(2) $\{(x, y) \in (\mathbb{R}^2, \mathbf{0}) \mid x^4 \pm y^2 = 0\}$ *(tacnode or point); (Figure 6),*

(3) $\{(x, y) \in (\mathbb{R}^2, \mathbf{0}) \mid x^5 + y^2 = 0\}$ *(rhamphoid cusp); (Figure 7),*

(4) $\{(x, y) \in (\mathbb{R}^2, \mathbf{0}) \mid x^3 - xy^2 = 0\}$ *(three lines); (Figure 8),*

(5) $\{(x, y) \in (\mathbb{R}^2, \mathbf{0}) \mid x^3 + y^3 = 0\}$ *(a line).*

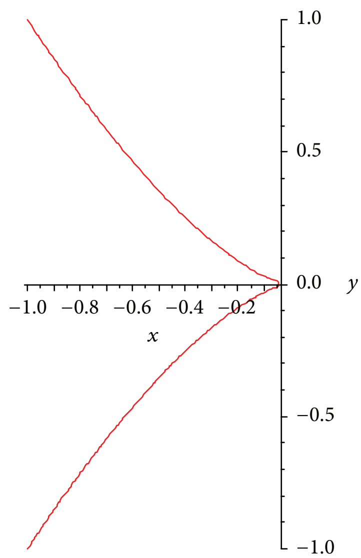

Figure 5: Ordinary cusp.

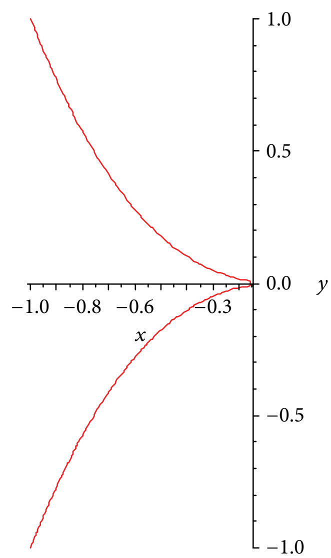

Figure 7: Rhamphoid cusp.

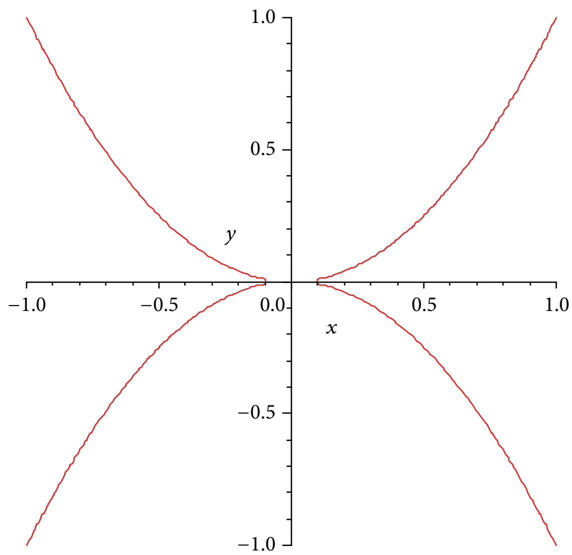

Figure 6: Tacnode.

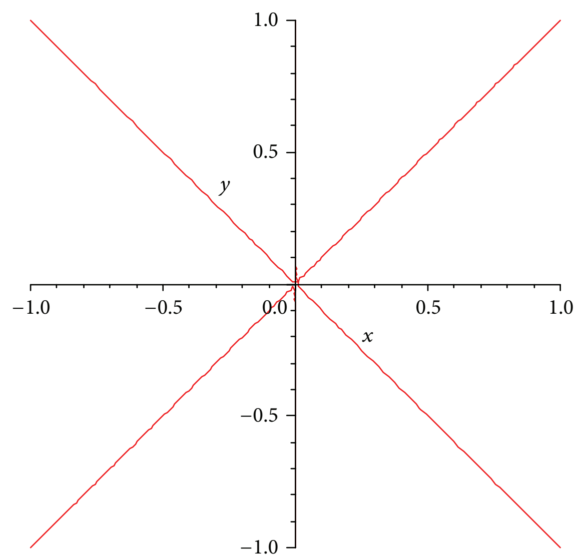

Figure 8: Three lines.

6. Canal Hypersurface of Lorentzian Surface

Canal hypersurfaces, which are generated by surfaces with codimension 2 along fixed direction, are envelopes of families of hyperspheres. Since the property of a hypersurface is to be a canal hypersurface is conformally invariant, canal hypersurfaces in a multidimensional Euclidean space were investigated in many papers, such as [15, 16]. In this section, we mainly consider the canal hypersurfaces in semi-Euclidean space with index 2. Let $\mathbf{X}(U)$ be a Lorentzian surface; the Mongle form is as follows:

$$\mathbf{X}(u_1, u_2) = \{u_1, u_2, f_1(u_1, u_2), f_2(u_1, u_2)\}. \tag{21}$$

The second fundamental form of $M = \mathbf{X}(U)$ is characterized by two quadratic forms. Their functional coefficients will be denoted by (a, b, c) and (e, f, g), respectively [15].

We have the following function:

$$\Delta(u) = \frac{1}{4}\det\begin{pmatrix} -a & -2b & c & 0 \\ e & 2f & g & 0 \\ 0 & a & 2b & c \\ 0 & e & 2f & g \end{pmatrix}(u). \tag{22}$$

The Gaussian curvature of M is

$$K(u) = \left(ac - b^2 + eg - f^2\right)(u), \tag{23}$$

and the matrix is

$$\alpha(u) = \begin{pmatrix} a & b & c \\ e & f & g \end{pmatrix}(u), \tag{24}$$

where $a = \partial^2 f_1/\partial u_1 \partial u_1$, $b = \partial^2 f_1/\partial u_1 \partial u_2$, $c = \partial^2 f_1/\partial u_2 \partial u_2$, $e = \partial^2 f_2/\partial u_1 \partial u_1$, $f = \partial^2 f_2/\partial u_1 \partial u_2$, and $g = \partial^2 f_2/\partial u_2 \partial u_2$.

Definition 22. Let $\mathbf{X}(U)$ be a Lorentzian surface in anti-de Sitter space and let $\mathbf{N}(u)$ be its spacelike normal vector; a hypersurface defined as $CM = \mathbf{X}(u) + \varepsilon v \mathbf{N}(u) \in \mathbb{R}^4_2$ is called canal hypersurface of $\mathbf{X}(U)$, where ε is a sufficiently small positive real number chosen such that CM is embedded in $\mathbb{R}^4_2$.

We denote by $\widehat{\mathbf{X}}$ the natural embedding of CM in $\mathbb{R}^4_2$ and by $(u, \mathbf{N}(u))$ the point $\mathbf{X}(u) + \varepsilon\mathbf{N}(u) \in CM$. From Looijenga's theorem [15], there is a residual subset of embeddings $\mathbf{X} : U \hookrightarrow \mathbb{R}^4_2$, for which the family of height functions $H : U \times \Lambda^3_1 \to \mathbb{R}$ by $H(u, \mathbf{v}) = \langle \mathbf{X}(u), \mathbf{v} \rangle$ is locally stable as a family of function on M with parameters on Λ^3_1. Moreover, the corresponding family $h(\widehat{\mathbf{X}})$ on the canal hypersurface is also generic. In fact the singularities of $h(\mathbf{X})$ and $h(\widehat{\mathbf{X}})$ are tightly related [16].

Thus, for a generic $\mathbf{X}$, those may be one of the following types: Morse (A_1), fold (A_2), cusp (A_3), swallowtail (A_4), and elliptic or hyperbolic umbilic ($D^{\pm}_4$). Moreover, the singularities of the lightcone Gauss indicatrix $\mathbb{L}^{\pm} : CM \to S^1_t \times S^{n-1}_s$ can be described in terms as follows [15].

Lemma 23 (see [15]). *Given a critical point $(u, \mathbf{v}) \in CM$ of the height function h_v, we have the following.*

(1) *u is a nondegenerate critical point of h_v if and only if $(u, \mathbf{v})$ is a regular point of $\mathbb{L}^{\pm}$.*

(2) *u is a degenerate critical point of h_v if and only if $(u, \mathbf{v})$ is singular point of $\mathbb{L}^{\pm}$.*

Let $\mathscr{K}_c : CM \to \mathbb{R}$ be the Gaussian curvature function on CM. The parabolic set, $\mathscr{K}_c(0)$ of CM is the singular set of $\mathbb{L}^{\pm}$. It can be shown that for a generic embedding of M, $\mathscr{K}_c(0)$ is a regular surface except by a finite number of points $(u, \mathbf{v})$, which are singularities of type $\Sigma^{2,0}$ of $\mathbb{L}^{\pm}$ or equivalently umbilic points $(D^{\pm}_4)$ of h_v [16].

Let $\mathfrak{p} : CM \to M$ be the natural projection of CM onto (i.e., $\mathfrak{p}(u, \mathbf{v}) = u$). The image of the set of parabolic points $\mathscr{K}_c(0)$ by p is the set $\Delta \le 0$.

Theorem 24. *(1) If $\Delta(u) > 0$, then u is a nondegenerate critical point of h_v for any $\mathbf{v} \in N_{X(u)}M$.*

(2) If $\Delta(u) < 0$, then there are exactly two vectors $\mathbf{v}_1, \mathbf{v}_2 \in N_{X(u)}M$, such that u is a degenerate critical point of h_{v_i}, $i = 1, 2$.

(3) If $\Delta(u) = 0$, then there is a unique vector $\mathbf{w} \in N_{X(u)}M$ such that u is a degenerate critical point of h_w.

Proof. Let $\mathbf{X}(u_1, u_2) = \{u_1, u_2, f_1(u_1, u_2), f_2(u_1, u_2)\}$ be the local expression of the embedding in Monge's form and let the height function in $\mathbf{v}$-direction be

$$h_v(u_1, u_2) = v_1 u_1 + v_2 u_2 + v_3 f_1(u_1, u_2) + v_4 f_2(u_1, u_2), \tag{25}$$

where $\mathbf{v} = (v_1, v_2, v_3, v_4) \in \mathbb{R}^4_2$. If $(0, 0)$ is a critical point of the height function h_v, then $\mathbf{v} = (0, 0, v_3, v_4)$ and the determinant of the Hessian matrix of h_v at $(0, 0)$ is given by

$$\begin{aligned} \det \mathbf{H}(h_v)(0,0) = {} & (ac - b^2) v_3^2 + (ag + ce - 2bf) v_3 v_4 \\ & + (eg - f^2) v_4^2, \end{aligned} \tag{26}$$

where (a, b, c), (e, f, g) are the above coefficients. Now,

$$\Delta = (ac - b^2)(eg - f^2) - \frac{1}{4}(ag + ce - 2bf)^2 \tag{27}$$

and the equation $\mathbf{H}(h_v)(0, 0) = 0$ has two, one, or zero solutions as $\Delta < 0$, $\Delta = 0$, or $\Delta > 0$, respectively. □

When u is a degenerate critical point of h_v, the hyperplane $\mathscr{H}_v$, orthogonal to $\mathbf{v}$, has a higher order contact with M at $\mathbf{X}(u)$. Therefore, we will say that $\mathbf{v}$ is a binormal vector of M at $\mathbf{X}(u)$ and $\mathscr{H}_v$ can be an osculating hyperplane [20].

At each point of $\mathscr{K}^{-1}_c(0) - \Sigma^2(\mathbb{L}^{\pm})$, there is a unique principal direction of zero curvature for CM. This direction is tangent to the surface $\mathscr{K}^{-1}_c(0)$ on a curve made of points of type $\Sigma^{1,1}(\mathbb{L}^{\pm})$. This curve is in turn tangent to a zero principal direction of curvature at isolated points [16].

Proposition 25. *The image of zero principal directions of curvature in $\mathscr{K}^{-1}_c(0) - \Sigma^2(\mathbb{L}^{\pm})$ under $\mathfrak{p}|_{\mathscr{K}^{-1}_c(0) - \Sigma^2(\mathbb{L}^{\pm})}$ are asymptotic directions on M.*

Proof. For the curvature vector $\eta(\theta)$ is given by

$$\begin{aligned} \eta(\theta) = {} & \left(\frac{1}{2}(a - c)\cos 2\theta + b \sin 2\theta\right) e_3 \\ & + \left(\frac{1}{2}(e - g)\cos 2\theta + f \sin 2\theta\right) e_4 + \mathbf{H}, \end{aligned} \tag{28}$$

where $\mathbf{H} = (1/2)(a+c)e_3 + (1/2)(e+g)e_4$ is the mean curvature vector; we can choose local coordinates for M such that

$$\alpha(u) = \begin{pmatrix} a & b & c \\ 0 & 0 & 1 \end{pmatrix}(u). \tag{29}$$

This choice will imply that $e_1 = (1, 0, 0, 0) \in TCM$ is the zero curvature direction and $T \circ \mathfrak{p}(u, \mathbf{v}) \cdot e_1 = e_1 \in T_{\mathbf{X}(u)}M$. Then, it follows easily that $\eta(\mathbf{0})$ and $(\partial\mathbf{N}(u)/\partial\theta)(\mathbf{0})$ are parallel. □

Therefore, we can have the singularities of canal hypersurfaces in the following theorem.

Theorem 26. *The canal hypersurfaces have the same singularities as Lorentzian surfaces in anti-de Sitter apace, so we can easily obtain the singularities of canal hypersurfaces as in Section 5.*

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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