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Research Article

Nonlinear Isometries on Schatten-p Class in Atomic Nest Algebras

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Let H be a complex Hilbert space; denote by $\operatorname{Alg}\mathcal{N}$ and $\mathcal{C}_p(H)$ the atomic nest algebra associated with the atomic nest $\mathcal{N}$ on H and the space of Schatten-p class operators on, H respectively. Let $\mathcal{C}_p(H)\cap\operatorname{Alg}\mathcal{N}$ be the space of Schatten-p class operators in $\operatorname{Alg}\mathcal{N}$. When $1 \le p < +\infty$ and $p \ne 2$, we give a complete characterization of nonlinear surjective isometries on $\mathcal{C}_p(H)\cap\operatorname{Alg}\mathcal{N}$. If $p = 2$, we also prove that a nonlinear surjective isometry on $\mathcal{C}_2(H)\cap\operatorname{Alg}\mathcal{N}$ is the translation of an orthogonality preserving map.

1. Introduction

Let X and Y be normed spaces and let ϕ be a map from X to Y. We say that ϕ is a nonlinear isometry (or a distance preserving map) if $\|\phi(x_1) - \phi(x_2)\| = \|x_1 - x_2\|$ for every pair x_1, x_2 in X. In particular, ϕ is an isometry if ϕ is linear and distance preserving. The question of characterizing isometries between operator algebras is very important in studying geometric structure of operator algebras. Many authors pay their attention to such a problem (see [1–11] and their references). One of well-known results is due to Kadison, who states that every isometry for the operator norm from a unital C^*-algebra onto another unital C^*-algebra is a C^*-isomorphism followed by left multiplication by a fixed unitary element (see [7]). Besides, of the operator norm, isometries for the Schatten-p norm also are studied extensively (see [1, 2, 5, 6, 8–11] and their references). Early in 1975, Arazy in [2] gave a characterization of isometries on the Schatten-p class ($p \ne 2$). In [1], Anoussis and Katavolos characterized isometries on the Schatten-p class in nest algebras and obtained the following theorem.

Theorem 1. *Let $\mathcal{N}$ and $\mathcal{M}$ be two nests of projections on a Hilbert space H and J a fixed involution on H. Assume that $1 \le p < +\infty$, $p \ne 2$. The surjective isometries $\Phi : \mathcal{C}_p(H)\cap\operatorname{Alg}\mathcal{N} \to \mathcal{C}_p(H)\cap\operatorname{Alg}\mathcal{M}$ have one of the following forms:*

$$\Phi(A) = VAV^*W \text{ or } \Phi(A) = VJA^*JV^*W, \tag{1}$$

where W is a unitary operator, and $E \mapsto VEV^*$ is an order isomorphism of $\mathcal{N}$ onto $\mathcal{M}$ ($E \mapsto VJEJV^*$ is an order isomorphism of $\mathcal{N}$ onto $\mathcal{M}^\perp$).

More generally, in recent years, many authors are devoted to characterizing distance preserving maps on operator algebras (see [3, 4, 12–15] and their references). In [14], Chan et al. showed that a nonlinear surjective isometry Φ for the unitarily invariant norm on $n \times m$ complex matrix algebras has one of the following forms.

(a) There are unitary matrices U, V and a $n \times m$ matrix S such that $\Phi(A) = UAV + S$ or $\Phi(A) = U\overline{A}V + S$ for each $n \times m$ matrix A.

(b) If $m = n$ and Φ has the form, there are unitary matrices U, V and a $n \times n$ matrix S such that $\Phi(A) = UA^tV + S$ or $\Phi(A) = UA^*V + S$ for each $n \times n$ matrix A.

(c) If the unitarily invariant norm is a multiple of the Frobenius norm, that is, $\|A\| = \lambda\,\mathrm{tr}(AA^*)^{1/2}$ for some $\lambda > 0$, then the map $A \mapsto \Phi(A) - S$ is a real orthogonal transformation with respect to the inner product $(A, B) = \mathrm{Retr}(AB^*)$ for each $n \times m$ matrix A.

Recall that a norm $\|\cdot\|$ of operators is a unitary invariant norm if $\|UAV\| = \|A\|$ for any unitary operators U, V. In [3, 4, 12, 13, 15], distance preserving maps on several kinds of operator algebras in the infinite dimensional case were characterized. Bai and Hou in [12] give a characterization of nonlinear numerical radius isometries on $\mathscr{B}(H)$. Cui and Hou in [13] characterize nonlinear numerical radius isometries on atomic nest algebras and diagonal algebras. Hou and He in [15] give a characterization of nonlinear isometries on the Schatten-p class. A nature problem is how to characterize nonlinear isometries on the Schatten-p class in nest algebras. The main purpose of this paper is to give a complete characterization of nonlinear surjective isometries on the Schatten-p class ($p \neq 2$) in atomic nest algebras acting on Hilbert spaces (Theorem 2). Such a result generalizes the linear map assumption in Theorem 1 to the nonlinear case. Also, the problem on orthogonality of nonlinear surjective isometries for Hilbert-Schmidt norms is discussed (Theorem 3).

By the classical Mazur-Ulam theorem (see [10]) which states that every distance preserving surjective map sending 0 to 0 between normed spaces is real linear, we essentially deal with the real linear isometries (i.e., the distance preserving real linear maps). One can not expect that each real isometry has the same structure as the complex isometry. Applicable examples are found in [3] (Example 0.2, 0.3 in [3]).

Following the idea of [3], a key step in our approach is to show that the distance preserving maps on the Schatten-p class ($p \neq 2$) in nest algebras also preserve rank-one operators in both directions. This leads to a demand for characterizing rank-1 preserving additive maps between nest algebras. Related results had been obtained in [16].

Before embarking upon our results, it is convenient here to introduce some notations. Denote by $\mathbb{R}$ or $\mathbb{C}$ the real or complex field. For an operator A on H, we denote the range of A by $\operatorname{ran} A$ and the adjoint of A by A^*. Let τ be an automorphism (or homomorphism) of $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. If a map A on H satisfies $A(x+y) = Ax + Ay$ and $A(\lambda x) = \tau(\lambda)Ax$ for every $x, y \in H$ and $\lambda \in \mathbb{F}$, then we say that A is τ-linear. If τ is a ring homomorphism, then we say that A is semilinear and in the case that $\mathbb{F} = \mathbb{C}$ and $\tau(\lambda) \equiv \overline{\lambda}$, we say A is conjugate linear. If U is a conjugate linear operator between Hilbert spaces and $U^*U = UU^* = I$, U is called conjugate unitary or antiunitary, where U^* is the Hilbert space conjugate operator of U. Denote by $\mathscr{K}(H)$ and $\mathscr{F}(H)$ the space of all compact operators and the space of all finite rank operators on the Hilbert space H. For any $A \in \mathscr{K}(H)$, the trace of A, $\operatorname{tr}(A) = \Sigma_i \langle Ae_i, e_i\rangle$, where $\{e_i\}_{(i\in I)}$ is a normal orthogonal base in the Hilbert space H. Let $|A| = (A^*A)^{1/2}$; the Schatten-p norm of A is as follows:

$$\|A\|_p = \operatorname{tr}\left(|A|^p\right)^{1/p}. \quad (2)$$

The Schatten-p class $\mathscr{C}_p(H)$ is the set of all Schatten-p class operators, that is, all compact operators with the finite Schatten-p norm. If $p = 1$, the set $\mathscr{C}_1(H)$ is called the trace class. If $p = \infty$, $\mathscr{C}_\infty(H) = \mathscr{K}(H)$. Recall that a nest on H is a chain $\mathscr{N}$ of closed (under norm topology) subspaces of H containing $\{0\}$ and H, which is closed under the formation of arbitrary closed linear span (denoted by $\bigvee$) and intersection (denoted by $\bigwedge$). $\operatorname{Alg}\mathscr{N}$ denotes the associated nest algebra, which is the set of all operators T in $\mathscr{B}(H)$ such that $TN \subseteq N$ for every element $N \in \mathscr{N}$. If $\mathscr{N}$ is a nest, $\mathscr{N}^\perp = \{N^\perp \mid N \in \mathscr{N}\}$ is a nest. If $\mathscr{N} \neq \{\{0\}, H\}$, we say that $\mathscr{N}$ is nontrivial. We denote $\operatorname{Alg}_{\mathscr{F}}\mathscr{N} = \operatorname{Alg}\mathscr{N} \cap \mathscr{F}(H)$. For any $N \in \mathscr{N}$, let $N_- = \bigvee\{M \in \mathscr{N} \mid M \subset N\}$, $N_+ = \bigwedge\{M \in \mathscr{N} \mid N \subset M\}$, and $N_-^\perp = (N_-)^\perp$. $0_- = 0$, $H_+ = H$. If $N \ominus N_- = N \cap (N_-)^\perp \neq 0$, we say $N \ominus N_-$ is an atom of $\mathscr{N}$. A nest $\mathscr{N}$ on H is said to be atomic if H is spanned by its atoms and to be maximal if $\mathscr{N}$ is atomic and all its atoms are one-dimensional. The rank-one operator $x \otimes f \in \operatorname{Alg}\mathscr{N}$ if and only if there is an $N \in \mathscr{N}$ such that $x \in N$ and $f \in N_-^\perp$. For each $x \in H$, $L_x = \{x \otimes f \mid f \in H\}$ and $f \in H$, $R_f = \{x \otimes f \mid x \in H\}$. If $\mathscr{N}$ is a nest and $N \in \mathscr{N}$, for $x \in N$, $L_x^N = \{x \otimes f \mid f \in N_-^\perp\}$; for $f \in N_-^\perp$, $R_f^N = \{x \otimes f \mid x \in N\}$. Assume that $\mathscr{E}_1(\mathscr{N}) = \cup\{N \in \mathscr{N} \mid \dim N_-^\perp > 1\}$, $\mathscr{E}_2(\mathscr{N}) = \cup\{N_-^\perp \mid N \in \mathscr{N}, \dim N > 1\}$, $\mathscr{D}_1(\mathscr{N}) = \cup\{N \in \mathscr{N} \mid N_- \neq H\}$, $\mathscr{D}_2(\mathscr{N}) = \cup\{N_-^\perp \mid N \in \mathscr{N}$ and $N \neq 0\}$, $E_1(\mathscr{N}) = \{N \mid N \in \mathscr{N}, \dim N_-^\perp > 1\}$, and $E_2(\mathscr{N}) = \{N \in \mathscr{N} \mid \dim N > 1\}$. If the nest is fixed, they are written briefly as $\mathscr{E}_1$, $\mathscr{E}_2$, $\mathscr{D}_1$, $\mathscr{D}_2$, E_1, E_2, respectively.

2. Main Results

In the following theorem, we give a characterization of nonlinear surjective isometries on $\mathscr{C}_p(H) \cap \operatorname{Alg}\mathscr{N}$, where $\mathscr{N}$ is an atomic nest.

Theorem 2. *Let H be a complex Hilbert space, $\mathscr{N}$ an atomic nest on H. Assume that $1 \leq p < +\infty$, $p \neq 2$, $\Phi : \mathscr{C}_p(H) \cap \operatorname{Alg}\mathscr{N} \to \mathscr{C}_p(H) \cap \operatorname{Alg}\mathscr{N}$ is a surjective map. Then Φ satisfies $\|\Phi(A) - \Phi(B)\|_p = \|A - B\|_p$ for all $A, B \in \mathscr{C}_p(H) \cap \operatorname{Alg}\mathscr{N}$ if and only if one of the following holds true.*

(1) *There exist an operator $S \in \mathscr{C}_p(H) \cap \operatorname{Alg}\mathscr{N}$, a dimension preserving order isomorphism $\theta : \mathscr{N} \to \mathscr{N}$, and unitary operators $U, V : H \to H$ satisfying $U(N) = \theta(N)$ and $V(N^\perp) = \theta(N)_-^\perp$ for every $N \in \mathscr{N}$, such that*

$$\Phi(A) = UAV + S, \quad \forall A \in \mathscr{C}_p(H) \cap \operatorname{Alg}\mathscr{N}. \quad (3)$$

(2) *There exist an operator $S \in \mathscr{C}_p(H) \cap \operatorname{Alg}\mathscr{N}$, a dimension preserving order isomorphism $\theta : \mathscr{N} \to \mathscr{N}$, and conjugate unitary operators $U, V : H \to H$ satisfying $U(N) = \theta(N)$ and $V(N^\perp) = \theta(N)_-^\perp$ for every $N \in \mathscr{N}$, such that*

$$\Phi(A) = UAV + S, \quad \forall A \in \mathscr{C}_p(H) \cap \operatorname{Alg}\mathscr{N}. \quad (4)$$

(3) *There exist an operator $S \in \mathscr{C}_p(H) \cap \operatorname{Alg}\mathscr{N}$, a dimension preserving order isomorphism $\theta : \mathscr{N}^\perp \to \mathscr{N}^\perp$, and unitary operators $U, V : H \to H$ satisfying $U(N^\perp) = \theta(N^\perp)$ and $V(N) = \theta(N_-^\perp)_-^\perp$ for every $N \in \mathscr{N}$, such that*

$$\Phi(A) = UA^*V + S, \quad \forall A \in \mathscr{C}_p(H) \cap \operatorname{Alg}\mathscr{N}. \quad (5)$$

(4) *There exist an operator $S \in \mathscr{C}_p(H) \cap \operatorname{Alg}\mathscr{N}$, a dimension preserving order isomorphism $\theta : \mathscr{N}^\perp \to \mathscr{N}^\perp$, and conjugate unitary operators $U, V : H \to H$ satisfying $U(N^\perp) = \theta(N^\perp)$ and $V(N) = \theta(N_-^\perp)_-^\perp$ for every $N \in \mathscr{N}$, such that*

$$\Phi(A) = UA^*V + S, \quad \forall A \in \mathscr{C}_p(H) \cap \operatorname{Alg}\mathscr{N}. \quad (6)$$

The problem on orthogonality of nonlinear surjective isometries for Hilbert-Schmidt norms is discussed in the following theorem.

Theorem 3. *Let H be a complex Hilbert space, $\mathcal{N}$ an atomic nest on H. Assume that $\Phi : \mathcal{C}_2(H) \cap \operatorname{Alg}\mathcal{N} \to \mathcal{C}_2(H) \cap \operatorname{Alg}\mathcal{N}$ is a surjective map. Then Φ satisfies $\|\Phi(A) - \Phi(B)\|_2 = \|A - B\|_2$ for all $A, B \in \mathcal{C}_2(H) \cap \operatorname{Alg}\mathcal{N}$ and then the map $A \mapsto \Phi(A) + \Phi(0)$ is a real linear and an orthogonal transformation on $\mathcal{C}_2(H) \cap \operatorname{Alg}\mathcal{N}$ with respect to the real inner product $\langle A, B\rangle = \operatorname{Retr}(AB^*)$.*

3. Proof of Main Results

To prove our main results, we need the following lemmas.

Lemma 4 (see [17]). *For arbitrary $A, B \in \mathcal{C}_p(H)$ and $1 \le p < +\infty$, $p \ne 2$, $A^*B = AB^* = 0$ if and only if*

$$\|A - B\|_p^p + \|A + B\|_p^p = 2\left(\|A\|_p^p + \|B\|_p^p\right). \tag{7}$$

In the following lemmas, we give a characterization of rank-oneness of operators by the relation of orthogonality between operators. Let $\{A\}^\perp = \{B \in \mathcal{C}_p(H) \cap \operatorname{Alg}\mathcal{N} \setminus \{0\} : A^*B = AB^* = 0\}$ for arbitrary $A \in \mathcal{C}_p(H) \cap \operatorname{Alg}\mathcal{N}$. The set $\{A\}^\perp$ is maximal, if for arbitrary operator $N \in \mathcal{C}_p(H) \cap \operatorname{Alg}\mathcal{N}$, $\{A\}^\perp \subseteq \{N\}^\perp \Rightarrow \{A\}^\perp = \{N\}^\perp$.

Lemma 5 (see Lemma 3 in [1]). *For $A = x \otimes f \in \mathcal{C}_p(H) \cap \operatorname{Alg}\mathcal{N}$, then*

(1) $\cap\{\ker T : T \in \{A\}^\perp\} = [f]$ *unless* $[x] = 0_+$, *in which case* $\cap\{\ker T : T \in \{A\}^\perp\} = [x, f]$;

(2) $\cap\{\ker T^* : T \in \{A\}^\perp\} = [x]$ *unless* $[f] = H_-^\perp$, *in which case* $\cap\{\ker T^* : T \in \{A\}^\perp\} = [x, f]$.

Lemma 6. *For any nonzero operator $A \in \mathcal{C}_p(H) \cap \operatorname{Alg}\mathcal{N}$ with the atomic nest $\mathcal{N}$, if the set $\{A\}^\perp$ is maximal and nonempty, then $\operatorname{rank} A = 1$. Conversely, if $\operatorname{rank} A = 1$, and either $0_+ \ne \operatorname{ran} A$ or $H_-^\perp \ne \operatorname{ran}(A^*)$, then the set $\{A\}^\perp$ is maximal and nonempty.*

Proof. If $\{A\}^\perp$ is maximal and nonempty, we show that $\operatorname{rank} A = 1$. If not, $\operatorname{rank} A \ge 2$, then there are two nonzero vectors x_1 and x_2 such that $Ax_1 \perp Ax_2$. Since the nest is atomic, let $P = Ax_1 \otimes A^*x_1 = Ax_1 \otimes x_1 A \in \mathcal{C}_p(H) \cap \operatorname{Alg}\mathcal{N}$ and one can find a vector $y_2 \perp A^*x_1$ such that $Q = Ax_2 \otimes y_2 \in \mathcal{C}_p(H) \cap \operatorname{Alg}\mathcal{N}$ (if necessary, interchanging x_1 for x_2). Now for any $T \in \{A\}^\perp$, it follows from the definition of $\{A\}^\perp$ that $A^*T = AT^* = 0$. So $P^*T = A^*x_1 \otimes Ax_1 T = A^*x_1 \otimes x_1 A^*T = 0$ and $PT^* = Ax_1 \otimes A^*x_1 T^* = Ax_1 \otimes TA^*x_1 = 0$; it follows that $T \in \{P\}^\perp$. So we have $\{P\}^\perp \supseteq \{A\}^\perp$. One can check $P^*Q = PQ^* = 0$ but $A^*Q \ne 0$. That is, $Q \in \{P\}^\perp$ but is not in $\{A\}^\perp$. It is a contradiction to the maximum of $\{A\}^\perp$. So $\operatorname{rank} A = 1$.

If $\operatorname{rank} A = 1$, let $A = x \otimes f$, and either $0_+ \ne \operatorname{ran} A$ or $H_-^\perp \ne \operatorname{ran}(A^*)$, and by Lemma 5, one of the following three cases happens.

Case 1. $\cap\{\ker T : T \in \{A\}^\perp\} = [f]$ and $\cap\{\ker T^* : T \in \{A\}^\perp\} = [x]$.

Case 2. $\cap\{\ker T : T \in \{A\}^\perp\} = [x, f]$ and $\cap\{\ker T^* : T \in \{A\}^\perp\} = [x]$.

Case 3. $\cap\{\ker T : T \in \{A\}^\perp\} = [f]$ and $\cap\{\ker T^* : T \in \{A\}^\perp\} = [x, f]$.

If for $N \in \mathcal{C}_p(H) \cap \operatorname{Alg}\mathcal{N}$, $\{A\}^\perp \subseteq \{N\}^\perp$, then either

$$\cap\{\ker T : T \in \{N\}^\perp\} \subseteq \cap\{\ker T : T \in \{A\}^\perp\} \subseteq [f] \tag{8}$$

or

$$\cap\{\ker T^* : T \in \{N\}^\perp\} \subseteq \cap\{\ker T : T \in \{A\}^\perp\} \subseteq [x]. \tag{9}$$

It follows that either $\operatorname{ran}(A^*) \subseteq \cap\{\ker T : T \in \{N\}^\perp\} \subseteq [f]$ or $\operatorname{ran} A \subseteq \cap\{\ker T^* : T \in \{N\}^\perp\} \subseteq [x]$. It implies that $\operatorname{rank} N = 1$ and N, A are linearly dependent. By computation, then $\{A\}^\perp = \{N\}^\perp$. So $\{A\}^\perp$ is maximal and nonempty. □

In the following lemma that is taken from [16], let X' be the dual of a Banach space X. Let $\mathcal{N}$ be a nest on X over real or complex field $\mathbb{F}$. If $\dim 0_+ = 1$, $\overline{\mathcal{E}_2(\mathcal{N})} \oplus [e_0] = X$, and if $\dim H_-^\perp = 1$, $\overline{\mathcal{E}_1(\mathcal{N})} \oplus [f_0] = X'$.

Lemma 7 (see [16]). *Let $\mathcal{N}$ and $\mathcal{M}$ be two nests on Banach spaces X and Y over real or complex field $\mathbb{F}$, respectively. Let $\Phi : \operatorname{Alg}_{\mathcal{F}}\mathcal{N} \to \operatorname{Alg}_{\mathcal{F}}\mathcal{M}$ be a continuous surjective additive map. Then Φ preserves rank-1 operators in both directions if and only if one of the following is true.*

(1) *There are linear or conjugate linear bounded bijective operators $A : X \to Y$, $C : X' \to Y'$, a dimension preserving order isomorphism $\theta : \mathcal{N} \to \mathcal{M}$, and vectors $y_0 \in Y$, $g_0 \in Y'$ such that $A(N) = \theta(N)$, $C(N_-^\perp) = \theta(N)_-^\perp$ for every $N \in \mathcal{N}$, and for each rank-1 operator $x \otimes f \in \operatorname{Alg}_{\mathcal{F}}\mathcal{N}$,*

$$\Phi(x \otimes f) = \begin{cases} Ax \otimes Cf \\ \quad \text{if } x \in \overline{\mathcal{E}_1(\mathcal{N})},\ f \in \overline{\mathcal{E}_2(\mathcal{N})}, \\ Ax \otimes Cf + \operatorname{Im} f(x)\, Ae_0 \otimes g_0 \\ \quad \text{if } x \in \overline{\mathcal{E}_1(\mathcal{N})},\ f \notin \overline{\mathcal{E}_2(\mathcal{N})}, \\ Ax \otimes Cf + \operatorname{Im} f(x)\, y_0 \otimes Cf_0 \\ \quad \text{if } x \notin \overline{\mathcal{E}_1(\mathcal{N})},\ f \in \overline{\mathcal{E}_2(\mathcal{N})}. \end{cases} \tag{10}$$

(2) *There are linear or conjugate linear bounded bijective operators $A : X' \to Y$, $C : X \to Y'$, a dimension preserving order isomorphism $\theta : \mathcal{N}^\perp \to \mathcal{M}$, and vectors $y_0 \in Y$, $g_0 \in Y'$ such that $A(N_-^\perp) = \theta(N_-^\perp)$,*

$C(N) = \theta(N_-^{\perp})_-^{\perp}$ for every $N \in \mathcal{N}$, and for each rank-1 operator $x \otimes f \in \mathrm{Alg}_{\mathcal{F}}\mathcal{N}$,

$$\Phi(x \otimes f) = \begin{cases} Af \otimes Cx \\ \quad \text{if } x \in \mathcal{E}_1(\mathcal{N}), f \in \mathcal{E}_2(\mathcal{N}), \\ Af \otimes Cx + \mathrm{Im}\, f(x)\, y_0 \otimes Ce_0 \\ \quad \text{if } x \in \mathcal{E}_1(\mathcal{N}), f \notin \mathcal{E}_2(\mathcal{N}), \\ Af \otimes Cx + \mathrm{Im}\, f(x)\, Af_0 \otimes g_0 \\ \quad \text{if } x \notin \mathcal{E}_1(\mathcal{N}), f \in \mathcal{E}_2(\mathcal{N}). \end{cases} \quad (11)$$

Moreover, in this case, X and Y are reflexive.

Lemma 8. *For any $A, B \in \mathcal{C}_2(H)$, the following are equivalent:*

(I) $\langle A, B\rangle = \mathrm{Retr}(AB^*) = 0$;

(II) $\|A + \lambda B\|_2 \geq \|A\|_2$ *for any real number* λ.

Proof. (I) $\Rightarrow$ (II) If $\langle A, B\rangle = \mathrm{Retr}(AB^*) = 0$, for any real number λ,

$$\|A\|_2^2 \leq \|A\|_2^2 + \|\lambda B\|_2^2 = \mathrm{tr}(AA^*) + \lambda^2 \mathrm{tr}(BB^*) + 2\lambda \mathrm{Retr}(AB^*) = \|A + \lambda B\|_2^2. \quad (12)$$

(II) $\Rightarrow$ (I) Without loss of generality, assume that $B \neq 0$, and by (II), we have, for any real number λ,

$$\|A\|_2^2 \leq \|A + \lambda B\|_2^2 = \mathrm{tr}(|A + \lambda B|) = \|A\|_2^2 + \|\lambda B\|_2^2 + \lambda\, \mathrm{tr}(AB^*) + \lambda\, \mathrm{tr}(BA^*); \quad (13)$$

that is, $\lambda^2 \|B\|_2^2 \geq -\lambda\, \mathrm{tr}(AB^*) - \lambda\, \mathrm{tr}(BA^*)$. So $\lambda \|B\|_2^2 \geq -\mathrm{Retr}(AB^*)$. It follows from arbitrariness of λ that $\mathrm{Retr}(AB^*) = 0$. We complete the proof. ☐

Proof of Theorem 2. Checking the "if" part is straightforward, so we will only deal with the "only if" part.

Let $\Psi(A) = \Phi(A) - \Phi(0)$ for any $A \in \mathcal{C}_p(H) \cap \mathrm{Alg}\,\mathcal{N}$; then $\Psi(0) = 0$, and $\|\Psi(A) - \Psi(B)\|_p = \|A - B\|_p$ for any $A, B \in \mathcal{C}_p(H) \cap \mathrm{Alg}\,\mathcal{N}$. By the Mazur-Ulam theorem (see [10]), we have that Ψ is an additive map. Furthermore, we have that $\|\Psi(A)\|_p = \|A\|_p$ and $\|\Psi(A) \pm \Psi(B)\|_p = \|A \pm B\|_p$ for any $A, B \in \mathcal{C}_p(H) \cap \mathrm{Alg}\,\mathcal{N}$. By Lemma 4, Ψ satisfies that $A^*B = AB^* = 0 \Leftrightarrow \Psi(A)^*\Psi(B) = \Psi(A)\Psi(B)^* = 0$ for all $A, B \in \mathcal{C}_p(H) \cap \mathrm{Alg}\,\mathcal{N}$.

Next we show that Ψ preserves rank-one operators in both directions. For any rank-one operator $A = x \otimes f$, by the above discussion, $\Psi(\{A\}^{\perp}) = \{\Psi(A)\}^{\perp}$. By Lemma 6, if either $0_+ \neq \mathrm{ran}\, A$ or $H_-^{\perp} \neq \mathrm{ran}(A^*)$, then $\Psi(A)$ has rank one. Ψ^{-1} has the same property as Ψ, and Ψ^{-1} preserves rank-one operators. So Ψ preserves rank-one operators in both directions. If both $0_+ = \mathrm{ran}\, A$ and $H_-^{\perp} = \mathrm{ran}(A^*)$, take rank-one operator $A_n = x \otimes ((1/n)x + f)$; then $A_n \to A$ $(n \to \infty)$. So we have $\Psi(A_n) \to \Psi(A)$. Since $\Psi(A_n)$ has rank one, then $\Psi(A)$ has rank one. As Ψ^{-1} has the same property as Ψ, so Ψ preserves rank-one operators in both directions.

Ψ preserves rank-one operators in both directions; then Ψ has the form in Lemma 7. In the case of complex Hilbert space, we have that one of the following is true.

(1) There are linear or conjugate linear bounded bijective operators $A, C : H \to H$, vectors $y_0, g_0 \in H$, and a dimension preserving order isomorphism $\theta : \mathcal{N} \to \mathcal{N}$ such that $A(N) = \theta(N)$, $C(N_-^{\perp}) = \theta(N)_-^{\perp}$ for every $N \in \mathcal{N}$, such that

$$\Phi(x \otimes y) = \begin{cases} Ax \otimes Cy \\ \quad \text{if } x \in \overline{\mathcal{E}_1}, y \in \overline{\mathcal{E}_2}, \\ Ax \otimes Cy + \mathrm{Im}\langle x, y\rangle\, Ae_0 \otimes g_0 \\ \quad \text{if } x \in \overline{\mathcal{E}_1}, y \notin \overline{\mathcal{E}_2}, \\ Ax \otimes Cy + \mathrm{Im}\langle x, y\rangle\, y_0 \otimes Cf_0 \\ \quad \text{if } x \notin \overline{\mathcal{E}_1}, y \in \overline{\mathcal{E}_2}. \end{cases} \quad (14)$$

(2) There are linear or conjugate linear bounded bijective operators $A, C : H \to H$, vectors $y_0, g_0 \in H$, and a dimension preserving order isomorphism $\theta : \mathcal{N}^{\perp} \to \mathcal{N}$ such that $A(N_-^{\perp}) = \theta(N_-^{\perp})$, $C(N) = \theta(N_-^{\perp})_-^{\perp}$ for every $N \in \mathcal{N}$, such that

$$\Phi(x \otimes y) = \begin{cases} Ay \otimes Cx \\ \quad \text{if } x \in \overline{\mathcal{E}_1}, y \in \overline{\mathcal{E}_2}, \\ Ay \otimes Cx + \mathrm{Im}\langle x, y\rangle\, y_0 \otimes Ce_0 \\ \quad \text{if } x \in \overline{\mathcal{E}_1}, y \notin \overline{\mathcal{E}_2}, \\ Ay \otimes Cx + \mathrm{Im}\langle x, y\rangle\, Af_0 \otimes g_0 \\ \quad \text{if } x \notin \overline{\mathcal{E}_1}, y \in \overline{\mathcal{E}_2}. \end{cases} \quad (15)$$

One can note that $\|T\| = \|T\|_p$ for all rank-one operator T, so is the same to the proof of Lemma 4.11 in [3], A, C can be chosen as unitary or conjugate unitary operators; denote A, C by U, W.

If case (1) occurs, next we claim $Ue_0 \otimes g_0 = 0$, and $y_0 \otimes Wf_0 = 0$. For case (2), similarly, we can show that $y_0 \otimes We_0 = 0$ and $Uf_0 \otimes g_0 = 0$. Assume that (1) occurs and Ψ has the second form, in fact $\dim 0_+ = 1$, $\overline{\mathcal{E}_2} = [e_0]^{\perp}$. Just like the discussion in Lemma 4.11 in [3], we have $g_0 = e_0$, $We_0 = e_0$. Assume on the contrary that $Ue_0 \otimes g_0 \neq 0$. Let $A = ie_0 \otimes e_0$, $\Psi(A) = \Psi(ie_0 \otimes e_0) = iUe_0 \otimes We_0 + Ue_0 \otimes e_0$; then $\|A\|_p = \|ie_0 \otimes e_0\|_p = \|ie_0\|\|e_0\| = 1$. By a computation, $\|\Psi(A)\|_p = \|iUe_0 \otimes We_0 + Ue_0 \otimes e_0\|_p = \|ie_0 \otimes e_0 + e_0 \otimes W^*e_0\|_p = \|ie_0 \otimes e_0 + e_0 \otimes e_0\|_p = \|(ie_0 + e_0) \otimes e_0\|_p \neq 1$. So $\|\Psi(A)\|_p \neq \|A\|_p$, a contradiction. So $Ue_0 \otimes g_0 = 0$. If Ψ has the third form, in this case $\dim H_-^{\perp} = 1$, $\overline{\mathcal{E}_1} = [f_0]^{\perp}$. Just like the discussion in Lemma 4.11 in [3] again, $y_0 = f_0 = Ue_0$. Assume on the contrary that $y_0 \otimes Wf_0 \neq 0$; let $A = if_0 \otimes f_0$, similar to the above discussion, and we get a contradiction again.

Every finite rank operator can be written as a sum of rank-one operators in the nest algebra, and the set of finite rank operators is dense in $\mathcal{C}_p(H) \cap \mathrm{Alg}\,\mathcal{N}$, so by addition of Ψ, we have that Theorem 2 holds true by replacing W^* by V. ☐

Proof of Theorem 3. Let $\Psi(A) = \Phi(A) - \Phi(0)$ for any $A \in \mathcal{C}_2(H)$; then $\Psi(0) = 0$, and $\|\Psi(A) - \Psi(B)\|_2 = \|A - B\|_2$ for

$A, B \in \mathscr{C}_2(H) \cap \operatorname{Alg} \mathcal{N}$. By the Mazur-Ulam theorem (see [10]), Ψ is real linear.

By real linearity of Ψ, we have that $\|A + \lambda B\|_2^2 = \|\Psi(A) + \lambda\Psi(B)\|_2^2$ for any real number λ. By Lemma 8, Ψ is real linear and the map Ψ preserves orthogonality with respect to $\langle A, B\rangle = \operatorname{tr}(AB^*)$. We complete the proof. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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