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Research Article

Strong Convergence of the Split-Step θ-Method for Stochastic Age-Dependent Capital System with Random Jump Magnitudes

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We develop a new split-step θ (SSθ) method for stochastic age-dependent capital system with random jump magnitudes. The main aim of this paper is to investigate the convergence of the SSθ method for a class of stochastic age-dependent capital system with random jump magnitudes. It is proved that the proposed method is convergent with strong order 1/2 under given conditions. Finally, an example is simulated to verify the results obtained from theory.

1. Introduction

Stochastic partial differential equations are becoming increasingly used to model real-world phenomena in different fields, such as economics, biology, and physics. Recently, the study of the stochastic age-dependent (vintage) capital system has received a great deal of attention. For example, Wang studied stability of solutions for stochastic investment system [1]. Zhang et al. studied the convergence and exponential stability of numerical solutions to the stochastic age-dependent capital system [2, 3].

In the stochastic age-dependent capital system, due to the effects of external environment for capital system, such as innovations in technique, introduction of new products, natural disasters, and changes in laws and government policies, the size of the capital systems increases or decreases drastically. So Poisson jumps with deterministic jump magnitude have been used in stochastic age-dependent population equations. For example, Li et al. [4] studied the Euler numerical method for stochastic age-dependent population equations with Poisson jumps. L. Wang and X. Wang [5] analysed the convergence of the semi-implicit Euler method for stochastic age-dependent population equations with Poisson jumps. Rathinasamy et al. [6] developed the numerical method for stochastic age-dependent population equations with Poisson jump and phase semi-Markovian switching. However, the random jump magnitude is now commonly seen in financial models [7–9]. In this paper, we will consider the following stochastic age-dependent capital system with random jump magnitudes as shown in [10]:

$$\begin{aligned}
\mathrm{d}K(t,a) &= -\mu(t,a)\,K(t,a)\,\mathrm{d}t \\
&\quad + f(t,K(t,a))\,\mathrm{d}t + g(t,K(t,a))\,\mathrm{d}W(t) \\
&\quad + h\left(t,K(t,a),\gamma_{N(t)+1}\right)\mathrm{d}N(t), \quad \text{in } D, \\
K(t,0) &= \varphi(t) \\
&= \alpha(t)\,B(t)\,F\left(L(t),\int_0^A K(t,a)\,\mathrm{d}a\right), \\
&\qquad \text{in } t\in[0,T], \\
K(0,a) &= K_0(a), \quad \text{in } a\in[0,A], \\
\mathbb{N}(t) &= \int_0^A K(t,a)\,\mathrm{d}a, \quad \text{in } t\in[0,T],
\end{aligned} \tag{1}$$

where $K(t,a)$ denotes the stock of capital goods of age a at time t, $\mathrm{d}K(t,a) = (\partial K(t,a)/\partial t + \partial K(t,a)/\partial a)\mathrm{d}t$, $\varphi(0) = K_0(0)$, and $D = [0,T]\times[0,A]$. $\mathbb{N}(t)$ is defined as total output produced in year t; also a is the age of the capital; the investment $\varphi(t)$ in the new capital. The f is the appreciation (when $f \geq 0$) or depreciation (when $f \leq 0$) of the production capacity, and g represents the volatility of the capital stock. The value of h is the actual jump and γ_i is the underlying random variables of the magnitudes, and often it is called "mark" of the jump.

Also $W(t)$ is a standard Wiener process. $N(t)$ is a scalar Poisson process with intensity λ_1. It is assumed that for some $q \geq 1$ there is a constant C such that $\mathbb{E}[|\gamma_i|^{2q}] \leq C$; that is, the $2q$th moment of the jump magnitude is bounded. The maximum physical lifetime of capital A, the planning interval of calendar time $[0, T)$, the depreciation rate $\mu(t, a)$ of capital, and the capital density $K_0(a)$ (the initial distribution of capital over age)are given. The $\alpha(t)$ denotes the accumulative rate of capital at the moment of t, $0 < \alpha(t) < 1$, and $B(t)$ is the technical progress at the moment of t. This makes that total output produced in year t be defined as $\mathbb{N}(t) = \int_0^A K(t, a)\mathrm{d}a$. In each sector all the firms have an identical neoclassical technology and produce output using labor and capital. The production function $F(L(t), \int_0^A K(t, a)\mathrm{d}a)$ is neoclassical, where $\int_0^A K(t, a)\mathrm{d}a$ is the total sum of capital at time t and $L(t)$ is the labor force.

The integral version of (1) is given by the equation

$$\begin{aligned} K_t = K_0 &- \int_0^t \frac{\partial K_s}{\partial a}\mathrm{d}s - \int_0^t \mu(s, a) K_s \mathrm{d}s + \int_0^t f(s, K_s)\,\mathrm{d}s \\ &+ \int_0^t g(s, K_s)\,\mathrm{d}W(s) + \int_0^t h(s, K_s, \gamma(s))\,\mathrm{d}N(s), \end{aligned} \tag{2}$$

where $K_t = K(t, a)$ for fixed a.

Since the system (1) does not have closed form solutions, it is necessary to develop numerical methods for (1). Recently, Zhang and Rathinasamy [10] first derived the numerical solutions for stochastic age-dependent capital system with random jump magnitudes. However, their method belongs to the classic explicit Euler method and has a lower accuracy in [10] if we do not consider the appropriate step sizes.

Higham and Kloeden [11] first constructed the split-step backward Euler (SSBE) method for nonlinear stochastic differential equations with Poisson jumps. Tan and Wang [12] studied the convergence and stability of the SSBE method for linear stochastic delay integrodifferential equations. Ding et al. [13] developed the split-step θ method for solving the stochastic differential equations. Rathinasamy [14] investigated the split-step θ methods for stochastic age-dependent population equations with Markovian switching. Thus, we can construct the SSθ method for stochastic age-dependent population equations with random Poisson jumps.

In this paper, we will investigate the convergence of the SSθ method for system (1). The outline of the paper is as follows. In Section 2, we will introduce some preliminary results which are essential for our analysis. Section 3 will show us the SSθ method for solving stochastic age-dependent population equations with random Poisson jumps. In Section 4, several lemmas which are useful for our main result are proved. We give the main result that the numerical solutions converge to the true solutions with strong order 1/2 in Section 5. At last, a numerical example is given to verify the results obtained from the theory.

2. Preliminaries

Throughout this paper, it will be denoted by $L^2([0, A])$ the space of functions that are square-integrable over the domain $[0, A]$. Let

$$V = H^1([0, A]) \equiv \left\{ \xi \mid \xi \in L^2([0, A]), \frac{\partial \xi}{\partial a} \in L^2([0, A]) \right\}, \tag{3}$$

where $\partial\xi/\partial a$ is generalized partial derivative with respect to age a and V is a Sobolev space. $H = L^2([0, A])$ such that $V \hookrightarrow H \equiv H' \hookrightarrow V'$. $V' = H^{-1}([0, A])$ is the dual space of V. We denote by $\|\cdot\|, |\cdot|$ and $\|\cdot\|_*$ the norms in V, H, and V', respectively, by $(\cdot, \cdot)$ the scalar product in H. $\langle\cdot, \cdot\rangle$, the duality product between V and V', is defined by

$$\langle\cdot, \cdot\rangle = \int_0^A u \cdot v \mathrm{d}a, \quad u \in V,\ v \in V'. \tag{4}$$

Let $W(t)$ be a Wiener process defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and taking its values in the separable Hilbert space S,

$$W(t) = \sum_{i=1}^{\infty} \beta_i(t) e_i, \tag{5}$$

where $\{e_i\}_{i\geq 0}$ is an orthonormal set of eigenvectors of G, $\beta_i(t)$ are mutually independent real Wiener processes with incremental covariance $\lambda_i > 0$, $Ge_i = \lambda_i e_i$, and $\operatorname{tr} G = \sum_{i=1}^{\infty} \lambda_i$ (tr denotes the trace of an operator). For an operator $g \in \mathcal{L}(S, H)$ to be the space of all bounded linear operators from S into H, it is denoted by $\|g\|_2$ the Hilbert-Schmidt norm; that is,

$$\|g\|_2^2 = \operatorname{tr}\left(gGg^T\right). \tag{6}$$

Let $C = C([0, T]; H)$ be the space of all continuous function from $[0, T]$ into H with sup-norm $\|\phi\|_c = \sup_{0\leq s\leq T}|\phi|(s)$, $L_V^p = L^p([0, T]; V)$, and $L_H^p = L^p([0, T]; H)$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$, satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets).

Definition 1. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ be the stochastic basis and W_t a Wiener process. Suppose that K_0 is a random variable such that $\mathbb{E}|K_0|^2 < \infty$. A stochastic process $K_t \equiv K(t, a)$ for fixed a is said to be a solution on Ω to the stochastic age-structured capital system for $t \in [0, T]$ if the following conditions are satisfied:

(D1) K_t is a $\mathcal{F}_t$-measurable random variable;

(D2) $K_t \in I^p([0, T]; V) \cap L^2(\Omega; C([0, T]; V))$, $p > 1$, $T > 0$, where $I^p([0, T]; V)$ denotes the space of all V-valued processes $(K_t)_{t\in[0,T]}$ (one will write K_t for short) measurable (from $[0,T]\times\Omega$ into V), and satisfying $\mathbb{E}\int_0^T \|K_t\|^p \mathrm{d}t < \infty$. Here $C([0,T];V)$ denotes the space of all continuous functions from $[0,T]$ to V;

(D3) it satisfies the following equation:

$$\begin{aligned}\langle K_t, v\rangle &+ \int_0^t \left\langle \frac{\partial K_s}{\partial a}, v\right\rangle \mathrm{d}s \\ &= \langle K_0, v\rangle - \int_0^t \langle \mu(s,a) K_s, v\rangle \mathrm{d}s \\ &\quad + \int_0^t \langle f(s,K_s), v\rangle \mathrm{d}s \\ &\quad + \int_0^t \langle g(s,K_s), v\rangle \mathrm{d}W(s) \\ &\quad + \int_0^t \langle h(s,K_s,\gamma_{N(s)+1}), v\rangle \mathrm{d}N(s),\end{aligned} \tag{7}$$

for all $v \in V$, $t \in [0,T]$, a.e. $\omega \in \Omega$, where the stochastic integrals are understood in the Itô sense.

The parameter A is the maximal age of the capital, so $K(t,a) \equiv K_t = 0, \forall a \geq A$.

As the standing hypotheses we always assume that the following conditions are satisfied.

($\mathbb{H}1$) $f(t,0) = 0$, $g(t,0) = 0$, and $h(t,0,\gamma_{N(t)+1}) = 0, t \in [0,T]$.

($\mathbb{H}2$) (Lipschitz condition) there exists a positive constant k such that $x, y \in H$ and $\forall t$,

$$\begin{aligned}&\text{(i) } |f(t,y)-f(t,x)| \vee \|g(t,y)-g(t,x)\|_2 \leq k|y-x|, \\ &\text{(ii) } |h(t,y,u)-h(t,x,v)| \leq k[|y-x|+|u-v|].\end{aligned}$$

($\mathbb{H}3$) $\mu(t,a)$ is nonnegative measurable in D such that

$$0 \leq \mu_0 \leq \mu(t,a) \leq \overline{\mu} < \infty, \tag{8}$$

and $B(t)$ is nonnegative continuous in $[0,T]$ such that

$$\alpha(t) B(t) \leq \eta; \quad \eta \text{ is a non-negativeconstant in } [0,T]. \tag{9}$$

($\mathbb{H}4$) $F(L,N) \geq 0$, $(F(L,0) = 0)$, $\partial F/\partial L > 0$, $0 < \partial F/\partial N \leq F_1$, where F_1 is a positive constant.

In an analogous way to the corresponding proof presented in [15], the following existence and uniqueness of solutions is established: under the conditions ($\mathbb{H}1$)–($\mathbb{H}4$), (1) has a unique continuous solution $K(t,a)$ on $(t,a) \in D$.

We note for the following jump process:

$$\gamma(t) := \gamma_{N(t)+1} = \sum_j \gamma_{j+1} I_{[\tau_j,\tau_{j+1})}(t), \tag{10}$$

where $\tau_0 = 0$ and I_A is the indicator function for the set A; that is, $I_A(x) = \left\{\begin{smallmatrix}1, & x\in A\\ 0, & x\notin A\end{smallmatrix}\right.$.

Then the following lemma can be found in [16, 17].

Lemma 2. *There exists a constant B for any $t \in [0,T]$ and $\mathbb{E}[|\gamma_i|^{2q}] \leq B$ for some $q > 1$ such that*

$$\mathbb{E}\left[\int_0^t \left|\overline{\gamma}(s) - \gamma(s)\right|^2 ds\right] \leq C\Delta^{1-(1/q)}. \tag{11}$$

3. The Split-Step θ-Method

Let τ_j denote the jth jump of $N(s)$ occurrence time. Suppose, for example, that jumps arrive at distinct, ordered times $\tau_1 < \tau_2 <, \ldots$; let $t_1, t_2, \ldots, t_m$ be a deterministic grid point of $[0,T]$. We construct approximate solutions to models of the form (1) at a discrete set of times τ_n. This set is the superposition of the random jump times of a Poisson process on $[0,T]$ and a deterministic grid $t_1, t_2, \ldots, t_m$ and satisfies $\max|\tau_{i+1} - \tau_i| < \Delta$. Let $\Delta = t_{n+1} - t_n$, $\Delta W_n := W(t_{n+1}) - W(t_n)$, and $\Delta N_n := N(t_{n+1}) - N(t_n)$ denote the increments of the time, Brownian motion, and the Poisson processes, respectively.

For system (1) the split-step θ approximate solution is defined by the iterative scheme

$$\begin{aligned}Q_n^* = Q_n - \frac{\partial Q_{n+1}}{\partial a} &+ \left[(1-\theta)\left[-\mu(t_n,a) Q_n + f(t_n,Q_n)\right]\right. \\ &\left. + \theta\left[-\mu(t_n,a) Q_n^* + f(t_n,Q_n^*)\right]\right]\Delta,\end{aligned} \tag{12a}$$

$$Q_{n+1} = Q_n^* + g(t_n,Q_n^*)\Delta W_n + h\left(t_n,Q_n^*,\gamma_{N(t_n)+1}\right)\Delta N_n, \tag{12b}$$

with initial values $Q_0 = K(0,A)$, $Q_n(t,0) = \int_0^A \beta(t,a) Q_n \mathrm{d}a$, $n > 1$; Q_n is the numerical approximation of $K(t_n,a)$ with $t_n = n\cdot\Delta$; the time increment is $\Delta = T/N \ll 1$. Where $\theta \in [0,1]$, if we give $\theta = 1$, the SSθ method becomes the SSBE method in [11]. If $\theta = 0$, the SSθ method is an explicit method.

To answer the question of the existence of numerical solution, we will give the following lemma.

Lemma 3. *Let conditions* ($\mathbb{H}2$) *and* ($\mathbb{H}3$) *hold, and let* $0 < \theta < 1$ *and* $0 < \Delta < 1/\theta(k+\overline{\mu})$*; then the implicit equation* (12a) *can be solved uniquely for* Q_n^**, with probability 1.*

Proof. Writing (12a) as $Q_n^* = F(Q_n^*) = y + \theta\Delta[-\mu(t_n,a)Q_n^* + f(t_n,Q_n^*)]$, and using condition ($\mathbb{H}2$) and ($\mathbb{H}3$), we have

$$\begin{aligned}|F(u) - F(v)| &\leq \theta\Delta\left[\left|f(t,u) - f(t,v)\right| + \left|\mu(t,a)(u-v)\right|\right] \\ &\leq \theta\Delta(k+\overline{\mu})|u-v|.\end{aligned} \tag{13}$$

Then the result follows from the classical Banach contraction mapping theorem [18]. □

When Lemma 3 followed, we find it is convenient to use continuous-time approximation solution in our strong convergence analysis; hence for $\in [t_n, t_{n+1})$, we can define the following step functions:

$$Z_1(t) = Z_1(t,a) = \sum_{n=0}^{N-1} Q_n I_{[n\Delta t,(n+1)\Delta t)}(t), \tag{14}$$

$$Z_2(t) = Z_2(t,a) \sum_{n=0}^{N-1} Q_n^* I_{[n\Delta t,(n+1)\Delta t)}(t), \tag{15}$$

$$\overline{\gamma}(t) = \sum_{n=0}^{N-1} \gamma(t_n) I_{[n\Delta,(n+1)\Delta)}(t), \tag{16}$$

where N is the largest number such that $N\Delta \le T$.

When $t \in [t_n, t_{n+1})$, Lemma 3 ensures the existence of Q_n^* by (12a); then we define

$$\begin{aligned} Q_t = Q_0 &- \int_0^t \frac{\partial Q_s}{\partial a} \mathrm{d}s \\ &+ \int_0^t (1-\theta)\left[-\mu(s,a) Z_1(s) + f(s, Z_1(s))\right] \mathrm{d}s \\ &+ \int_0^t \theta\left[-\mu(s,a) Z_2(s) + f(s, Z_2(s))\right] \mathrm{d}s \\ &+ \int_0^t g(s, Z_2(s))\,\mathrm{d}W_s + \int_0^t h(s, Z_2(s), \overline{\gamma}(s))\,\mathrm{d}N_s, \end{aligned} \tag{17}$$

with initial value $Q_0 = p(0,a)$, $Q(t,0) = \int_0^A \beta(t,a) Q_t \mathrm{d}a$, $Q_t = Q(t,a)$.

It is easy to verify that $Z_1(t_n, a) = Q_n = Q(t_n, a)$; that is, $Z_1(t,a)$ and $Q(t,a)$ coincide with the discrete solutions at the grid points. Hence we refer to $Q(t,a)$ as a continuous-time extension of the discrete approximation $\{Q_n\}$. So our plan is to prove a strong convergence result for $Q(t,a)$.

4. Several Lemmas

In this section, we will give several lemmas which are useful for the following main result.

Lemma 4. *Under the conditions* $(\mathbb{H}1)$–$(\mathbb{H}4)$, *there are constants* $p \ge 2$ *and* $C_1 > 0$ *such that*

$$\mathbb{E}\left[\sup_{0\le t\le T} |K_t|^p\right] \le C_1. \tag{18}$$

The proof is similar to that in [3].

Next lemma shows the relationship between $\mathbb{E}|Q_n^*|^2$ and $\mathbb{E}|Q_n|^2$.

Lemma 5. *Under conditions* $(\mathbb{H}1)$–$(\mathbb{H}4)$, *let* $0 < \theta < 1$, $0 < \Delta < \min\{1, 1/\theta(k+\overline{\mu}), 1/2\sqrt{3(\overline{\mu}^2 + k^2)}\}$, *and* $\mathbb{E}|\partial Q_{n+1}/\partial a|^2 < \infty$; *then there exist two positive constants* C_2 *and* C_3 *such that*

$$\mathbb{E}|Q_n^*|^2 \le C_2 \mathbb{E}|Q_n|^2 + C_3, \tag{19}$$

where Q_n^* *and* Q_n *are produced by* (12a) *and* (12b).

Proof. Squaring both sides of (12a) and using the elementary inequality $(a+b+c)^2 \le 3a^2 + 3b^2 + 3c^2$, we find

$$\begin{aligned} |Q_n^*|^2 \le{}& 3|Q_n|^2 + 3\left|\frac{\partial Q_{n+1}}{\partial a}\right|^2 \Delta^2 \\ &+ 3\left[(1-\theta)\left[-\mu(t_n,a)Q_n + f(t_n,Q_n)\right]\right. \\ &\left.+ \theta\left[-\mu(t_n,a)Q_n^* + f(t_n,Q_n^*)\right]\right]^2 \Delta^2. \end{aligned} \tag{20}$$

Using the elementary inequality $|(1-\theta)x + \theta y|^2 \le (1-\theta)|x|^2 + \theta|y|^2$, we obtain

$$\begin{aligned} |Q_n^*|^2 \le{}& 3|Q_n|^2 + 3\left|\frac{\partial Q_{n+1}}{\partial a}\right|^2 \Delta^2 \\ &+ 3(1-\theta)\left[-\mu(t_n,a)Q_n + f(t_n,Q_n)\right]^2 \Delta^2 \\ &+ 3\theta\left[-\mu(t_n,a)Q_n^* + f(t_n,Q_n^*)\right]^2 \Delta^2. \end{aligned} \tag{21}$$

Due to $0 < \theta < 1$ and conditions $(\mathbb{H}2)$–$(\mathbb{H}3)$, we can get

$$\begin{aligned} |Q_n^*|^2 \le{}& 3|Q_n|^2 + 3\left|\frac{\partial Q_{n+1}}{\partial a}\right|^2 \Delta^2 \\ &+ 6\left[|\mu(t_n,a)Q_n|^2 + |f(t_n,Q_n)|^2\right]\Delta^2 \\ &+ 6\left[|\mu(t_n,a)Q_n^*|^2 + |f(t_n,Q_n^*)|^2\right]\Delta^2 \\ \le{}& 3|Q_n|^2 + 3\left|\frac{\partial Q_{n+1}}{\partial a}\right|^2 \Delta^2 + 6\left(\overline{\alpha}^2 + k^2\right)|Q_n|^2 \Delta^2 \\ &+ 6\left(\overline{\alpha}^2 + k^2\right)|Q_n^*|^2\Big]\Delta^2. \end{aligned} \tag{22}$$

Taking mathematical expectation for both sides, we can obtain

$$\begin{aligned} \mathbb{E}|Q_n^*|^2 \le{}& 3\mathbb{E}\left|\frac{\partial Q_{n+1}}{\partial a}\right|^2 \Delta^2 + \left[3 + 6\left(\overline{\alpha}^2 + k^2\right)\Delta^2\right]\mathbb{E}|Q_n|^2 \\ &+ 6\left(\overline{\alpha}^2 + k^2\right)\Delta^2 \mathbb{E}|Q_n^*|^2. \end{aligned} \tag{23}$$

Since $6(\overline{\alpha}^2 + k^2)\Delta^2 < 1/2$, thus $1 - 6(\overline{\alpha}^2 + k^2)\Delta^2 \ge 1/2$; then by $0 < \Delta < 1$, we have

$$\begin{aligned} \mathbb{E}|Q_n^*|^2 &\le \frac{3 + 6\left(\overline{\alpha}^2 + k^2\right)\Delta^2}{1 - 6\left(\overline{\alpha}^2 + k^2\right)\Delta^2}\mathbb{E}|Q_n|^2 + \frac{3\mathbb{E}|\partial Q_{n+1}/\partial a|^2 \Delta^2}{1 - 6\left(\overline{\alpha}^2 + k^2\right)\Delta^2} \\ &\le 6\left[1 + 2\left(\overline{\alpha}^2 + k^2\right)\right]\mathbb{E}|Q_n|^2 + 6\mathbb{E}\left|\frac{\partial Q_{n+1}}{\partial a}\right|^2 \\ &= C_2 \mathbb{E}|Q_n|^2 + C_3, \end{aligned} \tag{24}$$

where $C_2 = 6 + 12(\overline{\alpha}^2 + k^2)$ and $C_3 = 6\mathbb{E}|\partial Q_{n+1}/\partial a|^2$. The proof is completed. □

The next lemma shows that the continuous numerical solutions Q_t have bounded second moments.

Lemma 6. *Under conditions* $(\mathbb{H}1)$–$(\mathbb{H}4)$, *let* $0 < \theta < 1$, $0 < \Delta < \min\{1, 1/\theta(k + \overline{\mu}), 1/2\sqrt{3(\overline{\mu}^2 + k^2)}\}$; *then there exists a constant* $C_4 > 0$ *such that*

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|Q_{t\wedge v_n}\right|^2\right] \le C_4, \tag{25}$$

where $\tau_n := \inf\{t \ge 0, |K_t| \ge n\}$, $\sigma_n := \inf\{t \ge 0, |Q_t| \ge n\}$, *and* $v_n = \tau_n \wedge \sigma_n$.

Proof. From (17), we can get

$$\begin{aligned} dQ_t = & -\frac{\partial Q_t}{\partial a}dt + (1-\theta)\left[-\mu(t,a)Z_1(t) + f(s, Z_1(t))\right]dt \\ & + \theta\left[-\mu(t,a)Z_2(t) + f(s, Z_2(t))\right]dt \\ & + g(s, Z_2(t))\,dW_t + h(s, Z_2(t), \overline{\gamma}(t))\,dN_t. \end{aligned} \tag{26}$$

Applying the Itô formula [19] to $|Q_{t\wedge v_n}|^2$ it yields

$$\begin{aligned} \left|Q_{t\wedge v_n}\right|^2 = & \left|Q_0\right|^2 + 2\int_0^{t\wedge v_n}\left\langle -\frac{\partial Q_s}{\partial a}, Q_s\right\rangle ds \\ & - 2\int_0^{t\wedge v_n}\big((1-\theta)\mu(s,a)Z_1(s) \\ & \qquad + \theta\mu(s,a)Z_2(s), Q_s\big)\,ds \\ & + 2\int_0^{t\wedge v_n}\big((1-\theta)f(s, Z_1(s)) \\ & \qquad + \theta f(s, Z_2(s)), Q_s\big)\,ds \\ & + 2\int_0^{t\wedge v_n}\left(Q_s, g(s, Z_s)\right)dW(s) \\ & + 2\int_0^{t\wedge v_n}\left(Q_s, h(s, Z_s, \overline{\gamma}(s))\right)dN(s) \\ & + \int_0^{t\wedge v_n}\left\|g(s, Z_s)\right\|_2^2 ds \\ & + \int_0^{t\wedge v_n}\left|h(s, Z_s, \overline{\gamma}(s))\right|^2 dN(s). \end{aligned} \tag{27}$$

Using conditions $(\mathbb{H}1)$–$(\mathbb{H}3)$ and the compensated Poisson process $\widetilde{N}(t) := N(t) - \lambda_1 t$, we have

$$\begin{aligned} \left|Q_{t\wedge v_n}\right|^2 \le & \left|Q_0\right|^2 + 2\int_0^{t\wedge v_n}\left\langle -\frac{\partial Q_s}{\partial a}, Q_s\right\rangle ds \\ & + 2\mu_0\int_0^{t\wedge v_n}\left|Q_s\right|\left|(1-\theta)Z_1(s) + \theta Z_2(s)\right| ds \\ & + 2\int_0^{t\wedge v_n}\left|Q_s\right|\big|(1-\theta)f(s, Z_1(s)) \\ & \qquad + \theta f(s, Z_2(s))\big|\,ds \\ & + 2\int_0^{t\wedge v_n}\left(Q_s, g(s, Z_2(s))\,dW_s\right) \\ & + \int_0^{t\wedge v_n}\left\|g(s, Z_2(s))\right\|_2^2 ds \\ & + 2\int_0^{t\wedge v_n}\left(Q_s, h(s, Z_2(s), \overline{\gamma}(s))\right)d\widetilde{N}_s \\ & + \int_0^{t\wedge v_n}\left|h(s, Z_2(s), \overline{\gamma}(s))\right|^2 d\widetilde{N}_s, \\ & + \lambda_1\int_0^{t\wedge v_n}\left|h(s, Z_2(s), \overline{\gamma}(s))\right|^2 ds \\ & + 2\lambda_1\int_0^{t\wedge v_n}\left|Q_s\right|\left|h(s, Z_2(s), \overline{\gamma}(s))\right| ds. \end{aligned} \tag{28}$$

Note

$$\begin{aligned} \left\langle -\frac{\partial Q_s}{\partial a}, Q_s\right\rangle & = -\int_0^A Q_s\cdot\frac{\partial Q_s}{\partial a}da \\ & = \frac{1}{2}\alpha^2(s)A^2(s)\left[F\left(L(s), \int_0^A Q_s da\right)\right. \\ & \qquad \left. - F(L(s), 0)\right]^2 \\ & \le \frac{1}{2}\eta^2\left(\left.\frac{\partial F(L,N)}{\partial N}\right|_y\right)^2\left(\int_0^A Q_s da\right) \\ & \le \frac{1}{2}AF_1^2\eta^2\left|Q_s\right|^2, \end{aligned} \tag{29}$$

where $y \in (0, \int_0^A Q_s da)$. Consider

$$\begin{aligned} & 2\mu_0\int_0^{t\wedge v_n}\left|Q_s\right|\left|(1-\theta)Z_1(s) + \theta Z_2(s)\right| ds \\ & \quad \le \mu_0\int_0^{t\wedge v_n}\left|Q_s\right|^2 ds + 2\mu_0\int_0^{t\wedge v_n}\left[\left|Z_1(s)\right|^2 + \left|Z_2(s)\right|^2\right] ds, \\ & 2\int_0^{t\wedge v_n}\left|Q_s\right|\left|(1-\theta)f(s, Z_1(s)) + \theta f(s, Z_2(s))\right| ds \\ & \quad \le \int_0^{t\wedge v_n}\left|Q_s\right|^2 ds \\ & \qquad + 2\int_0^{t\wedge v_n}\left[\left|f(s, Z_1(s))\right|^2 + \left|f(s, Z_2(s))\right|^2\right] ds \\ & \quad \le \int_0^{t\wedge v_n}\left|Q_s\right|^2 ds + 2k^2\int_0^{t\wedge v_n}\left[\left|Z_1(s)\right|^2 + \left|Z_2(s)\right|^2\right] ds, \\ & 2\lambda_1\int_0^{t\wedge v_n}\left|Q_s\right|\left|h(s, Z_2(s), \overline{\gamma}(s))\right| ds \end{aligned}$$

$$\leq \lambda_1 \int_0^{t\wedge v_n} |Q_s|^2 ds + \lambda_1 \int_0^{t\wedge v_n} |h(s, Z_2(s), \overline{\gamma}(s))|^2 ds,$$
$$\leq \lambda_1 \int_0^{t\wedge v_n} |Q_s|^2 ds + \lambda_1 k^2 \int_0^{t\wedge v_n} |Z_2(s)|^2 + |\overline{\gamma}(s) - \gamma(s)|^2 ds. \tag{30}$$

Taking (29)-(30) into (28), we compute that for some positive constants $K_1 = A^2F_1^2\eta^2 + 1 + \mu_0 + \lambda_1$, $K_2 = 2(k^2 + \mu_0)$, $K_3 = 3k^2 + 2\lambda_1 k^2 + 2\mu_0$,

$$\begin{aligned}
|Q_{t\wedge v_n}|^2 \leq{}& |Q_0|^2 + K_1 \int_0^{t\wedge v_n} |Q_s|^2 ds \\
&+ K_2 \int_0^{t\wedge v_n} |Z_1(s)|^2 ds + K_3 \int_0^{t\wedge v_n} |Z_2(s)|^2 ds \\
&+ \lambda_1 K^2 \int_0^{t\wedge v_n} |\overline{\gamma}(s) - \gamma(s)|^2 ds \\
&+ 2\int_0^{t\wedge v_n} (Q_s, g(s, Z_2(s))\, dW_s) \\
&+ 2\int_0^{t\wedge v_n} (Q_s, h(s, Z_2(s), \overline{\gamma}(s)))\, d\widetilde{N}_s \\
&+ \int_0^{t\wedge v_n} |h(s, Z_2(s), \overline{\gamma}(s))|^2 d\widetilde{N}_s.
\end{aligned} \tag{31}$$

Now, it follows that for any $t_1 \in [0, T]$ and by Lemma 5

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\in[0,t_1]} |Q_{t\wedge v_n}|^2\right]
&\leq \mathbb{E}|Q_0|^2 + (K_1 + K_2 + K_3C_2) \int_0^{t_1\wedge v_n} \mathbb{E}\left[\sup_{0\leq s\leq t} |Q_s|^2\right] dt \\
&\quad + K_3C_3T + \lambda_1 k^2 CT \\
&\quad + 2\mathbb{E}\left[\sup_{t\in[0,t_1]} \int_0^{t\wedge v_n} (Q_s, g(s, Z_2(s))\, dW_s)\right] \\
&\quad + 2\mathbb{E}\left[\sup_{t\in[0,t_1]} \int_0^{t\wedge v_n} (Q_s, h(s, Z_2(s), \overline{\gamma}(s)))\, d\widetilde{N}_s\right] \\
&\quad + \mathbb{E}\left[\sup_{t\in[0,t_1]} \int_0^{t\wedge v_n} |h(s, Z_2(s), \overline{\gamma}(s))|^2 d\widetilde{N}_s\right].
\end{aligned} \tag{32}$$

By Burkholder-Davis-Gundy inequality, we have

$$\begin{aligned}
&2\mathbb{E}\left[\sup_{t\in[0,t_1]} \int_0^{t\wedge v_n} (Q_s, g(s, Z_2(s))\, dW_s)\right] \\
&\quad\leq \frac{1}{4}\mathbb{E}\left[\sup_{t\in[0,t_1]} |Q_t|^2\right] + K_4 \int_0^{t_1\wedge v_n} \|g(t, Z_2(t))\|_2^2 dt \\
&\quad\leq \frac{1}{4}\mathbb{E}\left[\sup_{t\in[0,t_1]} |Q_t|^2\right] + K_4 k^2 \int_0^{t_1\wedge v_n} \mathbb{E}\|Z_2(t)\|^2 dt, \\
&\mathbb{E}\left[\sup_{t\in[0,t_1]} \int_0^{t\wedge v_n} |h(s, Z_2(s), \overline{\gamma}(s))|^2 d\widetilde{N}_s\right] \\
&\quad\leq 4\lambda_1 \mathbb{E}\int_0^{t_1\wedge v_n} |h(t, Z_2(t), \overline{\gamma}(s))|^2 dt \\
&\quad\leq K_5 k^2 \int_0^{t_1\wedge v_n} \mathbb{E}\|Z_2(t)\|^2 + \mathbb{E}|\overline{\gamma}(s) - \gamma(s)|^2 dt \\
&\quad\leq K_5 k^2 \int_0^{t_1\wedge v_n} \mathbb{E}\|Z_2(t)\|^2 dt + K_5 k^2 CT, \\
&2\mathbb{E}\left[\sup_{t\in[0,t_1]} \int_0^{t\wedge v_n} (Q_s, h(s, Z_2(s), \overline{\gamma}(s)))\, d\widetilde{N}_s\right] \\
&\quad\leq \frac{1}{4}\mathbb{E}\left[\sup_{t\in[0,t_1]} |Q_t|^2\right] + K_6 \int_0^{t_1\wedge v_n} |h(t, Z_2(t), \overline{\gamma}(s))|^2 dt \\
&\quad\leq \frac{1}{4}\mathbb{E}\left[\sup_{t\in[0,t_1]} |Q_t|^2\right] \\
&\qquad + K_6 k^2 \int_0^{t_1\wedge v_n} \mathbb{E}\|Z_2(t)\|^2 dt + K_6 k^2 CT,
\end{aligned} \tag{33}$$

for some positive constants $K_4, K_5, K_6 > 0$. Substituting (33) into (32) and then by Lemma 5, we can obtain that

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\in[0,t_1]} |Q_{t\wedge v_n}|^2\right]
&\leq 2\mathbb{E}|Q_0|^2 \\
&\quad + 2(K_1 + K_2 + K_3C_2) \int_0^{t_1\wedge v_n} \mathbb{E}\left[\sup_{0\leq s\leq t} |Q_s|^2\right] dt \\
&\quad + 2K_3C_3T + 2(K_4 + K_5 + K_6) k^2 \int_0^{t_1\wedge v_n} \mathbb{E}\|Z_2(t)\|^2 dt \\
&\quad + 2(\lambda_1 + K_5 + K_6) k^2 CT \\
&\leq 2\left[K_1 + K_2 + K_3C_2 + (K_4 + K_5 + K_6) k^2 C_2\right] \\
&\quad \times \int_0^{t_1\wedge v_n} \mathbb{E}\left[\sup_{0\leq s\leq t} |Q_s|^2\right] dt \\
&\quad + 2\mathbb{E}|Q_0|^2 + 2K_3C_3T + 2(K_4 + K_5 + K_6) k^2 C_3 T \\
&\quad + 2(\lambda_1 + K_5 + K_6) k^2 CT \\
&:= K_7 + K_8 \int_0^{t_1\wedge v_n} \mathbb{E}\left[\sup_{0\leq s\leq t} |Q_s|^2\right] dt,
\end{aligned} \tag{34}$$

where

$$K_7 = 2\mathbb{E}|Q_0|^2 + 2K_3C_3T + 2(K_4+K_5+K_6)k^2C_3T + 2(\lambda_1+K_5+K_6)k^2CT, \quad (35)$$
$$K_8 = 2\left[K_1+K_2+K_3C_2+(K_4+K_5+K_6)k^2C_2\right].$$

Applying the continuous Gronwall inequality, we can easy get

$$\mathbb{E}\left[\sup_{t\in[0,T]}|Q_{t\wedge v_n}|^2\right] \le C_4, \quad (36)$$

with $C_4 = K_7\exp(K_8T)$. □

The next lemma shows that the continuous-time approximation Q_t in (17) remains close to the step functions $Z_1(t)$ and $Z_2(t)$ in the mean square sense.

Lemma 7. *Under conditions* ($\mathbb{H}$1)–($\mathbb{H}$4), $\mathbb{E}|\partial Q_s/\partial a|^2 < \infty$, *and let* $0<\theta<1$, $0<\Delta<\min\{1, 1/\theta(k+\overline{\mu}), 1/2\sqrt{3(\overline{\mu}^2+k^2)}\}$; *then there exist two positive constants* C_5 *and* C_6 *that are independent of* Δ, *such that*

$$\mathbb{E}|Q_t - Z_1(t)|^2 \le C_5\Delta, \quad (37)$$

$$\mathbb{E}|Q_t - Z_2(t)|^2 \le C_6\Delta, \quad (38)$$

where $Z_1(t)$, $Z_2(t)$, *and* Q_t *are defined by* (14), (15), *and* (17), *respectively.*

Proof. Consider $t\in[n\Delta,(n+1)\Delta]\subseteq[0,T]$; we have

$$\begin{aligned} Q_t - Z_1(t) &= Q_t - Q_n \\ &= -\int_{n\Delta}^t \frac{\partial Q_s}{\partial a}\mathrm{d}s \\ &\quad + \int_{n\Delta}^t \left[(1-\theta)f(s,Z_1(s)) + \theta f(s,Z_2(s))\right]\mathrm{d}s \\ &\quad - \int_{n\Delta}^t \mu(s,a)\left[(1-\theta)Z_1(s)+\theta Z_2(s)\right]\mathrm{d}s \\ &\quad + \int_{n\Delta}^t g(s,Z_2(s))\,\mathrm{d}W_s + \int_{n\Delta}^t h(s,Z_2(s),\overline{\gamma}(s))\,\mathrm{d}N_s. \end{aligned} \quad (39)$$

Squaring both sides and using the element inequality $(a+b+c+e+f)^2 \le 5|a|^2+5|b|^2+5|c|^2+5|d|^2+5|e|^2$, we have

$$\begin{aligned} |Q_t - Z_1(t)|^2 &\le 5\left|\int_{n\Delta}^t \frac{\partial Q_s}{\partial a}\mathrm{d}s\right|^2 \\ &\quad + 5\left|\int_{n\Delta}^t \left[(1-\theta)f(s,Z_1(s))+\theta f(s,Z_2(s))\right]\mathrm{d}s\right|^2 \\ &\quad + 5\left|\int_{n\Delta}^t \mu(s,a)\left[(1-\theta)Z_1(s)+\theta Z_2(s)\right]\mathrm{d}s\right|^2 \\ &\quad + 5\left|\int_{n\Delta}^t g(s,Z_2(s))\,\mathrm{d}W(s)\right|^2 \\ &\quad + 5\left|\int_{n\Delta}^t h(s,Z_2(s),\overline{\gamma}(s))\,\mathrm{d}N(s)\right|^2. \end{aligned} \quad (40)$$

Now, the Cauchy-Schwarz inequality, condition ($\mathbb{H}$3), and the compensated Poisson process give

$$\begin{aligned} |Q_t - Z_1(t)|^2 &\le 5\Delta\int_{n\Delta}^t \left|\frac{\partial Q_s}{\partial a}\right|^2\mathrm{d}s \\ &\quad + 5\overline{\mu}^2\Delta\int_{n\Delta}^t |(1-\theta)Z_1(s)+\theta Z_2(s)|^2\mathrm{d}s \\ &\quad + 5\Delta\int_{n\Delta}^t |(1-\theta)f(s,Z_1(s))+\theta f(s,Z_2(s))|^2\mathrm{d}s \\ &\quad + 5\left|\int_{n\Delta}^t g(s,Z_2(s))\,\mathrm{d}W(s)\right|^2 \\ &\quad + 10\left|\int_{n\Delta}^t h(s,Z_2(s),\overline{\gamma}(s))\,\mathrm{d}\widetilde{N}(s)\right|^2 \\ &\quad + 10\lambda_1^2\left|\int_{n\Delta}^t h(s,Z_2(s),\overline{\gamma}(s))\,\mathrm{d}s\right|^2. \end{aligned} \quad (41)$$

Taking mathematical expectation, by element inequality $(a+b)^2 \le 2|a|^2+2|b|^2$, and using the martingale isometry, we have

$$\begin{aligned} \mathbb{E}|Q_t - Z_1(t)|^2 &\le 5\Delta\int_{n\Delta}^t \left|\frac{\partial Q_s}{\partial a}\right|^2\mathrm{d}s \\ &\quad + 5\overline{\mu}^2\Delta\int_{n\Delta}^t \mathbb{E}\left(|Z_1(s)|^2+|Z_2(s)|^2\right)\mathrm{d}s \end{aligned}$$

$$+ 10\Delta \int_{n\Delta}^{t} \mathbb{E}\left|f\left(s, Z_1(s)\right)\right|^2 + \mathbb{E}\left|f\left(s, Z_2(s)\right)\right|^2 \mathrm{d}s$$
$$+ 5 \int_{n\Delta}^{t} \mathbb{E}\left|g\left(s, Z_2(s)\right)\right|^2 \mathrm{d}s$$
$$+ 10\lambda_1 \int_{n\Delta}^{t} \mathbb{E}\left|h\left(s, Z_2(s), \overline{\gamma}(s)\right)\right|^2 \mathrm{d}s$$
$$+ 10\lambda_1^2 \Delta \int_{n\Delta}^{t} \mathbb{E}\left|h\left(s, Z_2(s), \overline{\gamma}(s)\right)\right|^2 \mathrm{d}s. \tag{42}$$

By conditions ($\mathbb{H}1$) and ($\mathbb{H}2$), we get

$$\begin{aligned}
&\mathbb{E}\left|Q_t - Z_1(t)\right|^2 \\
&\quad \le 5\Delta \int_{n\Delta}^{t} \left|\frac{\partial Q_s}{\partial a}\right|^2 \mathrm{d}s \\
&\qquad + 5\overline{\mu}^2 \Delta \int_{n\Delta}^{t} \mathbb{E}\left(\left|Z_1(s)\right|^2 + \left|Z_2(s)\right|^2\right) \mathrm{d}s \\
&\qquad + 10\Delta k^2 \int_{n\Delta}^{t} \mathbb{E}\left|Z_1(s)\right|^2 + \mathbb{E}\left|Z_2(s)\right|^2 \mathrm{d}s \\
&\qquad + 5k^2 \int_{n\Delta}^{t} \mathbb{E}\left|Z_2(s)\right|^2 \mathrm{d}s \\
&\qquad + \left(10\lambda_1 + 10\lambda_1^2\Delta\right) k^2 \int_{n\Delta}^{t} \mathbb{E}\left|Z_2(s)\right|^2 \\
&\qquad\qquad + \mathbb{E}\left|\overline{\gamma}(s) - \gamma(s)\right|^2 \mathrm{d}s. \\
&\quad = 5\Delta \int_{n\Delta}^{t} \left|\frac{\partial Q_s}{\partial a}\right|^2 \mathrm{d}s + \left(10\Delta k^2 + 5\overline{\mu}^2\Delta\right) \int_{n\Delta}^{t} \mathbb{E}\left|Z_1(s)\right|^2 \mathrm{d}s \\
&\qquad + \left(5 + 10\lambda_1 + 10\lambda_1^2\Delta\right) k^2 \int_{n\Delta}^{t} \mathbb{E}\left|Z_2(s)\right|^2 \mathrm{d}s \\
&\qquad + \left(10\lambda_1 + 10\lambda_1^2\Delta\right) k^2 C\Delta.
\end{aligned} \tag{43}$$

Since $Z_1(t) \equiv Q_n$ and $Z_2(t) \equiv Q_n^*$ on $[n\Delta t, (n+1)\Delta t)$, we have

$$\begin{aligned}
&\mathbb{E}\left|Q_t - Z_1(t)\right|^2 \\
&\quad \le 5\Delta \int_{n\Delta}^{t} \left|\frac{\partial Q_s}{\partial a}\right|^2 \mathrm{d}s + \left(10\Delta k^2 + 5\overline{\mu}^2\Delta\right) \int_{n\Delta}^{t} \mathbb{E}\left|Q_n\right|^2 \mathrm{d}s \\
&\qquad + \left(5 + 10\lambda_1 + 10\lambda_1^2\Delta\right) k^2 \int_{n\Delta}^{t} \mathbb{E}\left|Q_n^*\right|^2 \mathrm{d}s \\
&\qquad + \left(10\lambda_1 + 10\lambda_1^2\Delta\right) k^2 C\Delta.
\end{aligned} \tag{44}$$

Then by Lemmas 5 and 6, we can derive

$$\begin{aligned}
&\mathbb{E}\left|Q_t - Z_1(t)\right|^2 \\
&\quad \le \left[10\Delta k^2 + 5\overline{\mu}^2\Delta\right. \\
&\qquad\quad \left. + C_2\left(5 + 10\lambda_1 + 10\lambda_1^2\Delta\right) k^2\right] \int_{n\Delta}^{t} \mathbb{E}\left|Q_n\right|^2 \mathrm{d}s \\
&\qquad + \left(5 + 10\lambda_1 + 10\lambda_1^2\Delta\right) k^2 C_3 \Delta + 5\Delta \int_{n\Delta}^{t} \left|\frac{\partial Q_s}{\partial a}\right|^2 \mathrm{d}s \\
&\qquad + \left(10\lambda_1 + 10\lambda_1^2\Delta\right) k^2 C\Delta \\
&\quad \le \left[10\Delta k^2 + 5\overline{\mu}^2\Delta + C_2\left(5 + 10\lambda_1 + 10\lambda_1^2\Delta\right) k^2\right] C_4 \Delta \\
&\qquad + \left(5 + 10\lambda_1 + 10\lambda_1^2\Delta\right) k^2 C_3 \Delta + 5\Delta \int_{n\Delta}^{t} \left|\frac{\partial Q_s}{\partial a}\right|^2 \mathrm{d}s \\
&\qquad + \left(10\lambda_1 + 10\lambda_1^2\Delta\right) k^2 C\Delta \le C_5 \Delta,
\end{aligned} \tag{45}$$

where

$$\begin{aligned}
C_5 &= \left[10k^2 + 5\overline{\mu}^2 + C_2\left(5 + 10\lambda_1 + 10\lambda_1^2\right) k^2\right] C_4 \\
&\quad + \left(10\lambda_1 + 10\lambda_1^2\Delta\right) k^2 C + \left(5 + 10\lambda_1 + 10\lambda_1^2\right) k^2 C_3 \\
&\quad + 5 \int_{n\Delta}^{t} \left|\frac{\partial Q_s}{\partial a}\right|^2 \mathrm{d}s.
\end{aligned} \tag{46}$$

Thus we can prove (37), and a similar analysis gives the proof of (38). □

Lemma 8. *Under conditions* ($\mathbb{H}1$)–($\mathbb{H}4$), *for any* $p \ge 2$, *there exists a positive constant* C_7 *such that*

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left|Q_t\right|^p\right] \le C_7. \tag{47}$$

Proof. The proof is similar to that of Lemma 4. □

5. Main Result

Now we use the above lemmas to prove a strong convergent result.

Theorem 9. *Under condition* ($\mathbb{H}1$)–($\mathbb{H}4$), *let* $0 < \theta < 1$, $0 < \Delta < \min\{1, 1/\theta(k + \overline{\mu}), 1/2\sqrt{3(\overline{\mu}^2 + k^2)}\}$; *then there exists a constant* $C_8 > 0$ *such that*

$$\mathbb{E}\left[\sup_{0\le t\le T} \left|K_{t\wedge v_n} - Q_{t\wedge v_n}\right|^2\right] \le C_8 \Delta^{1-(1/q)}. \tag{48}$$

Proof. From (2) and (17), we have

$$\begin{aligned}
K_t - Q_t &= -\int_0^t \frac{\partial (K_s - Q_s)}{\partial a} \mathrm{d}s \\
&\quad - \int_0^t \mu(s,a)\left[(1-\theta)(K_s - Z_1(s)) + \theta(K_s - Z_2(s))\right]\mathrm{d}s \\
&\quad + \int_0^t \left[(1-\theta)(f(s,K_s) - f(s,Z_1(s))) + \theta(f(s,K_s) - f(s,Z_2(s)))\right]\mathrm{d}s \\
&\quad + \int_0^t \left[g(s,K_s) - g(s,Z_2(s))\right]\mathrm{d}W_s \\
&\quad + \int_0^t \left[h(s,K_s,\gamma(s)) - h(s,Z_2(s),\overline{\gamma}(s))\right]\mathrm{d}N_s.
\end{aligned} \tag{49}$$

Using the generalized Itô formula it yields

$$\begin{aligned}
|K_t - Q_t|^2 &= 2\int_0^t \left\langle K_s - Q_s, -\frac{\partial(K_s - Q_s)}{\partial a}\right\rangle \mathrm{d}s \\
&\quad - 2\int_0^t (K_s - Q_s, \mu(s,a) \\
&\qquad \times \left[(1-\theta)(K_s - Z_1(s)) + \theta(K_s - Z_2(s))\right]\mathrm{d}s) \\
&\quad + \int_0^t (K_s - Q_s, \left[(1-\theta)(f(s,K_s) - f(s,Z_1(s))) \right.\\
&\qquad \left. + \theta(f(s,P_s) - f(s,Z_2(s)))\right]\mathrm{d}s) \\
&\quad + \int_0^t \|g(s,K_s) - g(s,Z_2(s))\|_2^2 \mathrm{d}s \\
&\quad + 2\int_0^t (K_s - Q_s, \left[g(s,K_s) - g(s,Z_2(s))\right]\mathrm{d}W_s) \\
&\quad + 2\int_0^t (K_s - Q_s, h(s,K_s,\gamma(s)) \\
&\qquad - h(s,Z_2(s),\overline{\gamma}(s)))\,\mathrm{d}\widetilde{N}_s \\
&\quad + 2\lambda_1 \int_0^t (K_s - Q_s, h(s,K_s,\gamma(s)) \\
&\qquad - h(s,Z_2(s),\overline{\gamma}(s)))\,\mathrm{d}s \\
&\quad + \int_0^t |h(s,K_s,\gamma(s)) - h(s,Z_2(s),\overline{\gamma}(s))|^2 \mathrm{d}\widetilde{N}_s \\
&\quad + \lambda_1 \int_0^t |h(s,K_s,\gamma(s)) - h(s,Z_2(s),\overline{\gamma}(s))|^2 \mathrm{d}s.
\end{aligned} \tag{50}$$

By Cauchy-Schwartz inequality and condition ($\mathbb{H}2$), we have

$$\begin{aligned}
|K_t - Q_t|^2 &\le \left(A^2 F_1^2 \eta^2 + \mu_0 + \lambda_1 + 1\right)\int_0^t |K_s - Q_s|^2 \mathrm{d}s \\
&\quad + 2\left(K^2 + \mu_0\right)\int_0^t |K_s - Z_1(s)|^2 \mathrm{d}s \\
&\quad + 2\lambda_1 K^2 \int_0^t |\gamma(s) - \overline{\gamma}(s)|^2 \mathrm{d}s \\
&\quad + \left(3K^2 + 2\lambda_1 K^2 + 2\mu_0\right)\int_0^t |K_s - Z_2(s)|^2 \mathrm{d}s \\
&\quad + 2\int_0^t (K_s - Q_s, \left[g(s,K_s) - g(s,Z_2(s))\right]\mathrm{d}W_s) \\
&\quad + 2\int_0^t (K_s - Q_s, h(s,K_s,\gamma(s)) \\
&\qquad - h(s,Z_2(s),\overline{\gamma}(s)))\,\mathrm{d}\widetilde{N}_s \\
&\quad + \int_0^t |h(s,K_s,\gamma(s)) - h(s,Z_2(s),\overline{\gamma}(s))|^2 \mathrm{d}\widetilde{N}_s.
\end{aligned} \tag{51}$$

Hence for any $t \in [0,T]$

$$\begin{aligned}
\mathbb{E}\sup_{s\in[0,t]} \left|K_{s\wedge v_n} - Q_{s\wedge v_n}\right|^2 &\le \left(A^2 F_1^2 \eta^2 + \mu_0 + \lambda_1 + 1\right) \\
&\quad \times \int_0^{t\wedge v_n} \mathbb{E}\sup_{r\in[0,s]} \left|K_{r\wedge v_n} - Q_{r\wedge v_n}\right|^2 \mathrm{d}s \\
&\quad + 2\left(k^2 + \mu_0\right)\int_0^{t\wedge v_n} \mathbb{E}\|K_s - Z_1(s)\|_c^2 \mathrm{d}s \\
&\quad + 2\lambda_1 k^2 C \Delta^{1-(1/q)} \\
&\quad + \left(3k^2 + 2\lambda_1 k^2 + 2\mu_0\right)\int_0^{t\wedge v_n} \mathbb{E}\|K_s - Z_1(s)\|_c^2 \mathrm{d}s \\
&\quad + 2\mathbb{E}\sup_{s\in[0,t]} \int_0^{s\wedge v_n} (K_r - Q_r, \left[g(r,K_r) \right.\\
&\qquad \left. - g(r,Z_2(r))\right]\mathrm{d}W_r) \\
&\quad + 2\mathbb{E}\sup_{s\in[0,t]} \int_0^{s\wedge v_n} (K_r - Q_r, h(r,K_r,\gamma(r)) \\
&\qquad - h(s,Z_2(r),\overline{\gamma}(r)))\,\mathrm{d}\widetilde{N}_r \\
&\quad + \mathbb{E}\sup_{s\in[0,t]} \int_0^{s\wedge v_n} |h(r,K_r,\gamma(r)) \\
&\qquad - h(r,Z_2(r),\overline{\gamma}(r))|^2 \mathrm{d}\widetilde{N}_r.
\end{aligned} \tag{52}$$

Now using the Burkholder-Davis-Gundy inequality and Lemma 2, we have

$$
\begin{aligned}
&2\mathbb{E}\left[\sup_{s\in[0,t]}\int_0^{s\wedge v_n}\left(K_r-Q_r,\left[g\left(r,K_r\right)-g\left(r,Z_2\left(r\right)\right)\right]\mathrm{d}W_r\right)\right]\\
&\quad\le 8\mathbb{E}\left[\sup_{s\in[0,t]}\left|K_{s\wedge v_n}-Q_{s\wedge v_n}\right|\right.\\
&\qquad\left.\times\left(\int_0^{t\wedge v_n}\left\|g\left(s,K_s\right)-g\left(t,Z_2(s)\right)\right\|_2^2\mathrm{d}s\right)^{1/2}\right]\\
&\quad\le \frac{1}{4}\mathbb{E}\left[\sup_{s\in[0,t]}\left|K_{s\wedge v_n}-Q_{s\wedge v_n}\right|^2\right]\\
&\qquad+k_1\int_0^{t\wedge v_n}\mathbb{E}\left\|K_s-Z_2(s)\right\|^2\mathrm{d}s,\\
&2\mathbb{E}\left[\sup_{s\in[0,t]}\int_0^{s\wedge v_n}\left(K_r-Q_r,\left(h\left(r,K_r\right)-h\left(s,Z_2\left(r\right)\right)\right)\right)\mathrm{d}\widetilde{N}_r\right]\\
&\quad\le 8\lambda_1\mathbb{E}\left[\sup_{s\in[0,t]}\left|K_{s\wedge v_n}-Q_{s\wedge v_n}\right|\right.\\
&\qquad\times\left(\int_0^{t\wedge v_n}\left|h\left(s,K_s,\gamma\left(s\right)\right)\right.\right.\\
&\qquad\left.\left.\left.-h\left(s,Z_2\left(s\right),\overline{\gamma}\left(s\right)\right)\right|^2\mathrm{d}s\right)^{1/2}\right]\\
&\quad\le \frac{1}{4}\mathbb{E}\left[\sup_{s\in[0,t]}\left|K_{s\wedge v_n}-Q_{s\wedge v_n}\right|^2\right]\\
&\qquad+k_2\int_0^{t\wedge v_n}\mathbb{E}\left\|K_s-Z_2(s)\right\|^2+\mathbb{E}\left|\gamma(s)-\overline{\gamma}(s)\right|^2\mathrm{d}s\\
&\quad\le \frac{1}{4}\mathbb{E}\left[\sup_{s\in[0,t]}\left|K_{s\wedge v_n}-Q_{s\wedge v_n}\right|^2\right]\\
&\qquad+k_2\int_0^{t\wedge v_n}\mathbb{E}\left\|K_s-Z_2(s)\right\|^2\mathrm{d}s+k_2C\Delta^{1-(1/q)},\\
&\mathbb{E}\left[\sup_{s\in[0,t]}\int_0^{s\wedge v_n}\left|h\left(r,K_r,\gamma\left(r\right)\right)-h\left(r,Z_2\left(r\right),\overline{\gamma}\left(r\right)\right)\right|^2\mathrm{d}\widetilde{N}_r\right]\\
&\quad\le 4\lambda_1\int_0^{t\wedge v_n}\mathbb{E}\left|h\left(s,K_s,\gamma\left(s\right)\right)-h\left(s,Z_2\left(s\right),\overline{\gamma}\left(s\right)\right)\right|^2\mathrm{d}s\\
&\quad\le 4\lambda_1k^2\int_0^{t\wedge v_n}\mathbb{E}\left|K_s-Z_2(s)\right|^2+\mathbb{E}\left|\gamma(s)-\overline{\gamma}\left(s\right)\right|^2\mathrm{d}s\\
&\quad\le k_3\int_0^{t\wedge v_n}\mathbb{E}\left\|K_s-Z_2(s)\right\|^2\mathrm{d}s+k_3C\Delta^{1-(1/q)},
\end{aligned}
\tag{53}
$$

where k_1, k_2, k_3 are positive constants. Therefore, inserting (53) into (52) it has

$$
\begin{aligned}
&\mathbb{E}\sup_{s\in[0,t]}\left|K_{s\wedge v_n}-Q_{s\wedge v_n}\right|^2\\
&\quad\le\left(A^2F_1^2\eta^2+\mu_0+\lambda_1+1\right)\\
&\qquad\times\int_0^{t\wedge v_n}\mathbb{E}\sup_{r\in[0,s]}\left|K_{r\wedge v_n}-Q_{r\wedge v_n}\right|^2\mathrm{d}s\\
&\qquad+2\left(k^2+\mu_0\right)\int_0^{t\wedge v_n}\mathbb{E}\left\|K_s-Z_1(s)\right\|_c^2\mathrm{d}s\\
&\qquad+2\lambda_1k^2C\Delta^{1-(1/q)}\\
&\qquad+\left(3k^2+2\lambda_1k^2+2\mu_0\right)\int_0^{t\wedge v_n}\mathbb{E}\left\|K_s-Z_2(s)\right\|_c^2\mathrm{d}s\\
&\qquad+\left(k_1+k_2+k_3\right)\int_0^{t\wedge v_n}\mathbb{E}\left\|K_s-Q_s\right\|_c^2\mathrm{d}s\\
&\qquad+\left(k_2+k_3\right)C\Delta^{1-(1/q)}+\frac{1}{2}\mathbb{E}\left[\sup_{s\in[0,t]}\left|P_{s\wedge v_n}-Q_{s\wedge v_n}\right|^2\right].
\end{aligned}
\tag{54}
$$

Then using Lemma 7, we have

$$
\begin{aligned}
&\mathbb{E}\sup_{s\in[0,t]}\left|P_{s\wedge v_n}-Q_{s\wedge v_n}\right|^2\\
&\quad\le\left(A^2F_1^2\eta^2+\mu_0+\lambda_1+1\right)\\
&\qquad\times\int_0^{t\wedge v_n}\mathbb{E}\sup_{r\in[0,s]}\left|P_{r\wedge v_n}-Q_{r\wedge v_n}\right|^2\mathrm{d}s\\
&\qquad+4\left(k^2+\mu_0\right)\int_0^{t\wedge v_n}\mathbb{E}\left\|P_s-Q_s\right\|_c^2\mathrm{d}s\\
&\qquad+4\left(k^2+\mu_0\right)C_5\Delta\\
&\qquad+2\left(3k^2+2\lambda_1k^2+2\mu_0+k_1+k_2+k_3\right)\\
&\qquad\times\int_0^{t\wedge v_n}\mathbb{E}\left\|P_s-Q_s\right\|_c^2\mathrm{d}s\\
&\qquad+2\left(3k^2+2\lambda_1k^2+2\mu_0+k_1+k_2+k_3\right)C_6\Delta\\
&\qquad+\left(2\lambda_1k^2+k_2+k_3\right)C\Delta^{1-(1/q)}\\
&\qquad+\frac{1}{2}\mathbb{E}\left[\sup_{s\in[0,t]}\left|P_{s\wedge v_n}-Q_{s\wedge v_n}\right|^2\right].
\end{aligned}
\tag{55}
$$

That is,

$$
\begin{aligned}
&\mathbb{E}\sup_{s\in[0,t]}\left|P_{s\wedge v_n}-Q_{s\wedge v_n}\right|^2\\
&\quad\le 2\left(A^2F_1^2\eta^2+\mu_0+\lambda_1+1\right)\\
&\qquad\times\int_0^{t\wedge v_n}\mathbb{E}\sup_{r\in[0,s]}\left|P_{r\wedge v_n}-Q_{r\wedge v_n}\right|^2\mathrm{d}s\\
&\qquad+8\left(k^2+\mu_0\right)\int_0^{t\wedge v_n}\mathbb{E}\sup_{r\in[0,s]}\left|P_{r\wedge v_n}-Q_{r\wedge v_n}\right|^2\mathrm{d}s
\end{aligned}
$$

$$\begin{aligned}
&+4\left(3k^2+2\lambda_1K^2+2\mu_0+k_1+k_2+k_3\right)\\
&\quad\times\int_0^{t\wedge v_n}\mathbb{E}\sup_{r\in[0,s]}\left|P_{r\wedge v_n}-Q_{r\wedge v_n}\right|^2 ds\\
&+8\left(k^2+\mu_0\right)C_5\Delta\\
&+4\left(3k^2+2\lambda_1k^2+2\mu_0+k_1+k_2+k_3\right)C_6\Delta\\
&+2\left(2\lambda_1k^2+k_2+k_3\right)C\Delta^{1-(1/q)}\\
&\le d_1\Delta^{1-(1/q)}+d_2\int_0^t\mathbb{E}\sup_{r\in[0,s]}\left|P_{r\wedge v_n}-Q_{r\wedge v_n}\right|^2 ds,
\end{aligned}\tag{56}$$

where

$$\begin{aligned}
d_1&=8\left(k^2+\mu_0\right)C_5\\
&\quad+4\left(3k^2+2\lambda_1k^2+2\mu_0+k_1+k_2+k_3\right)C_6\\
&\quad+2\left(2\lambda_1k^2+k_2+k_3\right)C\Delta^{1-(1/q)},
\end{aligned}\tag{57}$$

$$\begin{aligned}
d_2&=2\left(A^2F_1^2\eta^2+\mu_0+\lambda_1+1\right)+8\left(k^2+\mu_0\right)\\
&\quad+4\left(3k^2+2\lambda_1k^2+2\mu_0+k_1+k_2+k_3\right).
\end{aligned}$$

Using the Gronwall inequality, we have

$$\mathbb{E}\sup_{s\in[0,t]}\left|P_{s\wedge v_n}-Q_{s\wedge v_n}\right|^2\le d_1\exp\left(d_2T\right)\Delta:=C_8\Delta^{1-(1/q)}.\tag{58}$$

Since for any $t\in[0,T]$, thus we have

$$\mathbb{E}\sup_{t\in[0,T]}\left|P_{t\wedge v_n}-Q_{t\wedge v_n}\right|^2\le C_8\Delta^{1-(1/q)}.\tag{59}$$

The proof is completed. □

Theorem 10. *Under conditions* ($\mathbb{H}$1)–($\mathbb{H}$4), *let* $0<\theta<1$, $0<\Delta<\min\{1,1/\theta(k+\overline{\mu}),1/2\sqrt{3(\overline{\mu}^2+k^2)}\}$; *then there exists a constant* $C_9>0$ *such that*

$$\mathbb{E}\left[\sup_{0\le t\le T}\left|K_t-Q_t\right|^2\right]\le C_9\Delta t+o(\Delta).\tag{60}$$

Proof. Let $e(t)=K_t-Q_t$; it is easy to see that

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\le t\le T}|e(t)|^2\right]\\
&\quad=\mathbb{E}\left[\sup_{0\le t\le T}|e(t)|^2I_{\{\tau_n>T,\sigma_n>T\}}\right]\\
&\qquad+\mathbb{E}\left[\sup_{0\le t\le T}|e(t)|^2I_{\{\tau_n\le T\text{ or }\sigma_n\le T\}}\right]\\
&\quad=\mathbb{E}\left[\sup_{0\le t\le T}|e(t)|^2I_{\{v_n>T\}}\right]\\
&\qquad+\mathbb{E}\left[\sup_{0\le t\le T}|e(t)|^2I_{\{\tau_n\le T\text{ or }\sigma_n\le T\}}\right]\\
&\quad\le\mathbb{E}\left[\sup_{0\le t\le T}|e(t\wedge v_n)|^2\right]\\
&\qquad+\mathbb{E}\left[\sup_{0\le t\le T}|e(t)|^2I_{\{\tau_n\le T\text{ or }\sigma_n\le T\}}\right].
\end{aligned}\tag{61}$$

Recall the Young inequality: for $(1/p)+(1/q)=1(p,q>0)$, we have

$$\begin{aligned}
ab&\le a\delta^{1/p}\frac{b}{\delta^{1/p}}\\
&\le\frac{\left(a\delta^{1/p}\right)^p}{p}+\frac{b^q}{q\delta^{q/p}}=\frac{a^p\delta}{p}+\frac{b^q}{q\delta^{q/p}},\quad\forall a,b,\delta>0.
\end{aligned}\tag{62}$$

Let $p=2,\delta=\Delta$; we have

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\le t\le T}|e(t)|^2I_{\{\tau_n\le T\text{ or }\sigma_n\le T\}}\right]\\
&\quad\le\frac{\Delta}{2}\mathbb{E}\left[\sup_{0\le t\le T}|e(t)|^4\right]+\frac{1}{2\Delta}P\{\tau_n\le T\text{ or }\sigma_n\le T\}.
\end{aligned}\tag{63}$$

Note

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\le t\le T}|e(t)|^4\right]&\le 8\left(\mathbb{E}\left[\sup_{0\le t\le T}\left|P_t\right|^4\right]+\mathbb{E}\left[\sup_{0\le t\le T}\left|Q_t\right|^4\right]\right)\\
&\le 8\left(C_1+C_7\right).
\end{aligned}\tag{64}$$

By Lemma 4, then

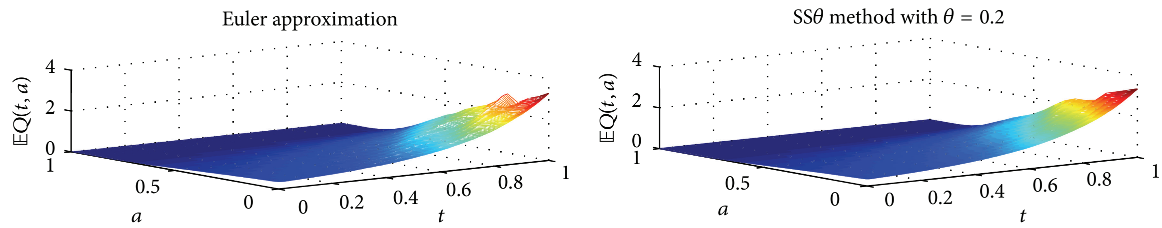
$$P\{\tau_n\le T\}=\mathbb{E}\left[I_{\{\tau_n\le T\}}\frac{\left|P_{\tau_n}\right|^p}{n^4}\right]\le\frac{1}{n^4}\mathbb{E}\left[\sup_{0\le t\le T}\left|P_t\right|^4\right]\le\frac{C_1}{n^4};\tag{65}$$

similarly, the result can be derived for σ_n

$$P\{\sigma_n\le T\}=\mathbb{E}\left[I_{\{\tau_n\le T\}}\frac{\left|Q_{\tau_n}\right|^p}{n^4}\right]\le\frac{1}{n^4}\mathbb{E}\left[\sup_{0\le t\le T}\left|Q_t\right|^4\right]\le\frac{C_7}{n^4},\tag{66}$$

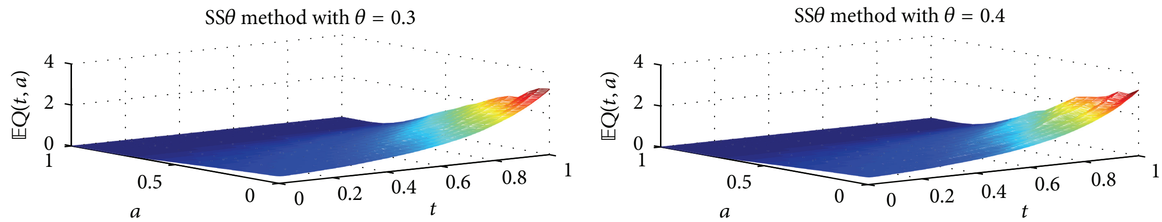
so that

$$\begin{aligned}
P\{\sigma_d\le T\text{ or }\nu_d\le T\}&\le P\{\sigma_d\le T\}+P\{\nu_d\le T\}\\
&\le\frac{\left(C_1+C_7\right)}{n^4}.
\end{aligned}\tag{67}$$



(a) (b)

Figure 1: Expectation of numerical solution with 1000 tests, where $\Delta = 0.005$ and $\Delta a = 0.05$.



(a) (b)

Figure 2: Expectation of numerical solution with 1000 tests, where $\Delta = 0.005$ and $\Delta a = 0.05$.

Then

$$\mathbb{E}\left[\sup_{0\le t\le T}|e(t)|^2 I_{\{\tau_n\le T \text{ or } \sigma_n\le T\}}\right] \le 4\Delta(C_1+C_7)+\frac{(C_1+C_7)}{2\Delta n^4}. \tag{68}$$

By Theorem 9, (61) becomes

$$\mathbb{E}\left[\sup_{0\le t\le T}|e(t)|^2\right] \le C_8\Delta + o(\Delta) + 4\Delta(C_1+C_7)+\frac{(C_1+C_7)}{2\Delta n^4}. \tag{69}$$

Let $n \ge 2(\Delta^2)^{-1/4}$; then

$$\begin{aligned}\mathbb{E}&\left[\sup_{0\le t\le T}|e(t)|^2\right]\\ &\le C_8\Delta + o(\Delta)+4\Delta(C_1+C_7)+\Delta(C_1+C_7)\\ &:= C_9\Delta + o(\Delta),\end{aligned} \tag{70}$$

where $C_9 = 5(C_1+C_7)+C_8$. The proof is completed. □

Theorem 11. *Under the conditions* ($\mathbb{H}1$) – ($\mathbb{H}4$) *there exists a constant C such that $\mathbb{E}[|\gamma_i|^{2q}] \le C$ for some $q > 1$, the numerical approximate solution* (1) *will converge to the exact solution to* (1) *in the sense*

$$\lim_{\Delta\to 0}\mathbb{E}\left[\sup_{0\le t\le T}\left|K_t - Q_t\right|^2\right] = 0. \tag{71}$$

Proof. The proof is easily deduced from Theorem 10. □

6. Numerical Example

In this section, we consider the following stochastic age-dependent capital system with random jump magnitudes age-dependent in [10]

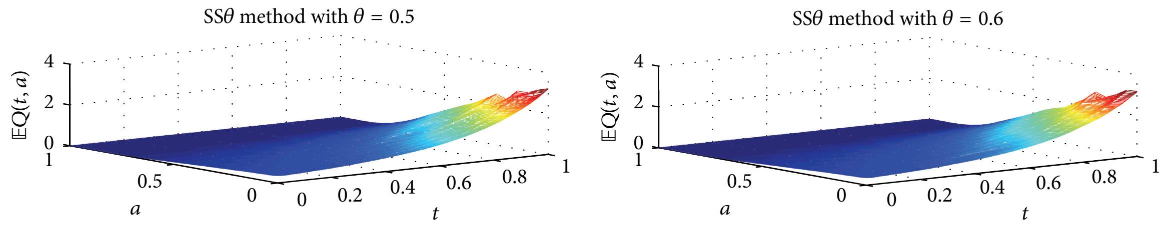
$$\begin{aligned} dK(t,a) &= -\frac{1}{(1-a)^2}K dt + 2tK dt - tK dW(t)\\ &\quad + \overline{\gamma}(t)K dN(t), \quad \text{in } D,\\ K(t,0) &= \frac{t^2}{(1-t)^2}\int_0^1 K(t,a)\,da, \quad \text{in } t\in[0,T],\\ K(0,a) &= e^{-1/(1-a)}, \quad \text{in } a\in(0,A),\\ \mathbb{N}(t) &= \int_0^1 K(t,a)\,da, \quad \text{in } t\in[0,T], \end{aligned} \tag{72}$$

where $W(t)$ is a scalar Brownian motion and $N(t)$ is a scalar Poisson process with intensity $\lambda_1 = 1$. Let $\overline{\gamma}(t)$ be defined in (16). Take $T = 1$, $A = 1$, and $\overline{\gamma}$ to be a lognormal random variable such that $\mathbb{E}[\overline{\gamma}] = 0.2$ and $\mathbb{E}[\overline{\gamma}^2] = 0.25$.

It is easy to verify that the conditions ($\mathbb{H}1$)–($\mathbb{H}4$) are satisfied. Then the approximate solution will converge to the true solution of (1) for any $(t,a)\in(0,1)\times(0,1)$ in the sense of Theorem 11. Obviously, $K(t,a)$ in (72) cannot be solved explicitly. It is necessary to know numerical approximation $Q(t,a)$ of $K(t,a)$.

Now we fix step sizes $\Delta = 0.005$ and $\Delta a = 0.05$ and change the parameter θ in all figures. Then we compare the expectation of the numerical solution, where $\mathbb{E}[Q(t,a)] = (1/1000)\sum_{n=1}^{1000} Q_n(t,a)$.

In Figure 1, we show the expectation of numerical solution of the system (72) by Euler method in [10] and split-step θ-method with $\theta = 0.2$, respectively. From the figure,



(a)

(b)

FIGURE 3: Expectation of numerical solution with 1000 tests, where $\Delta = 0.005$ and $\Delta a = 0.05$.

we can see that the split-step θ-method also reveals the age-dependent capital system tendency.

In Figures 2 and 3, the four pictures are simulation for numerical solution $Q_n(t, a)$ of the system (72) by split-step θ-method with $\theta = 0.3$, 0.4, 0.5, and 0.6, respectively.

7. Conclusion

Due to the complexity of this model, the comparison between the Euler approximation and split-step methods is not possible in this paper. Similarly, error analysis is also not obtained in this paper due to unavailability of closed form solutions. They will be discussed in the future researches. But the visualization of Figures 1, 2, and 3 provides the information about the split-step methods which coincide with the Euler approximation. The comparison between Euler approximation and split-step methods in the stochastic differential equations in [11–13] shows that the split-step methods are better than the Euler approximation. Similarly, we believe that this new approximation of the system (1) in this paper is a good approximation when compared with the Euler approximation.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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