Research Article

On Nonlinear Nonlocal Systems of Reaction Diffusion Equations

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The reaction diffusion system with anomalous diffusion and a balance law $u_t + (-\Delta)^{\alpha/2} u = -f(u,v)$, $v_t + (-\Delta)^{\beta/2} v = f(u,v)$, $0 < \alpha, \beta < 2$, is considered. The existence of global solutions is proved in two situations: (i) a polynomial growth condition is imposed on the reaction term f when $0 < \alpha \le \beta \le 2$; (ii) no growth condition is imposed on the reaction term f when $0 < \beta \le \alpha \le 2$.

1. Introduction

In this paper, we consider the system of nonlinear and nonlocal in space reaction diffusion equations

$$u_t + (-\Delta)^{\alpha/2} u = -f(u,v), \quad x \in \mathbb{R}^N, \ t > 0, \qquad (1)$$

$$v_t + (-\Delta)^{\beta/2} v = f(u,v), \quad x \in \mathbb{R}^N, \ t > 0, \qquad (2)$$

supplemented with the initial conditions

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in \mathbb{R}^N, \qquad (3)$$

where the initial data $u_0(x)$, $v_0(x)$ are given positive bounded functions.

Here the nonlocal operator $(-\Delta)^{\delta/2}$, $0 < \delta \le 2$ ($\delta = \alpha$ or β) accounts for anomalous diffusion (see, e.g., [1–3]) and can be defined via the Fourier transform pair $\mathscr{F}$ and $\mathscr{F}^{-1}$ as
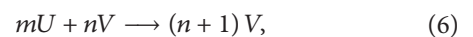
$$(-\Delta)^{\delta/2} u(x) = \mathscr{F}^{-1}\left(|\xi|^\delta \mathscr{F}(u)(\xi)\right)(x), \quad u \in \mathcal{S}\left(\mathbb{R}^N\right), \qquad (4)$$

where $\mathcal{S}(\mathbb{R}^N)$ is the Schwartz class of smooth real rapidly decreasing functions, or equivalently (see [4]) by the formula

$$(-\Delta)^{\delta/2} u(x) = C_N PV \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+\delta}} \ dy, \qquad (5)$$

with $C_N = (\delta 2^{\delta-1}/\pi^{N/2})(\Gamma((N+\delta)/2)/\Gamma(1-(\delta/2)))$ a normalizing constant, and $|\cdot|$ denotes the usual norm of $\mathbb{R}^N$.

A typical type of system under our consideration is given by the irreversible molecular combination

$$mU + nV \longrightarrow (n+1)V, \qquad (6)$$

where U and V are two chemical species. If u and v represent the concentrations of the species U and V, respectively, then according to the law of mass action due to Gulberg and Waage, the reaction diffusion system describing the chemical reaction can be written as

$$
\begin{aligned}
u_t - \Delta u &= -u^m v^n, \quad x \in \Omega, \ t > 0, \\
v_t - \Delta v &= u^m v^n, \quad x \in \Omega, \ t > 0,
\end{aligned}
\qquad (7)
$$

where $\Omega \subseteq \mathbb{R}^N$. This system has been studied by Masuda [5] via a judicious Lyapunov functional, Hollis et al. [6] by using the duality argument, Collet and Xin [7] in the case of the Euclidean space.

Let us now dwell for a while on the available literature concerning anomalous diffusion equations. Fractional differential equations have been used as effective mathematical tools for modeling diffusive processes associated with subdiffusion (fractional in time), superdiffusion (fractional in space), or both. Further examples can be found in physics, mathematical biology, or hydrology. These equations also

appear in finance because of the relationship with certain option pricing mechanisms and heavy tailed stochastic processes [8]. In water resources, fractional models have been used to describe chemical and contaminant transport in heterogeneous aquifers [9]. In spatial complex environment, reaction diffusion equation may not obey Fick's Law [10]. One idea is to replace the flux, say F, by its fractional counterpart [11]:

$$F = -\mathcal{K}\nabla^{\gamma}u, \quad 0 < \gamma \leq 1, \tag{8}$$

where $\mathcal{K}$ is the diffusion tensor and $\nabla^{\gamma} = (\partial^{\gamma}/\partial x^{\gamma}, \partial^{\gamma}/\partial y^{\gamma}, \partial^{\gamma}/\partial z^{\gamma})^{T}$ is the Riemann-Liouville fractional gradient, where

$$\frac{\partial^{\gamma}}{\partial x^{\gamma}}u(x,y,z) = \frac{1}{\Gamma(1-\gamma)}\frac{\partial}{\partial x}\int_{0}^{x}(x-\tilde{x})^{-\gamma}u(\tilde{x},y,z)\,d\tilde{x}, \tag{9}$$

with similar expressions for $\partial^{\gamma}/\partial y^{\gamma}$ and $\partial^{\gamma}/\partial z^{\gamma}$ [12]. The fractional Fick's Law for (8) implies nonlocality in space and in time. This modification, in the absence of external force, leads to the fractional diffusion equation

$$\frac{\partial}{\partial t}u - \nabla \cdot (\mathcal{K}\nabla^{\gamma}u) = 0. \tag{10}$$

Equivalently, in the isotropic setting [13], the space fractional reaction diffusion can be written as

$$\frac{\partial}{\partial t}u + \mathcal{K}(-\Delta)^{\gamma/2}u = f(u), \quad 0 < \gamma \leq 1, \tag{11}$$

where $(-\Delta)^{\gamma/2}$ is the fractional Laplacian operator; see also the valuable contribution of Douglas [14] for the use of the fractional Laplacian in polymer sciences.

In our consideration, we take into account the diffusion of two interacting species, diffusing at different rates.

The reaction term $f : [0,+\infty)^{2} \rightarrow [0,+\infty)$ is locally Lipschitz continuous, namely,

$$\left|f(u,v) - f(\tilde{u},\tilde{v})\right| \leq C(R)(|u-\tilde{u}| + |v-\tilde{v}|), \tag{12}$$

for all $0 \leq u, v, \tilde{u}, \tilde{v} \leq R$.

Further, it is assumed that there exist positive numbers $L(R)$, $M(R)$, and κ such that

$$\left|f(u,v)\right| \leq L(R)|u|^{\kappa} + M(R), \tag{13}$$

for all $u, v \geq 0$ with $|u| \leq R$, and

$$\forall u, v \geq 0, \quad f(0,v) = 0 \leq f(u,0). \tag{14}$$

(Note that $f(u,v) \geq 0$ for all $u, v \geq 0$.)

We first prove that system (1)–(3) admits global solutions for reaction terms of polynomial growth relying on the duality argument that has been used by Hollis et al. [6] for the case when the space variable belongs to a bounded domain and $\alpha = \beta = 2$. Notice that estimates obtained by this method have been recently improved by Cañizo et al. [15] in the same case $\alpha = \beta = 2$. In case of $x \in \mathbb{R}^{N}$, the duality method has

been used successfully by Fitzgibbon et al. [16] still in the case $\alpha = \beta = 2$.

A central role in the proof is played by a recent L^{p} regularity result due to Zhang [17] for the solution of the backward heat equation

$$-\varphi_{t} + (-\Delta)^{\beta/2}\varphi = \vartheta, \quad t \in (0,T), \ x \in \mathbb{R}^{N}, \ 0 < \beta < 2, \tag{15}$$

supplemented with the condition

$$\varphi(x,T) = 0, \quad x \in \mathbb{R}^{N}, \tag{16}$$

which will be stated in Section 2.

Next, we prove our second result; namely, global solutions of problem (1)–(3) exist for any growth of the reaction terms whenever $\beta \leq \alpha$.

Our second result has to be compared in some sense with that of Martin and Pierre [18]. It has been shown in [18] that the following problem admits global solutions for any nonlinearity f under the condition $b \leq a$:

$$\begin{aligned} u_{t} - a\Delta u &= -f(u,v), \quad x \in \mathbb{R}^{N}, \ t > 0, \\ v_{t} - b\Delta v &= f(u,v), \quad x \in \mathbb{R}^{N}, \ t > 0, \end{aligned} \tag{17}$$

supplemented with positive and bounded initial data.

The result of [18] is recalled in the appendix for the reader in order to compare our result with the result of Martin and Pierre.

The result of [18] has been extended by Kanel and Kirane [19] for the triangular system

$$\begin{aligned} u_{t} - a\Delta u &= -f(u,v), \quad x \in \Omega, \ t > 0, \\ v_{t} - c\Delta u - d\Delta v &= f(u,v), \quad x \in \Omega, \ t > 0, \\ \frac{\partial u}{\partial \nu} &= \frac{\partial v}{\partial \nu}, \quad x \in \partial\Omega, \ t > 0, \\ u(x,0) &= u_{0}(x), \quad u(x,0) = u_{0}(x), \quad x \in \Omega, \end{aligned} \tag{18}$$

where Ω is a bounded regular domain with boundary $\partial\Omega$, ν is the outward normal derivative to $\partial\Omega$, and $a, c, d > 0$ $(a > d)$ are the positive diffusion constants.

2. Preliminary Results

Notation. Consider $Q_{T} = \mathbb{R}^{N} \times (0,T)$, $\|u\|_{p} = \left(\int_{\mathbb{R}^{N}} |u(x)|^{p}dx\right)^{1/p}$, and $\|u\|_{p,T} = \left(\int_{Q_{T}} |u(x,t)|^{p}dx\,dt\right)^{1/p}$.

The proof of our first result is based on a recent lemma of Zhang [17] (Lemma 2) and a known interpolation inequality (Lemma 3).

Lemma 1. *Let $S_{\delta}(t) := e^{-t(-\Delta)^{\delta}}$ be the linear semigroup generated by the following linear anomalous diffusion problem:*

$$\begin{aligned} \Psi_{t} + (-\Delta)^{\delta}\Psi &= 0, \quad (x,t) \in Q_{\infty}, \\ \Psi(x,0) &= \psi(x), \quad x \in \mathbb{R}^{N}. \end{aligned} \tag{19}$$

Let $1 \le r \le p \le \infty$ and $\psi \in L^r(\mathbb{R}^N)$. Then the solution of (19) *satisfies the estimate*

$$\left\|(-\Delta)^{\gamma/2}S_\delta(t)\psi(x)\right\|_p \le Ct^{-(\nu/2\delta)-(N/2\delta)((1/r)-(1/p))}\|\psi\|_r \tag{20}$$

for $\delta > 0$ and $\nu \ge 0$.

The proof of this lemma follows from the Young inequality combined with scaling properties of the kernel

$$\mathcal{K}(x,t) = (2\pi)^{N/2} \int_{\mathbb{R}^N} e^{i\langle x,\xi\rangle - t|\xi|^{2\delta}}\, d\xi = t^{-(N/2\delta)}\mathcal{K}\left(\frac{x}{t^{1/2\delta}}\right) \tag{21}$$

with

$$\mathcal{K} = (2\pi)^{N/2} \int_{\mathbb{R}^N} e^{i\langle x,\xi\rangle - |\xi|^{2\delta}}\, d\xi, \tag{22}$$

where $\langle x,\xi\rangle$ is the ordinary inner product at the points $x,\xi \in \mathbb{R}^N$ and $|\xi|^2 = \langle \xi,\xi\rangle$.

The lemma is used for the local existence ($\nu = 0$), as well as for the global existence ($\nu \ge 0$).

Lemma 2. *Let $1 < q < \infty$ and suppose that $\vartheta \in L^q(Q_T)$. Then (15)-(16) has a unique positive solution φ such that $\varphi_t \in L^q(Q_T), (-\Delta)^{\beta/2}\varphi \in L^q(Q_T)$. Moreover, there exists a constant $C(p,T)$, independent of φ such that*

$$\|\varphi\|_{q,T} + \|\varphi\|_q + \left\|(-\Delta)^{\beta/2}\varphi\right\|_{q,T} + \|\varphi_0\|_q \le C\,\|\vartheta\|_{q,T}. \tag{23}$$

Lemma 3. *Let $(\mathcal{X}, \|\cdot\|)$ be a Banach space and A a positive operator on $\mathcal{X}$. Then, for $0 \le \varrho < \sigma < \gamma$, there exists a constant $C_{\varrho,\sigma,\gamma}$ such that for $x \in D(A^\gamma)$ (the domain of A^γ)*

$$\left\|A^\sigma x\right\| \le C_{\varrho,\sigma,\gamma}\left\|A^\gamma x\right\|^{(\sigma-\varrho)/(\gamma-\varrho)}\left\|A^\varrho x\right\|^{(\gamma-\sigma)/(\gamma-\varrho)}. \tag{24}$$

The proof of our second result is based on the following interesting lemma of Lopez-Mimbela and Morales [20].

Let $f_\delta(x,t)$ be the continuous transition density of the symmetric stable process $\{X(t); t \ge 0\}$ in $\mathbb{R}^N$ of index δ, $0 < \delta \le 2$, which is uniquely determined by

$$e^{-t|\xi|^\delta} = \int_{\mathbb{R}^N} e^{i\langle x,\xi\rangle} f_\delta(x,t)\, dx. \tag{25}$$

Lemma 4. *Let $f_{\alpha_i}(x,t)$, $t > 0$ be the transition density of the symmetric α_i-stable process in $\mathbb{R}^N$, $i = 1,2$. If $\alpha_1 \le \alpha_2$, then there exists a constant $K \ge 1$ such that, for every $t > 0$ and $x \in \mathbb{R}^N$,*

$$f_{\alpha_i}(x,t) \le Kf_{\alpha_1}\left(x, t^{\alpha_1/\alpha_i}\right), \quad i = 1,2. \tag{26}$$

If in addition $t \ge 1$, then

$$f_{\alpha_i}(x,t) = Kt^{N((1/\alpha_1)-(1/\alpha_2))}f_{\alpha_1}(x,t) \tag{27}$$
$$\le Kt^{N/\alpha_1}f_{\alpha_1}(x,t), \quad x \in \mathbb{R}^N.$$

As the proof is nice and instructive, we present it for the convenience of the reader.

Proof. By Theorem 2.1 [21], we have

$$\lim_{|x|\to\infty} |x|^{N+\alpha_i} f_{\alpha_i}(x,1)$$

$$= \alpha_i 2^{\alpha-1}\left(\frac{1}{\pi}\right)^{(N/2)+1} \sin\left(\frac{\alpha_i\pi}{2}\right)\Gamma\left(\frac{\alpha_i}{2}\right)\Gamma\left(\frac{N+\alpha_i}{2}\right)$$

$$= C_{\alpha_i} > 0, \quad (\alpha_i < 2). \tag{28}$$

If $\alpha_1 < \alpha_2$, then $f_{\alpha_2}(x,1)/f_{\alpha_1}(x,1) \sim (C_{\alpha_2}/C_{\alpha_1})|x|^{\alpha_1-\alpha_2} \to 0$ as $|x| \to \infty$. Hence, there exists a constant $C > 0$ such that $f_{\alpha_2}(x,1)/f_{\alpha_1}(x,1) \le 1$ for all $|x| > C$. Since $f_{\alpha_2}(x,1)/f_{\alpha_1}(x,1)$ is continuous and $\{y \in \mathbb{R}^N; |y| \le C\}$ is compact, there exists $M > 0$ such that $f_{\alpha_2}(x,1)/f_{\alpha_1}(x,1) \le M$ for all $|x| \le C$. Thus $f_{\alpha_2}(x,1) \le Kf_{\alpha_1}(x,1)$ for all $x \in \mathbb{R}^N$, where $K = \max\{1,M\}$. From scaling properties of stable densities, we get

$$f_{\alpha_2}(x,t) = t^{-N/\alpha_2}f_{\alpha_2}\left(xt^{-1/\alpha_2}, 1\right)$$

$$\le t^{-N/\alpha_2}Kf_{\alpha_1}\left(xt^{-1/\alpha_2}, 1\right)$$

$$= t^{-N/\alpha_2}K\left(t^{\alpha_1/\alpha_2}\right)^{N/\alpha_1}f_{\alpha_1}\left(x, t^{\alpha_1/\alpha_2}\right) \tag{29}$$

$$= Kf_{\alpha_1}\left(x, t^{\alpha_1/\alpha_2}\right),$$

which is (26).

Now assume that $t \ge 1$. Using (29) and the fact that $f_{\alpha_1}(x,1)$ is radially decreasing, we may write

$$f_{\alpha_2}(x,t) \le Kt^{-N/\alpha_2}f_{\alpha_1}\left(xt^{-1/\alpha_2}, 1\right)$$

$$= Kt^{N((1/\alpha_1)-(1/\alpha_2))}\left(t^{-(N/\alpha_1)}f_{\alpha_1}\left(xt^{-(1/\alpha_2)}, 1\right)\right) \tag{30}$$

$$\le Kt^{N((1/\alpha_1)-(1/\alpha_2))}f_{\alpha_1}(x,t).$$

$\square$

3. Main Results

Now, we are ready to announce and prove our main results.

Local existence of a classical nonnegative solution (u,v) of (1)–(3) on a maximal interval of existence $[0, T_{\max})$ is obtained as usual (see, e.g., [22]).

Theorem 5. *Assume $u_0, v_0 \in L^\infty(\mathbb{R}^N)$, $u_0, v_0 \ge 0$ a.e. on $\mathbb{R}^N$. Let the nonlinearity f satisfy* (12), (14), *and the polynomial growth condition* (13). *Then problem* (1)–(3) *admits a nonnegative classical solution on $\mathbb{R}^N \times (0, +\infty)$.*

Proof. First, as $u_0 \ge 0, v_0 \ge 0$ and f satisfies condition (14), we have $u \ge 0$ and $v \ge 0$.

In view of the maximum principle, we have the estimate

$$\|u\|_{p,T} \le \|u_0\|_{p,T}. \tag{31}$$

Case 1 ($0 < \alpha \le \beta \le 2$). From (1) and (2), we have

$$u_t + v_t + (-\Delta)^{\alpha/2}(u) + (-\Delta)^{\beta/2}(v) = 0, \tag{32}$$

which can alternatively be rewritten as

$$(u + v)_t + (-\Delta)^{\beta/2} (u + v) + (-\Delta)^{\alpha/2} (u) - (-\Delta)^{\beta/2} (u) = 0. \tag{33}$$

Now, we use the duality argument. By multiplying (33) throughout by φ, the solution of (15)-(16), and integrating by parts over Q_T, we obtain

$$\int_{Q_T} (u + v) \left(-\varphi_t + (-\Delta)^{\beta/2} (\varphi) \right)$$
$$= \int_{\mathbb{R}^N} (u_0 + v_0) \, \varphi (x, 0) \tag{34}$$
$$+ \int_{Q_T} u \left((-\Delta)^{\beta/2} (\varphi) - (-\Delta)^{\alpha/2} (\varphi) \right)$$

or

$$\int_{Q_T} (u + v) \, \vartheta = \int_{\mathbb{R}^N} (u_0 + v_0) \, \varphi (x, 0)$$
$$\tag{35}$$
$$+ \int_{Q_T} u \left((-\Delta)^{\beta/2} (\varphi) - (-\Delta)^{\alpha/2} (\varphi) \right).$$

Using Lemma 2, we have

$$\|\varphi\|_q + \|\varphi\|_{q,T} \le C \|\vartheta\|_{q,T},$$
$$\left\| (-\Delta)^{\beta/2} \varphi \right\|_{q,T} \le C \|\vartheta\|_{q,T}. \tag{36}$$

Making use of inequality (24) together with $\varrho = 0$, $\sigma = \alpha/2$, and $\gamma = \beta/2$, we obtain

$$\left\| (-\Delta)^{\alpha/2} \varphi \right\|_q \le C \left\| (-\Delta)^{\beta/2} \varphi \right\|_q^{\alpha/\beta} \|\varphi\|_q^{1-(\alpha/\beta)}. \tag{37}$$

Using estimates (36) and (37), we have

$$\left\| (-\Delta)^{\alpha/2} \varphi \right\|_{q,T} \le C \left(\|\vartheta\|_{q,T} \right)^{\alpha/\beta} \left(\|\vartheta\|_{q,T} \right)^{1-(\alpha/\beta)}$$
$$\le C \|\vartheta\|_{q,T}. \tag{38}$$

Now, we have the estimates

$$\int_{Q_T} u \left| (-\Delta)^{\beta/2} (\varphi) \right| \le \|u\|_{p,T} + \left\| (-\Delta)^{\beta/2} (\varphi) \right\|_{q,T},$$
$$p + q = pq,$$
$$\int_{Q_T} u \left| (-\Delta)^{\alpha/2} (\varphi) \right| \le \left(\|u\|_{p,T} + \left\| (-\Delta)^{\alpha/2} (\varphi) \right\|_{q,T} \right)$$
$$\le C \left(\|u\|_{p,T} + \|\vartheta\|_{q,T} \right), \quad p + q = pq. \tag{39}$$

Finally, we have

$$\int_{Q_T} (u + v) \, \vartheta \le C \left(\|u_0 + v_0\|_p + \|u\|_{p,T} + \|\vartheta\|_{q,T} + \|\varphi_0\|_q \right) \tag{40}$$

thanks to the above inequalities.

Since ϑ is arbitrarily nonnegative in $L^q(Q_T)$ and $u, v \ge 0$, therefore it follows by duality that

$$\|u + v\|_{p,T} \le C \left(\|u_0 + v_0\|_p + \|u\|_{p,T} + \|\vartheta\|_{q,T} + \|\varphi_0\|_q \right). \tag{41}$$

Therefore, for all $p < \infty$, the L^p-norm of u and v remains finite on $Q_{T_{\max}}$. From the polynomial growth assumption on the nonlinearity, it follows that $f(u, v)$ is also in $L^p(Q_{T_{\max}})$ for all $p \in (1, \infty)$. If we take $p > (N + 1)/2$, we deduce that $v \in L^p(Q_{T_{\max}})$:

$$\|v\|_{\infty, T_{\max}} \le C \|f(u, v)\|_{p, T_{\max}} < +\infty. \tag{42}$$

This implies that $T_{\max} = \infty$.

Case 2 ($0 < \beta \le \alpha \le 2$). This case can be treated in the same way by making use of inequality (20) with $\nu = 1$. □

The next theorem deals with the "no growth" restriction on f.

Theorem 6. *Assume* $0 < \beta \le \alpha \le 2$ *and* $u_0, v_0 \in L^\infty(\mathbb{R}^N)$, $u_0, v_0 \ge 0$ *a.e. on* $\mathbb{R}^N$. *Let the nonlinearity* f *satisfy* (12) *and* (14). *Then problem* (1)–(3) *has a classical solution on* $\mathbb{R}^N \times (0, +\infty)$.

Proof. Let $S_\delta(t)$ be the semigroup generated by $(-\Delta)^{\delta/2}$ on $\mathbb{R}^N$. Then we have

$$u(x, t) = S_\alpha(t) u_0(x) - \mathfrak{F}_\alpha(x, t),$$
$$v(x, t) = S_\beta(t) v_0(x) + \mathfrak{F}_\beta(x, t), \tag{43}$$

where $\mathfrak{F}_\delta(x, t) = \int_0^t \int_{\mathbb{R}^N} f_\delta(y, s) u(x - y, t - s) f(v(x - y, t - s)) \, dy \, ds$, $\delta = \alpha$, or $\delta = \beta$.

From (26), we have

$$0 \le \mathfrak{F}_\alpha(x, t) \le S_\alpha(t) u_0(x) \le C \|u_0\|_\infty. \tag{44}$$

Using Lemma 1 for $t \ge 1$, we obtain

$$\mathfrak{F}_\beta(x, t) = \int_0^t \int_{\mathbb{R}^N} f_\beta(y, s) u(x - y, t - s)$$
$$\times f(v(x - y, t - s)) \, dy \, ds$$
$$\le Kt^{N/\alpha} \int_0^t \int_{\mathbb{R}^N} f_\beta(y, s) u(x - y, t - s) \tag{45}$$
$$\times f(v(x - y, t - s)) \, dy \, ds$$
$$\le Kt^{N/\alpha} \mathfrak{F}_\alpha(t, x) \le CKt^{N/\alpha} \|u_0\|_\infty.$$

So $\mathfrak{F}_\beta(t, x)$ is bounded for any finite t, whereupon the solution is global. □

Remark 7. Our results remain valid when the reaction terms $f(u, v)$ in the first equation and $g(u, v)$ in the second equation satisfy

$$f(u, v) + g(u, v) \le L(u + v) + M, \quad \forall u, v \ge 0, \tag{46}$$

where L and M are nonnegative constants.

Appendix

Here we present the result of Martin and Pierre [18] concerning the determination of the bound on the component v of the system

$$\frac{\partial}{\partial t} u - a\Delta u = -f(u,v), \quad (x,t) \in \mathbb{R}^N \times (0,\infty),$$

$$\frac{\partial}{\partial t} v - b\Delta v = +f(u,v), \quad (x,t) \in \mathbb{R}^N \times (0,\infty), \qquad \text{(A.1)}$$

$$u(\cdot,0) = u_0(\cdot), \quad v(\cdot,0) = v_0(\cdot), \quad x \in \mathbb{R}^N,$$

where f satisfies hypotheses (12) and (14).

Theorem 8. *Assume*

$$b \le a,$$

$$u_0, v_0 \in L^\infty\left(\mathbb{R}^N\right), \quad u_0, v_0 \ge 0 \ a.e \ x \in \mathbb{R}^N. \qquad \text{(A.2)}$$

Then (A.1) has a classical solution on $\mathbb{R}^N \times (0,\infty)$.

Proof. For $t \in (0, T_{\max})$, we can write

$$u(t) = S_a(t) u_0 - \int_0^t S_a(t-s) f(u,v)(s) \, ds, \qquad \text{(A.3)}$$

$$v(t) = S_b(t) u_0 + \int_0^t S_b(t-s) f(u,v)(s) \, ds \qquad \text{(A.4)}$$

via the semigroups $S_a(t)$ and $S_b(t)$, where

$$S_\lambda(t) u_0(x) = (4\pi\lambda t)^{-N/2} \int_{\mathbb{R}^N} \exp\left(-\frac{|x-y|^2}{4\lambda t}\right) u_0(y) \, dy. \qquad \text{(A.5)}$$

It is not difficult to see that

$$\int_0^t S_b(t-s) f(u,v)(s) \, ds$$

$$\le \left(\frac{a}{b}\right)^{N/2} \int_0^t S_a(t-s) f(u,v)(s) \, ds. \qquad \text{(A.6)}$$

From (A.3)–(A.6) and $u \ge 0$, we deduce

$$v(t) \le S_b(t) v_0 + \left(\frac{a}{b}\right)^{N/2} S_a(t) u_0, \qquad \text{(A.7)}$$

which provides a uniform L^∞-bound for v. $\qquad \square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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