

Research Article

The Average Lower Connectivity of Graphs

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For a vertex v of a graph G , the *lower connectivity*, denoted by $s_v(G)$, is the smallest number of vertices that contains v and those vertices whose deletion from G produces a disconnected or a trivial graph. The average lower connectivity denoted by $\kappa_{av}(G)$ is the value $(\sum_{v \in V(G)} s_v(G))/|V(G)|$. It is shown that this parameter can be used to measure the vulnerability of networks. This paper contains results on bounds for the average lower connectivity and obtains the average lower connectivity of some graphs.

1. Introduction

In a communication network, the vulnerability parameters measure the resistance of the network to disruption of operation after the failure of certain stations or communication links. The best known and most useful measures of how well a graph is connected is the connectivity, defined to be the minimum number of vertices in a set whose deletion results in a disconnected or trivial graph. As the connectivity is the worst-case measure, it does not always reflect what happens throughout the graph. Recent interest in the vulnerability and reliability of networks (communication, computer, and transportation) has given rise to a host of other measures, some of which are more global in nature; see, for example, [1, 2].

Let G be a finite simple graph with vertex set $V(G)$ and edge set $E(G)$. In the graph G , n denotes the number of vertices. The minimum degree of a graph G is denoted by $\delta(G)$. A subset $S \subset V(G)$ of vertices is a dominating set if every vertex in $V(G) - S$ is adjacent to at least one vertex of S . The domination number $\gamma(G)$ is the minimum cardinality of a dominating set. A subset S of $V(G)$ is called an independent set of G if no two vertices of S are adjacent to G . An independent set S is maximum if G has no independent set S' with $|S'| > |S|$. The independence number of G , $\alpha(G)$, is the number of vertices in a maximum independent set of G .

Henning [3] introduced the concept of average independence and average domination. For a vertex v of a graph G , the *lower independence number*, denoted by $i_v(G)$, is the minimum cardinality of a maximal independent set of G that

contains v , and the *lower domination number*, denoted by $\gamma_v(G)$, is the minimum cardinality of a dominating set of G that contains v . The *average lower independence number* of G , denoted by $i_{av}(G)$, is the value $(\sum_{v \in V(G)} i_v(G))/|V(G)|$ and the *average lower domination number* of G , denoted by $\gamma_{av}(G)$, is the value $(\sum_{v \in V(G)} \gamma_v(G))/|V(G)|$. Since $\gamma_v(G) \leq i_v(G)$ holds for every vertex v , we have $\gamma_{av}(G) \leq i_{av}(G)$ for any graph G . Also, it is clear that $i(G) = \min\{i_v(G) \mid v \in V(G)\}$ and $\gamma(G) = \min\{\gamma_v(G) \mid v \in V(G)\}$ so $\gamma(G) \leq \gamma_{av}(G)$ and $i(G) \leq i_{av}(G)$.

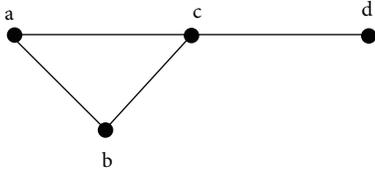
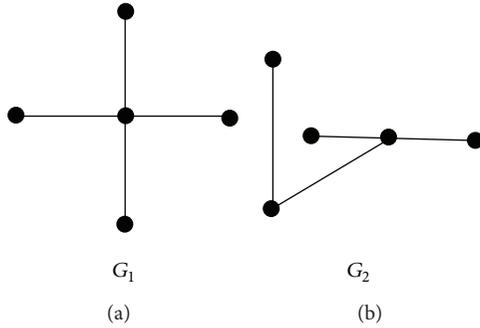
The (u, v) -connectivity of G , denoted by $\kappa_G(u, v)$, is defined to be the maximum value of k for which u and v are k -connected. It is a well-known fact that the connectivity $\kappa(G)$ equals $\min\{\kappa_G(u, v) \mid u, v \in V(G)\}$.

In 2002, Beineke et al. [4] introduced a parameter to give a more refined measure of the global “amount” of connectivity. If the order of G is n , then the *average connectivity* of G , denoted by $\bar{\kappa}(G)$, is defined to be $\bar{\kappa}(G) = (\sum_{u,v} \kappa_G(u, v)) / \binom{n}{2}$. The expression $\sum_{u,v} \kappa_G(u, v)$ is sometimes referred to as the total connectivity of G . Clearly, for any graph G , $\bar{\kappa}(G) \geq \kappa(G)$.

There are a lot of researches on the connectivity of a graph [5]. Many works provide sufficient conditions for a graph to be maximally connected [6–8]. The average connectivity has been extensively studied [4, 9].

2. The Average Lower Connectivity of a Graph

We introduce a new vulnerability parameter, the average lower connectivity. For a vertex v of a graph G , the *lower*

FIGURE 1: 3-cycle G : with one additional vertex and edge.FIGURE 2: The graphs G_1 and G_2 .

connectivity, denoted by $s_v(G)$, is the smallest number of vertices that contains v and those vertices whose deletion from G produces a disconnected or a trivial graph. We observe that

- (i) $1 \leq s_v(G) \leq n - 1$;
- (ii) $s_v(G) = 1$ if and only if v is a cut vertex;
- (iii) $s_v(G) = n - 1$ if and only if $G - v$ is complete.

The *average lower connectivity* denoted by $\kappa_{av}(G)$ is the value $(\sum_{v \in V(G)} s_v(G))/n$, where n will denote the number of vertices in graph G and $\sum_{v \in V(G)} s_v(G)$ will denote the sum over all vertices of G . For any graph G , $\kappa(G) = \min\{s_v(G) \mid v \in V(G)\}$ so $\kappa_{av}(G) \geq \kappa(G)$. We also observe that

- (i) $\kappa_{av}(G) \leq n - 1$;
- (ii) $\kappa_{av}(G) = n - 1$ if and only if G is complete.

Example 1. Let the graph G be 3-cycle with one additional vertex and edge, as shown in Figure 1. It is easy to see that $s_a(G) = 2$, $s_b(G) = 2$, $s_c(G) = 1$, and $s_d(G) = 3$ and we have $\kappa_{av}(G) = (2 + 2 + 1 + 3)/4 = 2$.

Let G_1 and G_2 be graphs. Now one can ask the following question: is the average lower connectivity a suitable parameter? In other words, does the average lower connectivity distinguish between G_1 and G_2 ?

For example, consider the graphs in Figure 2.

It can be easily seen that the connectivity and average connectivity of these graphs are equal:

$$\kappa(G_1) = \kappa(G_2) = \bar{\kappa}(G_1) = \bar{\kappa}(G_2) = 1. \quad (1)$$

On the other hand, the average lower connectivity of G_1 and G_2 is different:

$$\begin{aligned} \kappa_{av}(G_1) &= 1,8, \\ \kappa_{av}(G_2) &= 1,6. \end{aligned} \quad (2)$$

Thus, the average lower connectivity is a better parameter than the connectivity and average connectivity to distinguish these two graphs. The average parameters have been found to be more useful in some circumstances than the corresponding measures based on worst-case situations.

Theorem 2. Let G be a connected graph. Then,

$$\kappa_{av}(G) < \kappa(G) + 2. \quad (3)$$

Proof. For every vertex of G , $s_v(G) \leq \kappa(G) + 2$. For at least one vertex v , $s_v(G) = \kappa(G)$. Hence, the inequality is strict. Then,

$$\kappa_{av}(G) < \kappa(G) + 2. \quad (4)$$

The proof is completed. \square

Theorem 3. Let G be a connected graph. Then,

$$\kappa_{av}(G) \leq \kappa_{av}(G + e). \quad (5)$$

Proof. It is easy to see that $s_v(G) \leq s_v(G + e)$. Therefore,

$$\frac{\sum_{v \in V(G)} s_v(G)}{n} \leq \frac{\sum_{v \in V(G)} s_v(G + e)}{n}. \quad (6)$$

So, we have

$$\kappa_{av}(G) \leq \kappa_{av}(G + e). \quad (7)$$

The proof is completed. \square

Theorem 4. Let G be a k -connected and k -regular graph. Then,

$$\kappa_{av}(G) = k. \quad (8)$$

Proof. The cardinality of $s_v(G)$ -sets is always the same for every vertex of any graph G and equals k . Then, we have

$$\kappa_{av}(G) = \frac{\sum_{v \in V(G)} s_v(G)}{n} = \frac{n \cdot k}{n} = k. \quad (9)$$

This means that the proof is completed. \square

It is obvious that we can give the following equality for the average lower connectivity of the cycle C_n .

- (i) The average lower connectivity of the cycle C_n is 2.

Theorem 5. Let G be a connected graph. Then,

$$\kappa_{av}(G) \leq \delta(G) + 2. \quad (10)$$

Proof. For every vertex of G , $s_v(G) \leq \delta(G) + 2$. Thus,

$$\kappa_{av}(G) = \frac{\sum_{v \in V(G)} s_v(G)}{n} \leq \frac{n \cdot (\delta(G) + 2)}{n}, \quad (11)$$

$$\kappa_{av}(G) \leq \delta(G) + 2.$$

The proof is completed. \square

3. Average Lower Connectivity of Several Specific Graphs

In this section, we determine the average lower connectivity of several special graphs.

Theorem 6. *Let T be a tree with order n . If T has k vertices with degree 1, then*

$$\kappa_{av}(T) = \frac{n+k}{n}. \quad (12)$$

Proof. Assume that T has k vertices with degree 1 and $n - k$ vertices with degree at least 2. Let vertices set of T be $V(T) = V(G_1) \cup V(G_2)$ where in $V(G_1)$ the set contains k vertices with degree 1; in $V(G_2)$ the set contains $n - k$ vertices with degree at least 2. If $v \in V(G_1)$, then $s_v(G_1) = 2$. We have to repeat this process for k vertices with degree 1. If $v \in V(G_2)$, then $s_v(G_2) = 1$. We have to repeat this process for $n - k$ vertices with degree at least 2. Thus, we have

$$\kappa_{av}(T) = \frac{\sum_{v \in V(T)} s_v(T)}{|V(T)|} = \frac{k \cdot 2 + (n - k) \cdot 1}{n} = \frac{n+k}{n}. \quad (13)$$

The proof is completed. \square

Corollary 7. *The average lower connectivity of*

- (a) *the path P_n is $(n + 2)/n$;*
- (b) *the star $K_{1,n-1}$ is $(2n - 1)/n$;*
- (c) *the comet $C_{t,r}$ is $(2r + t + 1)/(t + r)$.*

Theorem 8. *Let $K_{r,s}$ be a complete bipartite graph. Then*

$$\kappa_{av}(K_{r,s}) = \begin{cases} \frac{s^2 + sr + r}{r + s}, & \text{if } s < r; \\ r, & \text{if } r = s. \end{cases} \quad (14)$$

Proof. Let the partite sets of $K_{r,s}$ be R and S with $|R| = r$ and $|S| = s$. We distinguish two cases.

Case 1. If $r = s$, then by Theorem 4 we have $\kappa_{av}(K_{r,s}) = r$.

Case 2 ($r < s$). For $x \in R$, a minimum disconnecting set of G that contains x must be $S \cup \{x\}$, so $s_x(G) = s + 1$. On the other hand, for $y \in S$, a minimum disconnecting set of G that contains y must be R , so $s_y(G) = r$. Elementary computation yields the result. \square

Definition 9. The wheel graph with $n - 1$ spokes, W_n , is the graph that consists of an $(n - 1)$ -cycle and one additional vertex, say u , that is adjacent to all the vertices of the cycle. In Figure 3, we display W_7 .

Theorem 10. *Let W_n be a wheel graph. Then,*

$$\kappa_{av}(W_n) = 3. \quad (15)$$

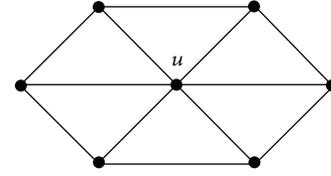


FIGURE 3: The wheel graph W_7 .

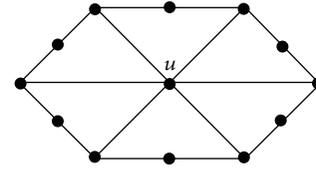


FIGURE 4: The gear graph G_6 .

Proof. The wheel graph W_n has n vertices. The cardinality of $s_v(G)$ -sets is always the same for every vertex of any W_n and equals 3. Then, we have

$$\kappa_{av}(W_n) = \frac{\sum_{v \in V(W_n)} s_v(W_n)}{n} = \frac{n \cdot 3}{n} = 3. \quad (16)$$

This means that the proof is completed. \square

Definition 11. The gear graph is a wheel graph with a vertex added between each pair adjacent to graph vertices of the outer cycle. The gear graph G_r has $2r + 1$ vertices and $3r$ edges. In Figure 4 we display G_6 .

Theorem 12. *Let G_r be a gear graph. Then,*

$$\kappa_{av}(G_r) = \frac{5r + 3}{2r + 1}. \quad (17)$$

Proof. Let vertices set of G_r be $V(G_r) = (H_1) \cup V(H_2) \cup V(H_3)$ where in $V(H_1)$ the set contains 1 vertex with degree n , $V(H_2)$ the set contains n vertices with degree 2, and $V(H_3)$ the set contains n vertices with degree 3. If $v \notin V(H_3)$, then $s_v(G) = 3$. If $v \in V(H_3)$, then $s_v(G_3) = 2$. Therefore,

$$\begin{aligned} \kappa_{av}(G_r) &= \frac{\sum_{v \in V(G_r)} s_v(G_r)}{2r + 1} \\ &= \frac{(r + 1) \cdot 3 + 2 \cdot r}{2r + 1} = \frac{5r + 3}{2r + 1}. \end{aligned} \quad (18)$$

The proof is completed. \square

Now we give the definition of Cartesian product.

Definition 13. The Cartesian product $G_1 \times G_2$ of graphs G_1 and G_2 has $V(G_1) \times V(G_2)$ as its vertex set and (u_1, u_2) is adjacent to (v_1, v_2) if either $u_1 = v_1$ and u_2 is adjacent to v_2 or $u_2 = v_2$ and u_1 is adjacent to v_1 .

Connectivity of graph products has already been studied by different authors. In [10] it is proved that $\kappa(G_1 \times G_2) \geq \kappa(G_1) + \kappa(G_2)$.

Theorem 14. Let G_1 and G_2 be two connected graphs; then

$$\kappa_{av}(G_1 \times G_2) \geq \kappa(G_1 \times G_2). \quad (19)$$

Proof. We know $\kappa_{av}(G) \geq \kappa(G)$. Therefore, we get

$$\kappa_{av}(G_1 \times G_2) \geq \kappa(G_1 \times G_2). \quad (20)$$

The proof is completed. \square

Proposition 15. For positive integer $m \geq 3$,

$$(i) \kappa_{av}(K_2 \times P_m) = 2;$$

$$(ii) \kappa_{av}(K_2 \times C_m) = 3.$$

Proposition 16. Let $r \geq 3$ and $t \geq 3$ be positive integers. Then

$$(i) \kappa_{av}(P_r \times P_t) = \begin{cases} \frac{23}{9}, & \text{if } r = 3, t = 3; \\ 3 - \frac{2}{t}, & \text{if } r = 3, t \geq 4; \\ 3 - \frac{8}{r \cdot t}, & \text{if } r \geq 4, t \geq 4. \end{cases} \quad (21)$$

$$(ii) \kappa_{av}(P_r \times C_t) = \begin{cases} 3, & \text{if } r = 3, 4, t \geq 3; \\ 4 - \frac{4}{r}, & \text{if } r \geq 5, t \geq 4. \end{cases}$$

$$(iii) \kappa_{av}(C_r \times C_t) = 4.$$

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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