Research Article

General Split Variational Inclusion Problem in Hilbert Spaces

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We consider a general split variational inclusion problem (GSFVIP) and propose an algorithm for finding the solutions of GSFVIP in Hilbert space. We establish the strong convergence of the proposed algorithm to a solution of GSFVIP. Our results extend and improve the related results in the literature.

1. Introduction

Let H be a real Hilbert space, and let B be a set-valued mapping with domain $D(B) := \{x \in H : B(x) \neq \emptyset\}$. Recall that B is called *monotone* if $\langle u - v, x - y \rangle \geq 0$ for any $u \in Bx$ and $v \in By$; B is *maximal monotone* if its graph $\{(x, y) : x \in D(B), y \in Bx\}$ is not properly contained in the graph of any other monotone mapping. An important problem for set-valued monotone mappings is to find $x^* \in H$ such that $0 \in Bx^*$. Here, x^* is called a *zero point* of B. A well-known method for approximating a zero point of a maximal monotone mapping defined in a real Hilbert space is *the proximal point algorithm* first introduced by Martinet [1] and generated by Rockafellar [2]. This is an iterative procedure, which generates $\{x_n\}$ by $x_1 = x \in H$ and

$$x_{n+1} = J_{\beta_n}^B x_n, \quad n \in N, \tag{1}$$

where $\{\beta_n\} \subset (0, \infty)$, B is a maximal monotone mapping in a real Hilbert space, and J_r^B is the *resolvent mapping* of B defined by $J_r^B = (I + rB)^{-1}$ for each $r > 0$. In 1976, Rockafellar [2] proved the following in the Hilbert space setting. If the solution set $B^{-1}(0)$ is nonempty and $\liminf_{n \to \infty} \beta_n > 0$, then the sequence $\{x_n\}$ in (1) converges weakly to an element of $B^{-1}(0)$. Later, many researchers have studied the convergence theorems of the proximal point algorithm in Hilbert spaces. For example, one can refer to [3–8] and references therein.

Let H_1 and H_2 be two real Hilbert spaces, $B_1 : H_1 \to H_1$ and $B_2 : H_2 \to H_2$ two set-valued maximal monotone mappings, $A : H_1 \to H_2$ a linear and bounded operator, and

A^* the adjoint of A. Chuang [9] considers the following *split variational inclusion problem*:

$$\text{(SFVIP) Find } x^* \in H_1 \text{ such that } 0 \in B_1(x^*),$$
$$0 \in B_2(Ax^*), \tag{2}$$

which was introduced by Moudafi [10]. In this paper, motivated by the works in Chuang [9] and related literature, we consider the following general split variational inclusion problem.

Definition 1. Let H_1 and H_2 be two real Hilbert spaces, $\{B_i : H_1 \to H_1\}_{i \in N}$ and $\{K_i : H_2 \to H_2\}_{i \in N}$ two families of set-valued maximal monotone mappings, $A : H_1 \to H_2$ a linear and bounded operator, and A^* the adjoint of A. The general split variational inclusion problem (GSFVIP) is formulated as the following problem:

$$\text{(GSFVIP) Find } x^* \in H_1 \text{ such that } x^* \in \bigcap_{i=1}^{\infty} B_i^{-1}(0),$$
$$Ax^* \in \bigcap_{i=1}^{\infty} K_i^{-1}(0). \tag{3}$$

In this paper, we propose an algorithm for finding the solutions of GSFVIP in a Hilbert space and prove that the sequence generated by the proposed method converges strongly to a solution of GSFVIP. Our results extend and improve the related results in the literature.

2. Preliminaries

Throughout this paper, let N be the set of positive integers. Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$, respectively. We also use "$\rightarrow$" to stand for strong convergence and "$\rightharpoonup$" to stand for weak convergence.

Lemma 2 (see [11]). *Let H be a real Hilbert space, and let x, $y \in H$. Then*

$$\|x + y\|^2 \leq \|x\|^2 + 2 \langle y, x + y \rangle, \quad \forall x, y \in H. \quad (4)$$

Lemma 3 (see [12]). *Let H be a Hilbert space, and let $\{x_n\}$ be a sequence in H. Then, for any given sequence $\{\alpha_n\}_{n=1}^{\infty} \subset (0, 1)$ with $\sum_{n=1}^{\infty} \alpha_n = 1$ and for any positive integer i, j with $i < j$,*

$$\left\| \sum_{n=1}^{\infty} \alpha_n x_n \right\|^2 \leq \sum_{n=1}^{\infty} \alpha_n \|x_n\|^2 - \alpha_i \alpha_j \|x_i - x_j\|^2. \quad (5)$$

Let D be a nonempty closed convex subset of a real Hilbert space H, and let $T : D \rightarrow H$ be a mapping. Then T is said to be a *nonexpansive mapping* if $\|Tx - Ty\| \leq \|x - y\|$ for every $x, y \in D$. It is easy to see that $\text{Fix}(T) := \{x \in D : Tx = x\}$ is a closed convex subset of D if T is a nonexpansive mapping. Besides, T is said to be a *firmly nonexpansive mapping* if $\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty \rangle$ for every $x, y \in D$.

Lemma 4 (see [13]). *Let D be a nonempty closed convex subset of a real Hilbert space H. Let $T : D \rightarrow H$ be a nonexpansive mapping, and let $\{x_n\}$ be a sequence in D. If $x_n \rightharpoonup w$ and $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$, then $Tw = w$.*

Let D be a nonempty closed convex subset of H. For every point $x \in H$, there exists a unique nearest point in D, denoted by $P_D x$, such that

$$\|x - P_D x\| \leq \|x - y\|, \quad \forall y \in D. \quad (6)$$

P_D is called the *metric projection* of H onto D. It is known that P_D is a nonexpansive mapping of H onto D.

Lemma 5 (see [14]). *Let D be a nonempty closed convex subset of a Hilbert space H. Let P_D be the metric projection from H onto D. Then, for each $x \in H$ and $z \in D$, we know that $z = P_D x$ if and only if $\langle x - z, z - y \rangle \geq 0$ for all $y \in D$.*

The following result is an important tool in this paper. For similar results, one can see [15].

Lemma 6. *Let H be a real Hilbert space. Let $B : H \rightarrow H$ be a set-valued maximal monotone mapping, $\beta > 0$, and let J_{β}^{B} be a resolvent mapping of B.*

(i) *For each $\beta > 0$, J_{β}^{B} is a single-valued and firmly nonexpansive mapping.*

(ii) *$D(J_{\beta}^{B}) = H$ and $\text{Fix}(J_{\beta}^{B}) = \{x \in D(B) : 0 \in Bx\}$.*

(iii) *$\|x - J_{\beta}^{B} x\| \leq \|x - J_{\alpha}^{B} x\|$ for all $0 < \beta \leq \alpha$ and for all $x \in H$.*

(iv) *$(I - J_{\beta}^{B})$ is a firmly nonexpansive mapping for each $\beta > 0$.*

(v) *Suppose that $B^{-1}(0) \neq \emptyset$. Then $\|x - J_{\beta}^{B} x\|^2 + \|J_{\beta}^{B} x - x^*\|^2 \leq \|x - x^*\|^2$ for each $x \in H$, each $x^* \in B^{-1}(0)$, and each $\beta > 0$.*

(vi) *Suppose that $B^{-1}(0) \neq \emptyset$. Then $\langle x - J_{\beta}^{B} x, J_{\beta}^{B} x - w \rangle \geq 0$ for each $x \in H$, each $w \in B^{-1}(0)$, and each $\beta > 0$.*

Lemma 7 (see [9]). *Let H_1 and H_2 be two real Hilbert spaces, let $A : H_1 \rightarrow H_2$ be a linear operator, let A^* be the adjoint of A, let $\beta > 0$ be fixed, and let $\gamma \in (0, 2/\|A\|^2)$. Let $K : H_2 \rightarrow H_2$ be a set-valued maximal monotone mapping and let J_{β}^{K} be a resolvent mapping of K. Then*

$$\left\| \left[x - \gamma A^* \left(I - J_{\beta}^{K} \right) Ax \right] - \left[y - \gamma A^* \left(I - J_{\beta}^{K} \right) Ay \right] \right\|^2$$

$$\leq \|x - y\|^2 - \left(2\gamma - \gamma^2 \|A\|^2 \right) \quad (7)$$

$$\times \left\| \left(I - J_{\beta}^{K} \right) Ax - \left(I - J_{\beta}^{K} \right) Ay \right\|^2,$$

for all $x, y \in H$. Furthermore, $I - \gamma A^* (I - J_{\beta}^{K})A$ is a nonexpansive mapping.

The following is a very important result for various strong convergence theorems. Recently, many researchers have studied Halpern's type strong convergence theorems by using the following lemma and get many generalized results. For example, one can see [9, 16, 17]. In this paper, we also use this result to get our strong convergence theorems.

Lemma 8 (see [18]). *Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in N$. Then there exists a nondecreasing sequence $\{\tau(n)\} \subset N$ such that $\tau(n) \rightarrow \infty$, $a_{\tau(n)} \leq a_{\tau(n)+1}$, and $a_n \leq a_{\tau(n)+1}$ are satisfied by all (sufficiently large) numbers $n \in N$. In fact, $\tau(n) = \max\{k \leq n : a_k < a_{k+1}\}$.*

Lemma 9 (see [19]). *Let $\{a_n\}_{n \in N}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ a sequence of real numbers in $(0, 1)$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{\eta_n\}$ a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty} \eta_n < \infty$, and $\{\delta_n\}$ a sequence of real numbers with $\limsup_{n \rightarrow \infty} \delta_n \leq 0$. Suppose that $a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n \delta_n + \eta_n$ for each $n \in N$. Then $\lim_{n \rightarrow \infty} a_n = 0$.*

3. Main Results

In this section, we first give the following result.

Lemma 10. *Let H_1 and H_2 be two real Hilbert spaces, let $A : H_1 \rightarrow H_2$ be a linear and bounded operator, and let A^* denote the adjoint of A. Let $\{B_i : H_1 \rightarrow H_1\}_{i \in N}$ and $\{K_i : H_2 \rightarrow H_2\}_{i \in N}$ be two families of set-valued maximal monotone mappings, and let $\beta_i > 0$ and $\gamma_i > 0$ for all $i \in N$. Given any $x^* \in H_1$, we have the following.*

(i) *If x^* is a solution of GSFVIP, then $J_{\beta_i}^{B_i}(x^* - \gamma_i A^*(I - J_{\beta_i}^{K_i})Ax^*) = x^*$, for all $i \in N$.*

(ii) *Suppose that $J_{\beta_i}^{B_i}(x^* - \gamma_i A^*(I - J_{\beta_i}^{K_i})Ax^*) = x^*$, for all $i \in N$, and that the solution set of GSFVIP is nonempty; then x^* is a solution of GSFVIP.*

Proof. (i) Suppose that $x^* \in H_1$ is a solution of GSFVIP. Then $x^* \in \bigcap_{i=1}^{\infty} B_i^{-1}(0)$ and $Ax^* \in \bigcap_{i=1}^{\infty} K_i^{-1}(0)$. By Lemma 6(ii), it is easy to see that

$$J_{\beta_i}^{B_i}\left(x^* - \gamma_i A^*\left(I - J_{\beta_i}^{K_i}\right)Ax^*\right)$$

$$= J_{\beta_i}^{B_i}\left(x^* - \gamma_i A^*\left(Ax^* - J_{\beta_i}^{K_i}Ax^*\right)\right) \tag{8}$$

$$= J_{\beta_i}^{B_i}x^* = x^*, \quad \forall i \in N.$$

(ii) Suppose that w^* is a solution of GSFVIP and $x^* = J_{\beta_i}^{B_i}(x^* - \gamma_i A^*(I - J_{\beta_i}^{K_i})Ax^*)$, for all $i \in N$. By Lemma 6(vi),

$$\left\langle \left(x^* - \gamma_i A^*\left(I - J_{\beta_i}^{K_i}\right)Ax^*\right) - x^*, x^* - w \right\rangle \geq 0, \tag{9}$$
$$\forall w \in B_i^{-1}(0), \quad \forall i \in N.$$

That is,

$$\left\langle A^*\left(I - J_{\beta_i}^{K_i}\right)Ax^*, x^* - w \right\rangle \leq 0, \quad \forall w \in B_i^{-1}(0), \; \forall i \in N. \tag{10}$$

By (10) and the fact that A^* is the adjoint of A,

$$\left\langle Ax^* - J_{\beta_i}^{K_i}Ax^*, Ax^* - Aw \right\rangle \leq 0, \quad \forall w \in B_i^{-1}(0), \; \forall i \in N. \tag{11}$$

On the other hand, by Lemma 6(vi) again,

$$\left\langle Ax^* - J_{\beta_i}^{K_i}Ax^*, v - J_{\beta_i}^{K_i}Ax^* \right\rangle \leq 0, \quad \forall v \in K_i^{-1}(0), \; \forall i \in N. \tag{12}$$

By (11) and (12),

$$\left\langle Ax^* - J_{\beta_i}^{K_i}Ax^*, v - J_{\beta_i}^{K_i}Ax^* + Ax^* - Aw \right\rangle \leq 0, \quad \forall i \in N \tag{13}$$

for each $w \in B_i^{-1}(0)$ and each $v \in K_i^{-1}(0)$, for all $i \in N$. That is,

$$\left\| Ax^* - J_{\beta_i}^{K_i}Ax^* \right\|^2 \leq \left\langle Ax^* - J_{\beta_i}^{K_i}Ax^*, Aw - v \right\rangle, \quad \forall i \in N \tag{14}$$

for each $w \in B_i^{-1}(0)$ and each $v \in K_i^{-1}(0)$, for all $i \in N$. Since w^* is a solution of GSFVIP, $w^* \in \bigcap_{i=1}^{\infty} B_i^{-1}(0)$ and $Aw^* \in \bigcap_{i=1}^{\infty} K_i^{-1}(0)$. So, it follows from (14) that $Ax^* = J_{\beta_i}^{K_i}Ax^*$, for all $i \in N$. Then, $Ax^* \in \mathrm{Fix}(J_{\beta_i}^{K_i}) = K_i^{-1}(0)$, for all $i \in N$. Therefore $Ax^* \in \bigcap_{i=1}^{\infty} K_i^{-1}(0)$. Further,

$$x^* = J_{\beta_i}^{B_i}\left(x^* - \gamma_i A^*\left(I - J_{\beta_i}^{K_i}\right)Ax^*\right) = J_{\beta_i}^{B_i}x^*, \quad \forall i \in N. \tag{15}$$

Then $x^* \in \mathrm{Fix}(J_{\beta_i}^{B_i}) = B_i^{-1}(0)$, for all $i \in N$. So, $x^* \in \bigcap_{i=1}^{\infty} B_i^{-1}(0)$. Therefore, x^* is a solution of GSFVIP. $\square$

Theorem 11. *Let H_1 and H_2 be two real Hilbert spaces, let $A : H_1 \rightarrow H_2$ be a linear and bounded operator, and let A^* denote the adjoint of A. Let $\{B_i : H_1 \rightarrow H_1\}_{i \in N}$ and $\{K_i : H_2 \rightarrow H_2\}_{i \in N}$ be two families of set-valued maximal monotone mappings. Let $\{a_n\}, \{b_n\}, \{c_{n,i}\}$ be sequences of real numbers in $(0,1)$ with $a_n + b_n + \sum_{i=1}^{\infty} c_{n,i} = 1$. Let $\{\beta_{n,i}\}$ be a sequence in $(0, \infty)$ and $\{\gamma_{n,i}\} \subset (0, 2/(\|A\|^2+1))$ for each $i \in N$. Let Ω be the solution set of GSFVIP and suppose that $\Omega \neq \emptyset$. Let f be a self k-contraction mapping of H_1, $k \in (0,1)$. Let $\{x_n\}$ be defined by*

$$x_{n+1} = a_n x_n + b_n f(x_n)$$
$$+ \sum_{i=1}^{\infty} c_{n,i} J_{\beta_{n,i}}^{B_i}\left(I - \gamma_{n,i} A^*\left(I - J_{\beta_{n,i}}^{K_i}\right)A\right)x_n, \quad n \geq 0. \tag{16}$$

If the sequences $\{a_n\}, \{b_n\}, \{c_{n,i}\}, \{\beta_{n,i}\}$, and $\{\gamma_{n,i}\}$ satisfy the following conditions:

(i) $\lim_{n \to \infty} b_n = 0$, $\sum_{n=0}^{\infty} b_n = \infty$,

(ii) *for each $i \in N$, $\liminf_{n \to \infty} a_n c_{n,i} > 0$, $\liminf_{n \to \infty} c_{n,i} \gamma_{n,i} > 0$, $\liminf_{n \to \infty} \beta_{n,i} > 0$, and $\limsup_{n \to \infty} \gamma_{n,i} < 2/(\|A\|^2 + 1)$,*

then the sequence $\{x_n\}$ converges strongly to $P_\Omega f(x^*)$.

Proof. First, we show that $\{x_n\}$ is bounded. In fact, let $z \in \Omega$; it follows from Lemmas 6(i) and 7 that $J_{\beta_{n,i}}^{B_i}(I - \gamma_{n,i}A^*(I - J_{\beta_{n,i}}^{K_i})A)$, for all $i \in N$, are nonexpansive, and by Lemma 10 we have

$$\|x_{n+1} - z\|$$

$$= \left\| a_n x_n + b_n f(x_n) \right.$$
$$\left. + \sum_{i=1}^{\infty} c_{n,i} J_{\beta_{n,i}}^{B_i}\left(I - \gamma_{n,i} A^*\left(I - J_{\beta_{n,i}}^{K_i}\right)A\right)x_n - z \right\|$$

$$\leq a_n \|x_n - z\| + b_n \|f(x_n) - z\|$$
$$+ \sum_{i=1}^{\infty} c_{n,i} \left\| J_{\beta_{n,i}}^{B_i}\left(I - \gamma_{n,i} A^*\left(I - J_{\beta_{n,i}}^{K_i}\right)A\right)x_n - z \right\|$$

$$\leq a_n \|x_n - z\| + b_n \|f(x_n) - z\| + \sum_{i=1}^{\infty} c_{n,i} \|x_n - z\|$$

$$\leq (1 - b_n) \|x_n - z\| + b_n \|f(x_n) - z\|$$

$$\leq (1 - b_n) \|x_n - z\|$$

$$\quad + b_n \|f(x_n) - f(z)\| + b_n \|f(z) - z\|$$

$$\leq (1 - b_n) \|x_n - z\| + b_n k \|x_n - z\| + b_n \|f(z) - z\|$$

$$\leq (1 - (1 - k) b_n) \|x_n - z\|$$

$$\quad + (1 - k) b_n \frac{1}{1 - k} \|f(z) - z\|$$

$$\leq \max \left\{ \|x_n - z\|, \frac{1}{1 - k} \|f(z) - z\| \right\}$$

$$\leq \cdots \leq \max \left\{ \|x_0 - z\|, \frac{1}{1 - k} \|f(z) - z\| \right\}, \tag{17}$$

which implies that $\{x_n\}$ is bounded, and we also obtain that $\{f(x_n)\}$ is bounded.

Next, we show that there exists a unique $x^* \in \Omega$ such that $x^* = P_\Omega f(x^*)$.

Since, for all $i \in N$, $\liminf_{n \to \infty} \beta_{n,i} > 0$, we may assume that $\beta_{n,i} > \beta_i > 0$ for each $n \in N$. Since, for all $i \in N$, $\{\gamma_{n,i}\}$ is bounded, there exists a converge subsequence. Without loss of generality, we can assume that $\gamma_{n,i} \to \gamma_i \in (0, 2/(\|A\|^2 + 1))$ for each $i \in N$.

It follows from Lemma 10 that $x^* \in \Omega$ solves the GSFVIP if and only if x^* solves the fixed point equation

$$x^* = J_{\beta_i}^{B_i} \left(I - \gamma_i A^* \left(I - J_{\beta_i}^{K_i} \right) A \right) x^*, \quad i \in N \tag{18}$$

that is, the solution sets of fixed point equation (18) and GSFVIP are the same. By Lemmas 6(i) and 7, the operators $J_{\beta_i}^{B_i}(I - \gamma_i A^* (I - J_{\beta_i}^{K_i}) A)$, for all $i \in N$, are nonexpansive. Since the fixed point set of nonexpansive operators is closed and convex, the projection onto the solution set Ω is well defined whenever $\Omega \neq \emptyset$. We observe that $P_\Omega f$ is a contraction of H_1 into itself. Indeed, since P_Ω is nonexpansive and f is a self k-contraction mapping H_1,

$$\|P_\Omega (f)(x) - P_\Omega (f)(y)\| \leq \|f(x) - f(y)\| \leq k \|x - y\|. \tag{19}$$

Hence, there exists a unique element $x^* \in \Omega$ such that $x^* = P_\Omega f(x^*)$.

In order to prove that $x_n \to x^*$ as $n \to \infty$, we consider two possible cases.

Case 1. There exists a natural number n_0 such that $\|x_{n+1} - x^*\| \leq \|x_n - x^*\|$ for each $n \geq n_0$. Since $\{\|x_n - x^*\|\}$ is bounded, we have $\{\|x_n - x^*\|\}$ is convergent.

Next, we show that, for each $i \in N$,

$$\lim_{n \to \infty} \left\| J_{\beta_i}^{B_i} \left(I - \gamma_i A^* \left(I - J_{\beta_i}^{K_i} \right) A \right) x_n - x_n \right\| = 0. \tag{20}$$

By Lemmas 6(i) and 7, for every $z \in \Omega$ and $i \in N$, we have

$$\left\| J_{\beta_{n,i}}^{B_i} \left(I - \gamma_{n,i} A^* \left(I - J_{\beta_{n,i}}^{K_i} \right) A \right) x_n - z \right\|$$

$$\leq \left\| \left(x_n - \gamma_{n,i} A^* \left(I - J_{\beta_{n,i}}^{K_i} \right) A x_n \right) \right.$$

$$\left. - \left(z - \gamma_{n,i} A^* \left(I - J_{\beta_{n,i}}^{K_i} \right) A z \right) \right\|$$

$$\leq \|x_n - z\|^2 - \left(2\gamma_{n,i} - \gamma_{n,i}^2 \|A\|^2 \right) \tag{21}$$

$$\times \left\| \left(I - J_{\beta_{n,i}}^{K_i} \right) A x_n - \left(I - J_{\beta_{n,i}}^{K_i} \right) A z \right\|^2$$

$$= \|x_n - z\|^2 - \left(2\gamma_{n,i} - \gamma_{n,i}^2 \|A\|^2 \right)$$

$$\times \left\| A x_n - J_{\beta_{n,i}}^{K_i} A x_n \right\|^2.$$

By using Lemma 3 and (21), for every $z \in \Omega$ and $i \in N$, we have

$$\|x_{n+1} - z\|^2$$

$$= \left\| a_n x_n + b_n f(x_n) \right.$$

$$\left. + \sum_{j=1}^{\infty} c_{n,j} J_{\beta_{n,j}}^{B_j} \left(I - \gamma_{n,j} A^* \left(I - J_{\beta_{n,j}}^{K_j} \right) A \right) x_n - z \right\|^2$$

$$\leq a_n \|x_n - z\|^2 + b_n \|f(x_n) - z\|^2$$

$$+ \sum_{j=1}^{\infty} c_{n,j} \left\| J_{\beta_{n,j}}^{B_j} \left(I - \gamma_{n,j} A^* \left(I - J_{\beta_{n,j}}^{K_j} \right) A \right) x_n - z \right\|^2$$

$$- a_n c_{n,i} \left\| J_{\beta_{n,i}}^{B_i} \left(I - \gamma_{n,i} A^* \left(I - J_{\beta_{n,i}}^{K_i} \right) A \right) x_n - x_n \right\|^2$$

$$\leq a_n \|x_n - z\|^2 + b_n \|f(x_n) - z\|^2 \tag{22}$$

$$+ \sum_{j=1}^{\infty} c_{n,j} \left(\|x_n - z\|^2 \right.$$

$$\left. - \left(2\gamma_{n,j} - \gamma_{n,j}^2 \|A\|^2 \right) \left\| A x_n - J_{\beta_{n,j}}^{K_j} A x_n \right\|^2 \right)$$

$$- a_n c_{n,i} \left\| J_{\beta_{n,i}}^{B_i} \left(I - \gamma_{n,i} A^* \left(I - J_{\beta_{n,i}}^{K_i} \right) A \right) x_n - x_n \right\|^2$$

$$\leq (1 - b_n) \|x_n - z\|^2 + b_n \|f(x_n) - z\|^2$$

$$- c_{n,i} \left(2\gamma_{n,i} - \gamma_{n,i}^2 \|A\|^2 \right) \left\| A x_n - J_{\beta_{n,i}}^{K_i} A x_n \right\|^2$$

$$- a_n c_{n,i} \left\| J_{\beta_{n,i}}^{B_i} \left(I - \gamma_{n,i} A^* \left(I - J_{\beta_{n,i}}^{K_i} \right) A \right) x_n - x_n \right\|^2.$$

Hence, for each $i \in N$, we have

$$a_n c_{n,i} \left\| J_{\beta_{n,i}}^{B_i} \left(I - \gamma_{n,i} A^* \left(I - J_{\beta_{n,i}}^{K_i} \right) A \right) x_n - x_n \right\|^2$$
$$\leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + b_n \|f(x_n) - z\|^2,$$

$$c_{n,i} \left(2\gamma_{n,i} - \gamma_{n,i}^2 \|A\|^2 \right) \left\| Ax_n - J_{\beta_{n,i}}^{K_i} Ax_n \right\|^2$$
$$\leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + b_n \|f(x_n) - z\|^2. \tag{23}$$

Since $\lim_{n \to \infty} b_n = 0$ and $f(x_n)$ is bounded, from (23) we get that

$$\lim_{n \to \infty} a_n c_{n,i} \left\| J_{\beta_{n,i}}^{B_i} \left(I - \gamma_{n,i} A^* \left(I - J_{\beta_{n,i}}^{K_i} \right) A \right) x_n - x_n \right\|^2 = 0,$$
$$\forall i \in N, \tag{24}$$

$$\lim_{n \to \infty} c_{n,i} \left(2\gamma_{n,i} - \gamma_{n,i}^2 \|A\|^2 \right) \left\| Ax_n - J_{\beta_{n,i}}^{K_i} Ax_n \right\|^2 = 0, \quad \forall i \in N. \tag{25}$$

By assuming that, for all $i \in N$, $\liminf_{n \to \infty} a_n c_{n,i} > 0$, it follows from (24) that

$$\lim_{n \to \infty} \left\| J_{\beta_{n,i}}^{B_i} \left(I - \gamma_{n,i} A^* \left(I - J_{\beta_{n,i}}^{K_i} \right) A \right) x_n - x_n \right\| = 0, \quad \forall i \in N. \tag{26}$$

Further, for all $i \in N$, we have

$$\left\| J_{\beta_{n,i}}^{B_i} \left(I - \gamma_i A^* \left(I - J_{\beta_{n,i}}^{K_i} \right) A \right) x_n - x_n \right\|$$
$$\leq \left\| J_{\beta_{n,i}}^{B_i} \left(I - \gamma_i A^* \left(I - J_{\beta_{n,i}}^{K_i} \right) A \right) x_n \right.$$
$$\left. - J_{\beta_{n,i}}^{B_i} \left(I - \gamma_{n,i} A^* \left(I - J_{\beta_{n,i}}^{K_i} \right) A \right) x_n \right\|$$
$$+ \left\| J_{\beta_{n,i}}^{B_i} \left(I - \gamma_{n,i} A^* \left(I - J_{\beta_{n,i}}^{K_i} \right) A \right) x_n - x_n \right\|$$
$$\leq \left\| \left(I - \gamma_i A^* \left(I - J_{\beta_{n,i}}^{K_i} \right) A \right) x_n \right.$$
$$\left. - \left(I - \gamma_{n,i} A^* \left(I - J_{\beta_{n,i}}^{K_i} \right) A \right) x_n \right\| \tag{27}$$
$$+ \left\| J_{\beta_{n,i}}^{B_i} \left(I - \gamma_{n,i} A^* \left(I - J_{\beta_{n,i}}^{K_i} \right) A \right) x_n - x_n \right\|$$
$$\leq |\gamma_i - \gamma_{n,i}| \left\| A^* \left(I - J_{\beta_{n,i}}^{K_i} \right) Ax_n \right\|$$
$$+ \left\| J_{\beta_{n,i}}^{B_i} \left(I - \gamma_{n,i} A^* \left(I - J_{\beta_{n,i}}^{K_i} \right) A \right) x_n - x_n \right\| \longrightarrow 0$$
$$(n \longrightarrow \infty).$$

Clearly, for all $i \in N$, $c_{n,i}(2\gamma_{n,i} - \gamma_{n,i}^2 \|A\|^2) \geq c_{n,i}\gamma_{n,i}/(\|A\|^2 + 1)$. Since, for all $i \in N$, $\liminf_{n \to \infty} c_{n,i}\gamma_{n,i} > 0$, it follows from (25) that

$$\lim_{n \to \infty} \left\| Ax_n - J_{\beta_{n,i}}^{K_i} Ax_n \right\| = 0, \quad \forall i \in N, \tag{28}$$

and it follows from Lemma 6(iii) that

$$\lim_{n \to \infty} \left\| Ax_n - J_{\beta_i}^{K_i} Ax_n \right\| = 0, \quad \forall i \in N. \tag{29}$$

Besides, by Lemma 6(i) and (28), for all $i \in N$, we have

$$\left\| J_{\beta_{n,i}}^{B_i} \left(I - \gamma_i A^* \left(I - J_{\beta_i}^{K_i} \right) A \right) x_n - J_{\beta_{n,i}}^{B_i} x_n \right\|$$
$$\leq \gamma_i \|A\| \left\| Ax_n - J_{\beta_{n,i}}^{K_i} Ax_n \right\| \longrightarrow 0 \quad (n \longrightarrow \infty). \tag{30}$$

By (27) and (30), for all $i \in N$, we obtain

$$\left\| x_n - J_{\beta_{n,i}}^{B_i} x_n \right\|$$
$$\leq \left\| x_n - J_{\beta_{n,i}}^{B_i} \left(I - \gamma_i A^* \left(I - J_{\beta_i}^{K_i} \right) A \right) x_n \right\|$$
$$+ \left\| J_{\beta_{n,i}}^{B_i} \left(I - \gamma_i A^* \left(I - J_{\beta_i}^{K_i} \right) A \right) x_n - J_{\beta_{n,i}}^{B_i} x_n \right\| \longrightarrow 0$$
$$(n \longrightarrow \infty). \tag{31}$$

It follows from Lemma 6(iii) that

$$\lim_{n \to \infty} \left\| x_n - J_{\beta_i}^{B_i} x_n \right\| = 0, \quad \forall i \in N. \tag{32}$$

By Lemma 6(i) and (29), for all $i \in N$, we have

$$\left\| J_{\beta_i}^{B_i} \left(I - \gamma_i A^* \left(I - J_{\beta_i}^{K_i} \right) A \right) x_n - J_{\beta_i}^{B_i} x_n \right\|$$
$$\leq \gamma_i \|A\| \left\| Ax_n - J_{\beta_i}^{K_i} Ax_n \right\| \longrightarrow 0 \quad (n \longrightarrow \infty). \tag{33}$$

By (32) and (33), for all $i \in N$, we obtain

$$\lim_{n \to \infty} \left\| J_{\beta_i}^{B_i} \left(I - \gamma_i A^* \left(I - J_{\beta_i}^{K_i} \right) A \right) x_n - x_n \right\| = 0. \tag{34}$$

Now, we show that

$$\limsup_{n \to \infty} \left\langle f(x^*) - x^*, x_n - x^* \right\rangle \leq 0. \tag{35}$$

To show this inequality, we choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\lim_{k \to \infty} \left\langle f(x^*) - x^*, x_{n_k} - x^* \right\rangle$$
$$= \limsup_{n \to \infty} \left\langle f(x^*) - x^*, x_n - x^* \right\rangle. \tag{36}$$

Since $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ which converges weakly to w. Without loss of generality, we can assume that $x_{n_k} \rightharpoonup w$. Notice that, for each $i \in N$, $J_{\beta_i}^{B_i}(I - \gamma_i A^*(I - J_{\beta_i}^{K_i})A)$ is nonexpansive. Thus, from Lemma 4 and (34), we have $w \in \Omega$. Therefore, it follows from Lemma 5 that

$$\limsup_{n \to \infty} \left\langle f(x^*) - x^*, x_n - x^* \right\rangle$$
$$= \lim_{k \to \infty} \left\langle f(x^*) - x^*, x_{n_k} - x^* \right\rangle \tag{37}$$
$$= \left\langle f(x^*) - x^*, w - x^* \right\rangle \leq 0.$$

Finally, we show that $x_n \to x^*$. Applying Lemma 2, we have that

$$\|x_{n+1} - x^*\|^2$$

$$= \left\| a_n x_n + b_n f(x_n) \right.$$

$$\left. + \sum_{i=1}^{\infty} c_{n,i} J_{\beta_{n,i}}^{B_i} \left(I - \gamma_{n,i} A^* \left(I - J_{\beta_{n,i}}^{K_i} \right) A \right) x_n - x^* \right\|^2$$

$$\leq \left\| a_n (x_n - x^*) \right.$$

$$+ \sum_{i=1}^{\infty} c_{n,i} \left(J_{\beta_{n,i}}^{B_i} \left(I - \gamma_{n,i} A^* \left(I - J_{\beta_{n,i}}^{K_i} \right) A \right) x_n - x^* \right)$$

$$\left. + b_n (f(x_n) - x^*) \right\|^2$$

$$\leq \left\| a_n (x_n - x^*) \right.$$

$$\left. + \sum_{i=1}^{\infty} c_{n,i} \left(J_{\beta_{n,i}}^{B_i} \left(I - \gamma_{n,i} A^* \left(I - J_{\beta_{n,i}}^{K_i} \right) A \right) x_n - x^* \right) \right\|^2$$

$$+ 2b_n \langle f(x_n) - x^*, x_{n+1} - x^* \rangle$$

$$\leq \left(a_n \|x_n - x^*\| + \sum_{i=1}^{\infty} c_{n,i} \|x_n - x^*\| \right)^2$$

$$+ 2b_n \langle f(x_n) - x^*, x_{n+1} - x^* \rangle$$

$$\leq (1 - b_n)^2 \|x_n - x^*\|^2 + 2b_n \langle f(x_n) - f(x^*), x_{n+1} - x^* \rangle$$

$$+ 2b_n \langle f(x^*) - x^*, x_{n+1} - x^* \rangle$$

$$\leq (1 - b_n)^2 \|x_n - x^*\|^2 + 2b_n k \|x_n - x^*\| \|x_{n+1} - x^*\|$$

$$+ 2b_n \langle f(x^*) - x^*, x_{n+1} - x^* \rangle$$

$$\leq (1 - b_n)^2 \|x_n - x^*\|^2 + b_n k \left(\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2 \right)$$

$$+ 2b_n \langle f(x^*) - x^*, x_{n+1} - x^* \rangle.$$

(38)

This implies that

$$\|x_{n+1} - x^*\|^2$$

$$\leq \frac{(1 - b_n)^2 + b_n k}{1 - b_n k} \|x_n - x^*\|^2$$

$$+ \frac{2b_n}{1 - b_n k} \langle f(x^*) - x^*, x_{n+1} - x^* \rangle$$

$$= \frac{1 - 2b_n + b_n k}{1 - b_n k} \|x_n - x^*\|^2 + \frac{b_n^2}{1 - b_n k} \|x_n - x^*\|^2$$

$$+ \frac{2b_n}{1 - b_n k} \langle f(x^*) - x^*, x_{n+1} - x^* \rangle$$

$$= \left(1 - \frac{2(1-k) b_n}{1 - b_n k} \right) \|x_n - x^*\|^2$$

$$+ \frac{2(1-k) b_n}{1 - b_n k} \left\{ \frac{b_n M}{2(1-k)} \right.$$

$$\left. + \frac{1}{1-k} \langle f(x^*) - x^*, x_{n+1} - x^* \rangle \right\}$$

$$= (1 - \alpha_n) \|x_n - x^*\|^2 + \alpha_n \delta_n,$$

(39)

where $\delta_n = b_n M / 2(1-k) + (1/(1-k)) \langle f(x^*) - x^*, x_{n+1} - x^* \rangle$, $M = \sup\{\|x_n - x^*\|^2 : n \geq 0\}$, and $\alpha_n = 2(1-k) b_n / (1 - b_n k)$. It is easy to see that $\lim_{n \to \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\lim \sup_{n \to \infty} \delta_n \leq 0$. Hence, by Lemma 9, the sequence $\{x_n\}$ converges strongly to $x^* = P_\Omega f(x^*)$.

Case 2. Assume that $\{\|x_n - x^*\|\}$ is not a monotone sequence. Then, we can define an integer sequence $\{\tau(n)\}$ for all $n \geq n_0$ (for some n_0 large enough) by

$$\tau(n) = \max \{ k \in N; k \leq n : \|x_k - x^*\| < \|x_{k+1} - x^*\| \}.$$

(40)

Clearly, $\{\tau(n)\}$ is a nondecreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ for all $n \geq n_0$:

$$\|x_{\tau(n)} - x^*\| < \|x_{\tau(n)+1} - x^*\|.$$

(41)

Following a similar argument as the proof of Case 1, we have

$$\lim_{n \to \infty} \sup \langle f(x^*) - x^*, x_{\tau(n)} - x^* \rangle \leq 0.$$

(42)

And by similar argument, we have

$$\|x_{\tau(n)+1} - x^*\|^2 = (1 - \alpha_{\tau(n)}) \|x_{\tau(n)} - x^*\|^2 + \alpha_{\tau(n)} \delta_{\tau(n)},$$

(43)

where $\lim_{n \to \infty} \alpha_{\tau(n)} = 0$, $\sum_{n=1}^{\infty} \alpha_{\tau(n)} = \infty$, and $\lim \sup_{n \to \infty} \delta_{\tau(n)} \leq 0$. Hence, by Lemma 9, we obtain $\lim_{n \to \infty} \|x_{\tau(n)} - x^*\| = 0$ and $\lim_{n \to \infty} \|x_{\tau(n)+1} - x^*\| = 0$. Now, from Lemma 8, we have

$$0 \leq \|x_n - x^*\| \leq \|x_{\tau(n)+1} - x^*\|.$$

(44)

Therefore, the sequence $\{x_n\}$ converges strongly to $x^* = P_\Omega f(x^*)$. This completes the proof. □

Corollary 12. *Let H_1 and H_2 be two real Hilbert spaces, let $A : H_1 \to H_2$ be a linear and bounded operator, and let A^* denote the adjoint of A. Let $\{B_i : H_1 \to H_1\}_{i \in N}$ and $\{K_i : H_2 \to H_2\}_{i \in N}$ be two families of set-valued maximal monotone mappings for all $i \in N$. Let $\{a_n\}, \{c_{n,i}\}$ be sequences of real numbers in $(0,1)$ with $a_n + \sum_{i=1}^{\infty} c_{n,i} \subset (0,1)$. Let $\{\beta_{n,i}\}$*

be a sequence in $(0, \infty)$ and $\{\gamma_{n,i}\} \subset (0, 2/(\|A\|^2 + 1))$ for each $i \in N$. Let Ω be the solution set of GSFVIP and suppose that $\Omega \neq \emptyset$. Let $\{x_n\}$ be defined by

$$x_{n+1} = a_n x_n + \sum_{i=1}^{\infty} c_{n,i} J_{\beta_{n,i}}^{B_i} \left(I - \gamma_{n,i} A^* \left(I - J_{\beta_{n,i}}^{K_i} \right) A \right) x_n, \tag{45}$$
$$n \geq 0.$$

If the sequences $\{a_n\}$, $\{c_{n,i}\}$, $\{\beta_{n,i}\}$, and $\{\gamma_{n,i}\}$ satisfy the following conditions:

(i) $\lim_{n \to \infty}(a_n + \sum_{i=1}^{\infty} c_{n,i}) = 1$, $\sum_{n=0}^{\infty}(1 - a_n - \sum_{i=1}^{\infty} c_{n,i}) = \infty$,

(ii) for each $i \in N$, $\liminf_{n\to\infty} a_n c_{n,i} > 0$, $\liminf_{n\to\infty} c_{n,i}\gamma_{n,i} > 0$, $\liminf_{n\to\infty} \beta_{n,i} > 0$, and $\limsup_{n\to\infty}\gamma_{n,i} < 2/(\|A\|^2 + 1)$,

then the sequence $\{x_n\}$ converges strongly to a point x^* which is the minimum norm solution of GSFVIP.

Proof. Let $f = 0$ in Theorem 11. We have proved that the sequence $\{x_n\}$ converges strongly to $x^* = P_\Omega f(x^*)$. Then, for all $z \in \Omega$, we have

$$\langle f(x^*) - x^*, z - x^* \rangle$$
$$= \langle f(x^*) - P_\Omega f(x^*), z - P_\Omega f(x^*) \rangle \leq 0. \tag{46}$$

Since $f = 0$, then $\langle -x^*, z - x^* \rangle \leq 0$, for all $z \in \Omega$. Hence, we obtain

$$\|x^*\|^2 \leq |\langle x^*, z \rangle| \leq \|x^*\| \|z\| \tag{47}$$

that is, $\|x^*\| \leq \|z\|$. Thus, x^* is the minimum norm solution of GSFVIP. This completes the proof. $\qquad\square$

In Theorem 11, if we set $f(x) = u$, then we get the following result.

Corollary 13. *Let H_1 and H_2 be two real Hilbert spaces, let $A : H_1 \to H_2$ be a linear and bounded operator, and let A^* denote the adjoint of A. Let $\{B_i : H_1 \to H_1\}_{i \in N}$ and $\{K_i : H_2 \to H_2\}_{i \in N}$ be two families of set-valued maximal monotone mappings for all $i \in N$. Let $\{a_n\}, \{b_n\}, \{c_{n,i}\}$ be sequences of real numbers in $(0, 1)$ with $a_n + b_n + \sum_{i=1}^{\infty} c_{n,i} = 1$. Let $\{\beta_{n,i}\}$ be a sequence in $(0, \infty)$ and $\{\gamma_{n,i}\} \subset (0, 2/(\|A\|^2 + 1))$ for each $i \in N$. Let $u \in H_1$ be fixed. Let Ω be the solution set of GSFVIP and suppose that $\Omega \neq \emptyset$. Let $\{x_n\}$ be defined by*

$$x_{n+1} = a_n x_n + b_n u$$
$$+ \sum_{i=1}^{\infty} c_{n,i} J_{\beta_{n,i}}^{B_i} \left(I - \gamma_{n,i} A^* \left(I - J_{\beta_{n,i}}^{K_i} \right) A \right) x_n, \quad n \geq 0.$$
$$\tag{48}$$

If the sequences $\{a_n\}, \{b_n\}, \{c_{n,i}\}, \{\beta_{n,i}\}$, and $\{\gamma_{n,i}\}$ satisfy the following conditions:

(i) $\lim_{n\to\infty} b_n = 0$, $\sum_{n=0}^{\infty} b_n = \infty$,

(ii) for each $i \in N$, $\liminf_{n\to\infty} a_n c_{n,i} > 0$, $\liminf_{n\to\infty} c_{n,i}\gamma_{n,i} > 0$, $\liminf_{n\to\infty} \beta_{n,i} > 0$, and $\limsup_{n\to\infty}\gamma_{n,i} < 2/(\|A\|^2 + 1)$,

then the sequence $\{x_n\}$ converges strongly to $P_\Omega u$.

Conflict of Interests

The authors declare that they have no conflict of interests.

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