Research Article

An Offline Formulation of MPC for LPV Systems Using Linear Matrix Inequalities

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An offline model predictive control (MPC) algorithm for linear parameter varying (LPV) systems is presented. The main contribution is to develop an offline MPC algorithm for LPV systems that can deal with both time-varying scheduling parameter and persistent disturbance. The norm-bounding technique is used to derive an offline MPC algorithm based on the parameter-dependent state feedback control law and the parameter-dependent Lyapunov functions. The online computational time is reduced by solving offline the linear matrix inequality (LMI) optimization problems to find the sequences of explicit state feedback control laws. At each sampling instant, a parameter-dependent state feedback control law is computed by linear interpolation between the precomputed state feedback control laws. The algorithm is illustrated with two examples. The results show that robust stability can be ensured in the presence of both time-varying scheduling parameter and persistent disturbance.

1. Introduction

Model predictive control (MPC), also known as receding horizon control, is an effective multivariable control algorithm in which a dynamic optimization problem is solved online. At each sampling time, MPC solves a finite horizon optimal control problem based on an explicit model of the plant. Although an optimal control sequence is determined, only the first control action is applied to the plant. Due to its ability to guarantee optimality while ensuring the satisfaction of constraints on input and state, MPC has received much interest in both industry and academia [1–3].

An explicit linear model is typically used in the MPC formulation because the online optimization can be reduced to either a linear program or a quadratic program. Since an industrial process is inherently nonlinear to a certain extent, the control performance of linear MPC can deteriorate as operating conditions significantly change [4, 5]. For this reason, MPC for linear parameter varying (LPV) systems has been widely developed. LPV systems are linear systems whose dynamics depend on the scheduling parameter that can be measured online. The analysis and synthesis of LPV systems play an important role in control theory since nonlinear systems can be dealt within the framework of LPV systems [6, 7].

In the context of MPC for LPV systems, one of the main approaches is to solve a semidefinite problem under linear matrix inequality (LMI) [8]. Quasi-Min-Max MPC algorithm for LPV systems was developed by Lu and Arkun [9]. Although the scheduling parameter is included in the controller design, it is assumed that there is no disturbance present in the problem formulation, so the algorithm cannot deal with disturbance. MPC for LPV systems using parameter-dependent Lyapunov functions was developed by [10]. It is shown that the proposed MPC algorithm can achieve less conservative results as compared with a robust MPC algorithm derived by using a single Lyapunov function [4]. However, this algorithm includes only time-varying scheduling parameter in the problem formulation so it cannot ensure robust stability in the presence of disturbance. The

bound on the rate of variation of the scheduling parameter can also be taken into account in the MPC formulation [11, 12]. However, this technique is not applicable to the case where there is the disturbance acting on the system.

MPC for LPV systems can be designed by using ellipsoidal set prediction [13, 14]. At each sampling instant, the predicted future states on the finite horizon are bounded by using a sequence of ellipsoids. The terminal ellipsoid is contained in a target set guaranteeing stability. The main drawback of this approach lies in the fact that the computational load increases with the length of prediction horizon. In order to reduce online computational time, an offline MPC algorithm for LPV systems was developed [15]. The real-time state feedback gain is calculated by linear interpolation between the precomputed state feedback gains. Although the online computational time is significantly reduced, the disturbance is not taken into account in the offline MPC formulation so robust stability cannot be guaranteed in the presence of disturbance. Explicit MPC for LPV systems was proposed by Besselmann et al. [16]. Only time-varying scheduling parameter is included in the problem formulation and it is also assumed that there is no disturbance. In Ding [17, 18], both time-varying scheduling parameter and disturbance are included in the problem formulation. However, the optimization problem contains a lot of decision variables and constraints, so the algorithm is computationally prohibitive in practical situations.

In the context of tube-based MPC [19–21], the disturbance is explicitly taken into account in the MPC design. The basic concept of robust tube-based MPC is to compute the region around the nominal prediction that contains any possible states of the uncertain system. One of the main advantages is that its online computational complexity increases only linearly with the prediction horizon. However, the time-varying scheduling parameter is not incorporated into the MPC design, so robust stability cannot be ensured in the presence of parametric uncertainty.

In this paper, an offline MPC algorithm for LPV systems is presented. Unlike Wan and Kothare [22] where only time-varying scheduling parameter is considered in the offline MPC formulation, the main contribution of this paper is to develop an offline MPC algorithm for LPV systems that can deal with both persistent disturbance and time-varying scheduling parameter. The norm-bounding technique [23] is used to derive an offline MPC algorithm based on the parameter-dependent state feedback control law and the parameter-dependent Lyapunov functions. Most of the optimization problems are solved offline, so the developed MPC algorithm can be applied to fast processes. This article is organized as follows. Section 2 concerns with problem statement and control objectives. The proposed algorithm is described in Section 3. In Section 4, the effectiveness of the proposed MPC algorithm is illustrated. Finally, Section 5 presents some conclusions.

Notation. For any vector x and positive-definite matrix P, $\|x\|_P^2 = x^T P x$. $x(k)$ is the state measured at real-time k and $x(k + i \mid k)$ is the state at prediction time $k + i$ predicted at real-time k. The symbol $*$ denotes symmetric blocks

in matrices. An element belonging to a convex hull Co$\{\cdot\}$ means that it is a convex combination of the elements in $\{\cdot\}$. The time-dependence (k) of the MPC decision variables is often dropped for simplicity. I is the identity matrix with appropriate dimension.

2. Problem Statement

Consider the following discrete-time LPV system:

$$x(k + 1) = A(p(k))x(k) + Bu(k) + D(\alpha(k))v(k), \quad (1)$$

$$y(k) = C(p(k))x(k), \quad (2)$$

where $x(k) \in \mathfrak{R}^{n_x}$ denotes the state, $u(k) \in \mathfrak{R}^{n_u}$ denotes the control input, $v(k) \in \mathfrak{R}^{n_v}$ denotes the disturbance, and $y(k) \in \mathfrak{R}^{n_y}$ denotes the output. The superscripts n_x, n_u, n_v, and n_y denote the dimensions of $x(k)$, $u(k)$, $v(k)$, and $y(k)$, respectively. It is assumed that the time-varying scheduling parameter $p(k) = [p_1(k), p_2(k), \ldots, p_L(k)] \in \mathfrak{R}^L$ is measurable at each sampling time but its future values are considered to be uncertain. The input and output constraints are

$$|u(k)| \le \bar{u}, \quad \bar{u}_h > 0, \quad h \in \{1, 2, \ldots, n_u\}, \quad (3)$$

$$|y(k)| \le \bar{y}, \quad \bar{y}_r > 0, \quad r \in \{1, 2, \ldots, n_y\}. \quad (4)$$

Any $A(p(k))$ and $C(p(k))$ belong to a convex polytope Ω defined by

$$\Omega = \text{Co}\{[A_1, C_1], [A_2, C_2], \ldots, [A_L, C_L]\}, \quad (5)$$

so they can be written as

$$[A(p(k)), C(p(k))] = \sum_{j=1}^{L} p_j(k)[A_j, C_j],$$

$$\sum_{j=1}^{L} p_j(k) = 1, \quad 0 \le p_j(k) \le 1, \quad (6)$$

where $[A_j, C_j]$ are the vertices of Ω and L is the number of the vertices of Ω. Any $D(\alpha(k))$ belongs to a convex polytope Ω_D defined by

$$\Omega_D = \text{Co}\{D_1, D_2, \ldots, D_{n_D}\}, \quad (7)$$

so it can be written as

$$D(\alpha(k)) = \sum_{t=1}^{n_D} \alpha_t(k)D_t,$$

$$\sum_{t=1}^{n_D} \alpha_t(k) = 1, \quad 0 \le \alpha_t(k) \le 1, \quad (8)$$

where D_t are the vertices of Ω_D, n_D is the number of the vertices of Ω_D, and $\alpha_t(k)$ is the time-varying parameter that is not necessary to be measurable. The disturbance $v(k)$ is unmeasurable and persistent. It is assumed to lie in a convex polytope Ω_v defined by

$$\Omega_v = \mathrm{Co}\left\{v_1, v_2, \ldots, v_{m_v}\right\}, \tag{9}$$

where m_v is the number of the vertices of Ω_v. The objective is to find a parameter-dependent state feedback control law $K(p(k+i)) = \sum_{j=1}^{L} p_j(k+i)K_j$, where $K_j, j \in \{1,2,\ldots,L\}$ are the state feedback gains corresponding to the vertices of Ω that is able to guarantee both robust stability and constraint satisfaction within a positively invariant set.

Definition 1. The set Z is said to be positively invariant set if it has the property that whenever the current state is contained in this set $x(k) \in Z$, all possible predicted states must be contained in this set $x(k+i \mid k) \in Z$ for all admissible realizations of $p(k+i)$, $\alpha(k+i)$ and $v(k+i)$, $i \geq 0$.

Remark 2. In this paper, the positively invariant set is $Z = \{x(k+i \mid k) \in \mathfrak{R}^{n_x} \mid \|x(k+i \mid k)\|^2_{P(p(k+i))} \leq \gamma\}$, where $P(p(k+i)) = \sum_{j=1}^{L} p_j(k+i)P_j$ is a parameter-dependent Lyapunov matrix, $P_j, j \in \{1,2,\ldots,L\}$ are the Lyapunov matrices corresponding to the vertices of Ω, and γ is an upper bound on the infinite horizon cost. From the convexity of the polytopic description, Z is an intersection area of $\varepsilon_j = \{x(k+i \mid k) \in \mathfrak{R}^{n_x} \mid \|x(k+i \mid k)\|^2_{P_j} \leq \gamma\}$.

Considering the discrete-time LPV system (1) to (9) at each sampling time k, a parameter-dependent state feedback control law $u(k+i \mid k) = K(p(k+i))x(k+i \mid k)$ that (i) minimizes an upper bound γ on $J_\infty(k)$ and (ii) guarantees both robust stability and robust constraint satisfaction within a positively invariant set Z can be calculated by solving the following optimization problem:

$$\min_{\gamma, K(p(k+i)), P(p(k+i)),\, i \geq 0} \quad \max_{[A(p(k+i)),C(p(k+i))] \in \Omega,}$$

$$D(\alpha(k+i)) \in \Omega_D, v(k+i) \in \Omega_v$$

$$J_\infty(k) = \sum_{i=0}^{\infty} \left[\left\|x_n(k+i \mid k)\right\|^2_\psi + \left\|K(p(k+i))x_n(k+i \mid k)\right\|^2_\sigma\right] \tag{10}$$

s.t.

$$\left\|x_n(k+i+1 \mid k)\right\|^2_{P(p(k+i+1))} - \left\|x_n(k+i \mid k)\right\|^2_{P(p(k+i))} \tag{11}$$

$$\leq -\left[\left\|x_n(k+i \mid k)\right\|^2_\psi + \left\|K(p(k+i))x_n(k+i \mid k)\right\|^2_\sigma\right]$$

$$\|x(k)\|^2_{P(p(k))} \leq \gamma, \quad x(k) = x_n(k \mid k) = x(k \mid k) \tag{12}$$

$$\|x(k+i+1 \mid k)\|^2_{P(p(k+i+1))} \leq \gamma \tag{13}$$

$$\left|K(p(k+i))x(k+i \mid k)\right| \leq \overline{u},$$

$$\overline{u}_h > 0, \quad h \in \{1,2,\ldots,n_u\} \tag{14}$$

$$\left|C(p(k+i+1))x(k+i+1 \mid k)\right| \leq \overline{y},$$

$$\overline{y}_r > 0, \quad r \in \{1,2,\ldots,n_y\}, \tag{15}$$

where $x_n(k+i+1 \mid k) = [A(p(k+i)) + BK(p(k+i))]x_n(k+i \mid k)$ is the predicted state not corrupted by disturbance, $x(k+i+1 \mid k) = [A(p(k+i)) + BK(p(k+i))]x(k+i \mid k) + D(\alpha(k+i))v(k+i)$ is the predicted state with disturbance, and ψ and σ are symmetric weighting matrices. The cost monotonicity is guaranteed by (11). A positively invariant set containing the state $x(k)$ at each sampling time is computed by (12). All predicted states $x(k+i+1 \mid k)$ are restricted to lie in a positively invariant set by (13). The input and output constraints are guaranteed by (14) and (15), respectively.

3. Offline MPC for LPV Systems with Persistent Disturbances

First of all, we will begin with the preliminary results of Wada et al. [10] where only time-varying scheduling parameter is considered in the problem formulation. Then, the proposed algorithm that can deal with both time-varying scheduling parameter and persistent disturbance will be developed. By following Wada et al. [10], (11) and (12) are satisfied and the cost monotonicity is guaranteed if there exist matrices Y_j, G_j, symmetric matrices Q_j and a positive scalar γ such that the following LMIs are satisfied:

$$\begin{bmatrix} G_j + G_j^T - Q_j & * & * & * \\ A_j G_j + BY_j & Q_l & * & * \\ \psi^{1/2}G_j & 0 & \gamma I & * \\ \sigma^{1/2}Y_j & 0 & 0 & \gamma I \end{bmatrix} \geq 0, \quad j, l \in \{1,2,\ldots,L\}, \tag{16}$$

$$\begin{bmatrix} 1 & * \\ x(k) & Q_j \end{bmatrix} \geq 0, \quad j \in \{1,2,\ldots,L\}. \tag{17}$$

Then, it follows that γ is the upper bound on $J_\infty(k)$. Moreover, a parameter-dependent state feedback gain is given by $K(p(k+i)) = \sum_{j=1}^{L} p_j(k+i)K_j, K_j = Y_j G_j^{-1}$.

Next, we will present the results of this paper that can deal with both time-varying scheduling parameter and disturbance.

Proposition 3 (Robust stability in the presence of both time-varying scheduling parameter and disturbance). (13) *is*

satisfied if there exist matrices Y_j, G_j and symmetric matrices Q_j such that the following LMIs are satisfied:

$$\begin{bmatrix} \theta \left(G_j + G_j^T - Q_j\right) & * \\ A_j G_j + BY_j & Q_l \end{bmatrix} \geq 0, \quad j,l \in \{1,2,\dots,L\}, \quad (18)$$

$$\begin{bmatrix} \xi & * \\ D_t v_s & Q_l \end{bmatrix} \geq 0, \quad t \in \{1,2,\dots,n_D\}, \ s \in \{1,2,\dots,m_v\},$$

$$l \in \{1,2,\dots,L\}, \ \xi = \left(1 - \theta^{1/2}\right)^2, \quad (19)$$

where $0 < \theta < 1$ is a prespecified scalar. Then, all predicted states are restricted to lie in a positively invariant set which is an intersection area of $\varepsilon_j = \{x(k+i \mid k) \in \Re^{n_x} \mid \|x(k+i \mid k)\|^2_{Q_j^{-1}} \leq 1\}$.

Proof. Equation (13) is guaranteed by (18) and (19). The proof detail can be found in Appendix A. $\square$

Remark 4. θ is a parameter that bounds disturbance-free state trajectories. The value of θ should be chosen such that $\theta \to 1$ as $v_s \to 0$.

Proposition 5 (Robust constraint satisfaction in the presence of both time-varying scheduling parameter and disturbance). *Robust constraint satisfaction is guaranteed if there exist matrices Y_j, G_j and symmetric matrices Q_j such that the following LMIs are satisfied:*

$$\begin{bmatrix} \chi & * \\ Y_j^T & G_j + G_j^T - Q_j \end{bmatrix} \geq 0,$$

$$\chi_{hh} \leq \bar{u}_h^2, \quad h \in \{1,2,\dots,n_u\}, \ j \in \{1,2,\dots,L\}, \quad (20)$$

$$\begin{bmatrix} \Gamma & * \\ \left(A_j G_j + BY_j\right)^T C_l^T & G_j + G_j^T - Q_j \end{bmatrix} \geq 0,$$

$$\Gamma_{rr} \leq \Xi_r,$$

$$\Xi_r = \left(\bar{y}_r - \Phi_r^{1/2}\right)^2, \quad r \in \{1,2,\dots,n_y\}, \quad j,l \in \{1,2,\dots,L\}, \quad (21)$$

where Φ_r is a scalar that can be obtained by solving the following optimization problem

$$\min_{\Phi_r} \quad \Phi_r \quad (22)$$

$$\text{s.t.} \quad \begin{bmatrix} \Phi_r & * \\ C_{r,l} D_t v_s & 1 \end{bmatrix} \geq 0,$$

$$r \in \{1,2,\dots,n_y\}, \ l \in \{1,2,\dots,L\}, \quad (23)$$

$$t \in \{1,2,\dots,n_D\}, \ s \in \{1,2,\dots,m_v\},$$

where $C_{r,l}$ is the rth row of C_l.

Proof. The input constraint (14) is guaranteed by (20). The proof details can be found in Appendix B. The output constraint (15) is guaranteed by (21). The proof details can be found in Appendix C. $\square$

By considering Propositions 3 and 5, a parameter-dependent state feedback control law that guarantees both robust stability and robust constraint satisfaction can be calculated. Consider the discrete-time LPV system (1) to (9) at each sampling instant k, a parameter-dependent state feedback control law $u(k+i \mid k) = K(p(k+i))x(k+i \mid k)$, $K(p(k+i)) = \sum_{j=1}^L p_j(k+i)K_j$, $K_j = Y_j G_j^{-1}$ that guarantees both robust stability and robust constraint satisfaction within a positively invariant set Z is obtained by solving the following optimization problem:

$$\min_{\gamma,Y_j,G_j,Q_j} \quad \gamma \quad (24)$$

$$\text{s.t.} \quad (16) \text{ to } (23). \quad (25)$$

It is computationally demanding to solve the optimization problem (24) at each sampling time. Inspired by Bumroongsri and Kheawhom [15], we propose an offline MPC algorithm for LPV systems that transfer most of the computations offline.

Algorithm 6.

Offline. Choose a sequence of states x_i, $i \in \{1,2,\dots,N\}$. For each i, substitute $x(k)$ in (17) by x_i and solve the optimization problem (24) to obtain the corresponding feedback gains $K_{i,j} = Y_{i,j} G_{i,j}^{-1}$ and ellipsoids $\varepsilon_{i,j} = \{x \in \Re^{n_x} \mid x^T Q_{i,j}^{-1} x \leq 1\}$, where the subscript i,j denote the solution of (24) for each x_i, $i \in \{1,2,\dots,N\}$ and vertices j, $j \in \{1,2,\dots,L\}$. Note that (i) the positively invariant set for each i is an intersection area of $\varepsilon_{i,j}$, $j \in \{1,2,\dots,L\}$ (ii) the inequalities (26) should be satisfied for each $i \neq N$ to ensure robust stability of the offline algorithm

$$\theta Q_{i,j}^{-1} - \left(A_j + BK_{i+1,j}\right)^T Q_{i,l}^{-1} \left(A_j + BK_{i+1,j}\right) \geq 0. \quad (26)$$

Remark 7. A sequence of states x_i, $i \in \{1,2,\dots,N\}$ should be chosen such that for each j, the ellipsoids $\varepsilon_{i,j}$, $\forall i \neq N$ obtained are nested ($\varepsilon_{i+1,j} \subset \varepsilon_{i,j}$, $Q_{i+1,j} < Q_{i,j}$). This is to guarantee that the state is kept within $\varepsilon_{i,j}$ and driven towards $\varepsilon_{i+1,j}$, and so on.

Online. At each sampling time k, measure $p(k)$ and $x(k)$. If $x(k) \in \varepsilon_{i,j}, x(k) \notin \varepsilon_{i+1,j}, j \in \{1,2,\dots,L\}, i \neq N$, adopt the following control law:

$$u(k) = K(\lambda(k))x(k), \quad (27)$$

$$K(\lambda(k)) = \lambda(k)\left[\sum_{j=1}^L p_j(k)K_{i,j}\right]$$

$$+ (1-\lambda(k))\left[\sum_{j=1}^L p_j(k)K_{i+1,j}\right], \quad (28)$$

where $\lambda(k) \in (0, 1]$ is calculated by solving

$$x(k)^T \left(\lambda(k) \left[\sum_{j=1}^{L} p_j(k) Q_{i,j}^{-1} \right] \right.$$
$$\left. + (1 - \lambda(k)) \left[\sum_{j=1}^{L} p_j(k) Q_{i+1,j}^{-1} \right] \right) x(k) = 1. \quad (29)$$

If $x(k) \in \varepsilon_{N,j}, j \in \{1, 2, \ldots, L\}$, adopt the following control law:

$$u(k) = K_N x(k), \qquad K_N = \sum_{j=1}^{L} p_j(k) K_{N,j}. \quad (30)$$

An intersection area of $\varepsilon_{1,j}, j \in \{1, 2, \ldots, L\}$ is the largest positively invariant set where robust stability and robust constraint satisfaction are guaranteed. If $x(k) \in \varepsilon_{1,j}, j \in \{1, 2, \ldots, L\}$, then $x(k + i) \in \varepsilon_{1,j}, j \in \{1, 2, \ldots, L\}, i \in \{0, 1, \ldots, \infty\}$. Thus, Algorithm 6 is always feasible if it is initially feasible.

Remark 8. The proposed offline MPC algorithm can deal with both time-varying scheduling parameter and disturbance. In comparison, an offline MPC algorithm [22] cannot deal with disturbance.

4. Examples

The numerical simulations have been performed in Intel Core 2 Duo (2.53 GHz), 2 GB RAM, using SeDuMi [24] and YALMIP [25] within the Matlab R2008a environment.

Example 1. The first example is a continuous stirred tank reactor (CSTR) adapted from Ding and Huang [26] where an exothermic reaction $A \rightarrow B$ takes place. The dynamic model based on a component balance and an energy balance can be written as

$$\dot{C}_A = \frac{q}{V}(C_{AF} - C_A) - k_o e^{(-Ea/RT)} C_A + D_C \nu, \quad (31)$$

$$\dot{T} = \frac{q}{V}(T_f - T) + \frac{-\Delta H}{\rho C_p} k_o e^{(-Ea/RT)} C_A$$
$$+ \frac{UA}{V\rho C_p}(T_c - T) + D_T \nu, \quad (32)$$

where C_A denotes the concentration of A in the reactor, T denotes the reactor temperature, T_c denotes the temperature of the coolant stream, and ν denotes the disturbance acting on the system. The operating parameters are shown in Table 1.

By defining $\overline{C}_A = C_A - C_{A,eq}$, $\overline{T} = T - T_{eq}$ and $\overline{T}_c = T_c - T_{c,eq}$, (31) and (32) can be transformed into the flowing nonlinear model:

$$\begin{bmatrix} \dot{\overline{C}}_A \\ \dot{\overline{T}} \end{bmatrix} = \begin{cases} \begin{bmatrix} -\frac{q}{V} - \varphi_1(\overline{T}) & -\varphi_2(\overline{T}) \\ \left(\frac{-\Delta H}{\rho C_p}\right)\varphi_1(\overline{T}) & -\frac{q}{V} - \frac{UA}{V\rho C_p} + \left(\frac{-\Delta H}{\rho C_p}\right)\varphi_2(\overline{T}) \end{bmatrix} \begin{bmatrix} \overline{C}_A \\ \overline{T} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{UA}{V\rho C_p} \end{bmatrix} \overline{T}_c + \begin{bmatrix} D_C \\ D_T \end{bmatrix} \nu, & \overline{T} \neq 0 \\[20pt] \begin{bmatrix} -\frac{q}{V} - \varphi_1(\overline{T}) & 0 \\ \left(\frac{-\Delta H}{\rho C_p}\right)\varphi_1(\overline{T}) & -\frac{q}{V} - \frac{UA}{V\rho C_p} \end{bmatrix} \begin{bmatrix} \overline{C}_A \\ \overline{T} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{UA}{V\rho C_p} \end{bmatrix} \overline{T}_c + \begin{bmatrix} D_C \\ D_T \end{bmatrix} \nu, & \overline{T} = 0 \end{cases}, \quad (33)$$

where $\varphi_1(\overline{T}) = k_o \exp(-(E_a/R)/(\overline{T} + T_{eq}))$, $\varphi_2(\overline{T}) = k_o[\exp(-(E_a/R)/(\overline{T} + T_{eq})) - \exp(-(E_a/R)/T_{eq})]C_{A,eq}/\overline{T}$. For

$|\overline{T}| \leq \beta$, (33) is transformed into the following model in order to balance the submodels:

$$\begin{bmatrix} \dot{\overline{C}}_A \\ \dot{\overline{T}} \end{bmatrix}$$
$$= \begin{cases} \begin{bmatrix} -\frac{q}{V} - \varphi_1^0 - g_1(\overline{T}) & -\varphi_2^0 - g_2(\overline{T}) \\ \left(\frac{-\Delta H}{\rho C_p}\right)\varphi_1^0 + \left(\frac{-\Delta H}{\rho C_p}\right)g_1(\overline{T}) & -\frac{q}{V} - \frac{UA}{V\rho C_p} + \left(\frac{-\Delta H}{\rho C_p}\right)\varphi_2^0 + \left(\frac{-\Delta H}{\rho C_p}\right)g_2(\overline{T}) \end{bmatrix} \begin{bmatrix} \overline{C}_A \\ \overline{T} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{UA}{V\rho C_p} \end{bmatrix} \overline{T}_c + \begin{bmatrix} D_C \\ D_T \end{bmatrix} \nu, & \overline{T} \neq 0 \\[20pt] \begin{bmatrix} -\frac{q}{V} - \varphi_1^0 - g_1(\overline{T}) & 0 \\ \left(\frac{-\Delta H}{\rho C_p}\right)\varphi_1^0 + \left(\frac{-\Delta H}{\rho C_p}\right)g_1(\overline{T}) & -\frac{q}{V} - \frac{UA}{V\rho C_p} \end{bmatrix} \begin{bmatrix} \overline{C}_A \\ \overline{T} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{UA}{V\rho C_p} \end{bmatrix} \overline{T}_c + \begin{bmatrix} D_C \\ D_T \end{bmatrix} \nu, & \overline{T} = 0 \end{cases}, \quad (34)$$

where $g_1(\overline{T}) = \varphi_1(\overline{T}) - \varphi_1^0$, $g_2(\overline{T}) = \varphi_2(\overline{T}) - \varphi_2^0$, $\varphi_1^0 = [\varphi_1(-\beta) + \varphi_1(\beta)]/2$ and $\varphi_2^0 = [\varphi_2(-\beta) + \varphi_2(\beta)]/2$. Since $g_1(\overline{T})$ and $g_2(\overline{T})$ can vary between $g_1(-\beta) \leq g_1(\overline{T}) \leq g_1(\beta)$ and $g_2(-\beta) \leq g_2(\overline{T}) \leq g_2(\beta)$, we have that all the solutions of (34) are also

the solutions of the following differential inclusion:

$$\begin{bmatrix} \dot{\overline{C}}_A \\ \dot{\overline{T}} \end{bmatrix} \in \left(\sum_{j=1}^{4} p_j A_j \right) \begin{bmatrix} \overline{C}_A \\ \overline{T} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{UA}{V\rho C_p} \end{bmatrix} \overline{T}_c + \begin{bmatrix} D_C \\ D_T \end{bmatrix} \nu, \quad (35)$$

where A_j are given by

$$A_1$$

$$= \begin{bmatrix} -\dfrac{q}{V} - \varphi_1^0 - 2g_1(\beta) & -\varphi_2^0 \\ \left(\dfrac{-\Delta H}{\rho C_p}\right)\varphi_1^0 + 2\left(\dfrac{-\Delta H}{\rho C_p}\right)g_1(\beta) & -\dfrac{q}{V} - \dfrac{UA}{V\rho C_p} + \left(\dfrac{-\Delta H}{\rho C_p}\right)\varphi_2^0 \end{bmatrix},$$

$$A_2$$

$$= \begin{bmatrix} -\dfrac{q}{V} - \varphi_1^0 - 2g_1(-\beta) & -\varphi_2^0 \\ \left(\dfrac{-\Delta H}{\rho C_p}\right)\varphi_1^0 + 2\left(\dfrac{-\Delta H}{\rho C_p}\right)g_1(-\beta) & -\dfrac{q}{V} - \dfrac{UA}{V\rho C_p} + \left(\dfrac{-\Delta H}{\rho C_p}\right)\varphi_2^0 \end{bmatrix},$$

$$A_3$$

$$= \begin{bmatrix} -\dfrac{q}{V} - \varphi_1^0 & -\varphi_2^0 - 2g_2(\beta) \\ \left(\dfrac{-\Delta H}{\rho C_p}\right)\varphi_1^0 & -\dfrac{q}{V} - \dfrac{UA}{V\rho C_p} + \left(\dfrac{-\Delta H}{\rho C_p}\right)\varphi_2^0 + 2\left(\dfrac{-\Delta H}{\rho C_p}\right)g_2(\beta) \end{bmatrix},$$

$$A_4$$

$$= \begin{bmatrix} -\dfrac{q}{V} - \varphi_1^0 & -\varphi_2^0 - 2g_2(-\beta) \\ \left(\dfrac{-\Delta H}{\rho C_p}\right)\varphi_1^0 & -\dfrac{q}{V} - \dfrac{UA}{V\rho C_p} + \left(\dfrac{-\Delta H}{\rho C_p}\right)\varphi_2^0 + 2\left(\dfrac{-\Delta H}{\rho C_p}\right)g_2(-\beta) \end{bmatrix}$$

$$(36)$$

and p_j are given by

$$p_1 = \frac{1}{2}\frac{\left(g_1(\overline{T}) - g_1(-\beta)\right)}{\left(g_1(\beta) - g_1(-\beta)\right)},$$

$$p_2 = \frac{1}{2}\frac{\left(g_1(\beta) - g_1(\overline{T})\right)}{\left(g_1(\beta) - g_1(-\beta)\right)},$$

$$p_3 = \frac{1}{2}\frac{\left(g_2(\overline{T}) - g_2(-\beta)\right)}{\left(g_2(\beta) - g_2(-\beta)\right)},$$

$$p_4 = \frac{1}{2}\frac{\left(g_2(\beta) - g_2(\overline{T})\right)}{\left(g_2(\beta) - g_2(-\beta)\right)}.$$

$$(37)$$

The objective is to regulate $\overline{C}_A$ and $\overline{T}$ by manipulating $\overline{T}_c$. The input and output constraints are $|\overline{C}_A| \leq 0.5\,\text{mol/L}$, $|\overline{T}| \leq 5\,\text{K}$ and $|\overline{T}_c| \leq 40\,\text{K}$. The discrete-time LPV model is obtained by discretizing (35) with a sampling period of 0.01 min and it is omitted here for brevity. The symmetric weighting matrices in (10) are $\Psi = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\sigma = 0.1$. The value of θ in (18) has been fixed at 0.97 which has negligible influence on the control performance.

Figure 1(a) shows the ellipsoids computed offline by Algorithm 6. In this example, four sequences of ellipsoids are computed because the polytope Ω has four vertices. A

sequence of positively invariant sets computed by intersection among four sequences of ellipsoids is shown in Figure 1(b).

Figure 2 shows the closed-loop responses of the system when the disturbances are varied as $v(t) = 0.1\sin(10t)$, $0.05\sin(10t)$, and $0.001\sin(10t)$, respectively. It can be observed that $\overline{C}_A$ and $\overline{T}$ are bounded for all values of disturbances so robust stability is ensured by applying Algorithm 6.

The online computational time of Algorithm 6 is very low as shown in Table 2 so it is applicable to fast systems.

Example 2. The second example is an angular positioning system adapted from Kothare et al. [4]. The system consists of an electric motor driving a rotating antenna so that is always points in the direction of a moving object. The motion of the antenna can be described by the following discrete-time LPV model:
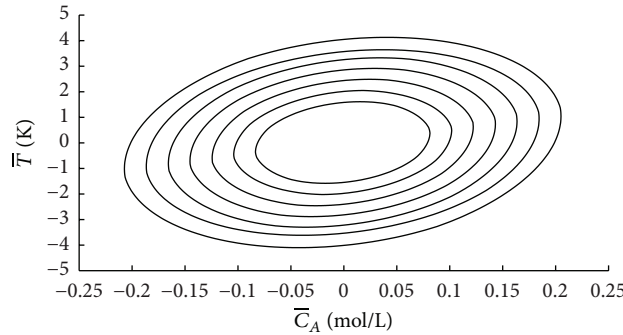
$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 0.1 \\ 0 & 1 - 0.1\Delta(k) \end{bmatrix}\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0.0787 \end{bmatrix}u(k) + \begin{bmatrix} 0.15 \\ 0.15 \end{bmatrix}v(k),$$

$$(38)$$

$$y(k) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix},$$

$$(39)$$

where $x_1(k)$ is the angular position of the antenna, $x_2(k)$ is the angular velocity of the antenna, $u(k)$ is the input voltage to the
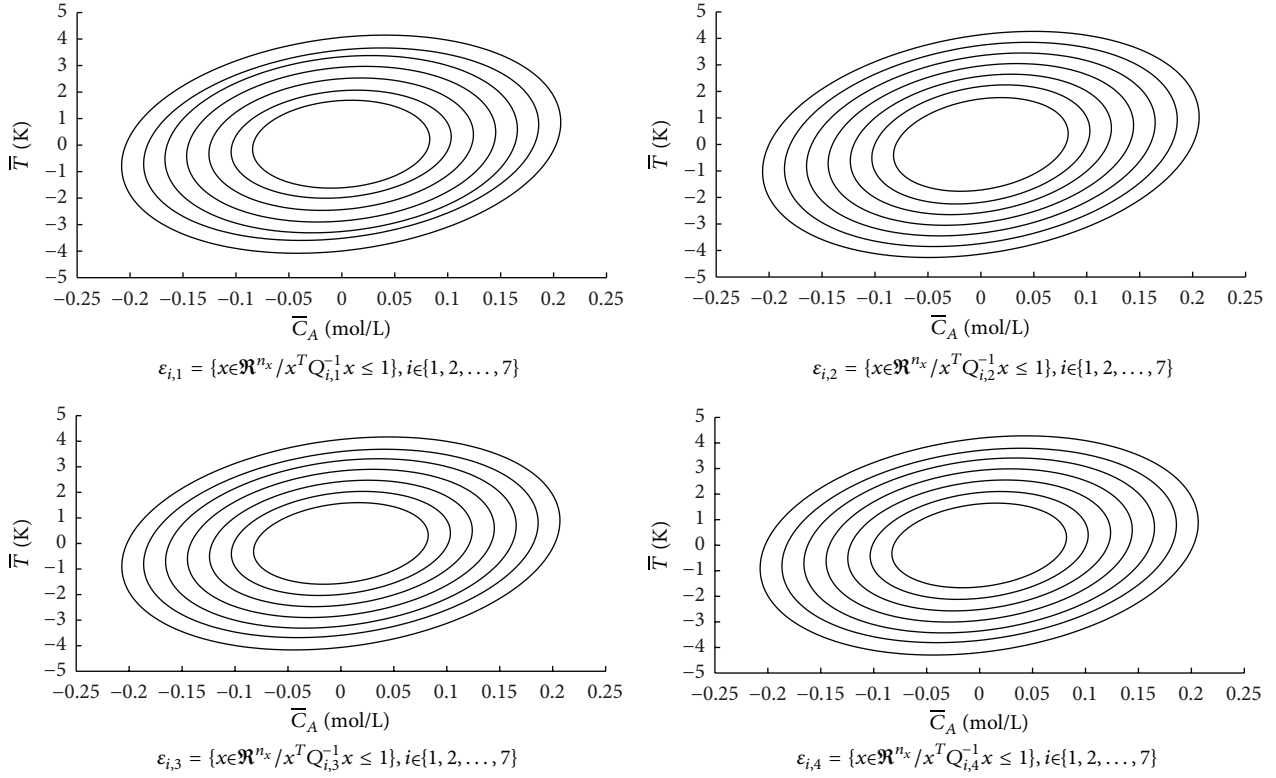
(a) Four sequences of ellipsoids corresponding to A_j

(b) A sequence of positively invariant sets

FIGURE 1: The ellipsoids computed offline in Example 1 (a) four sequences of ellipsoids corresponding to A_j (b) a sequence of positively invariant sets.

motor, and $v(k)$ is the disturbance acting on the system. The scheduling parameter $\Delta(k)$ is measurable at each sampling time and its value is varied between 0.1 and 10. We have that all the solutions of (38) are also the solutions of the following differential inclusion:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} \in \left(\sum_{j=1}^{2} p_j A_j \right) \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$
$$+ \begin{bmatrix} 0 \\ 0.0787 \end{bmatrix} u(k) + \begin{bmatrix} 0.15 \\ 0.15 \end{bmatrix} v(k), \tag{40}$$

where A_j are given by $A_1 = \begin{bmatrix} 1 & 0.1 \\ 0 & 0.99 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 0.1 \\ 0 & 0 \end{bmatrix}$ and p_j are given by $p_1 = (10 - \Delta(k))/0.9$, $p_2 = (\Delta(k) - 0.1)/0.9$.

The objective is to regulate $x_1(k)$ and $x_2(k)$ by manipulating $u(k)$. The input constraint is $|u(k)| \leq 2$ volts. The symmetric weighting matrices in (10) are $\Psi = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\sigma = 0.00002$. The value of θ in (18) has been fixed at 0.98 which has negligible influence on the control performance. A sampling time is 0.1 s.

Figure 3(a) shows two sequences of ellipsoids corresponding to A_j. A sequence of positively invariant sets computed by intersection between two sequences of ellipsoids is shown in Figure 3(b).

Figure 4 shows 100 state trajectories of the closed-loop system when $v(k)$ and $\Delta(k)$ are arbitrarily time-varying in the ranges of $-0.01 \leq v(k) \leq 0.01$ and $0.1 \leq \Delta(k) \leq 10$. Two initial points $(0.9, 0)$ and $(-0.8, 0.8)$ are chosen. It
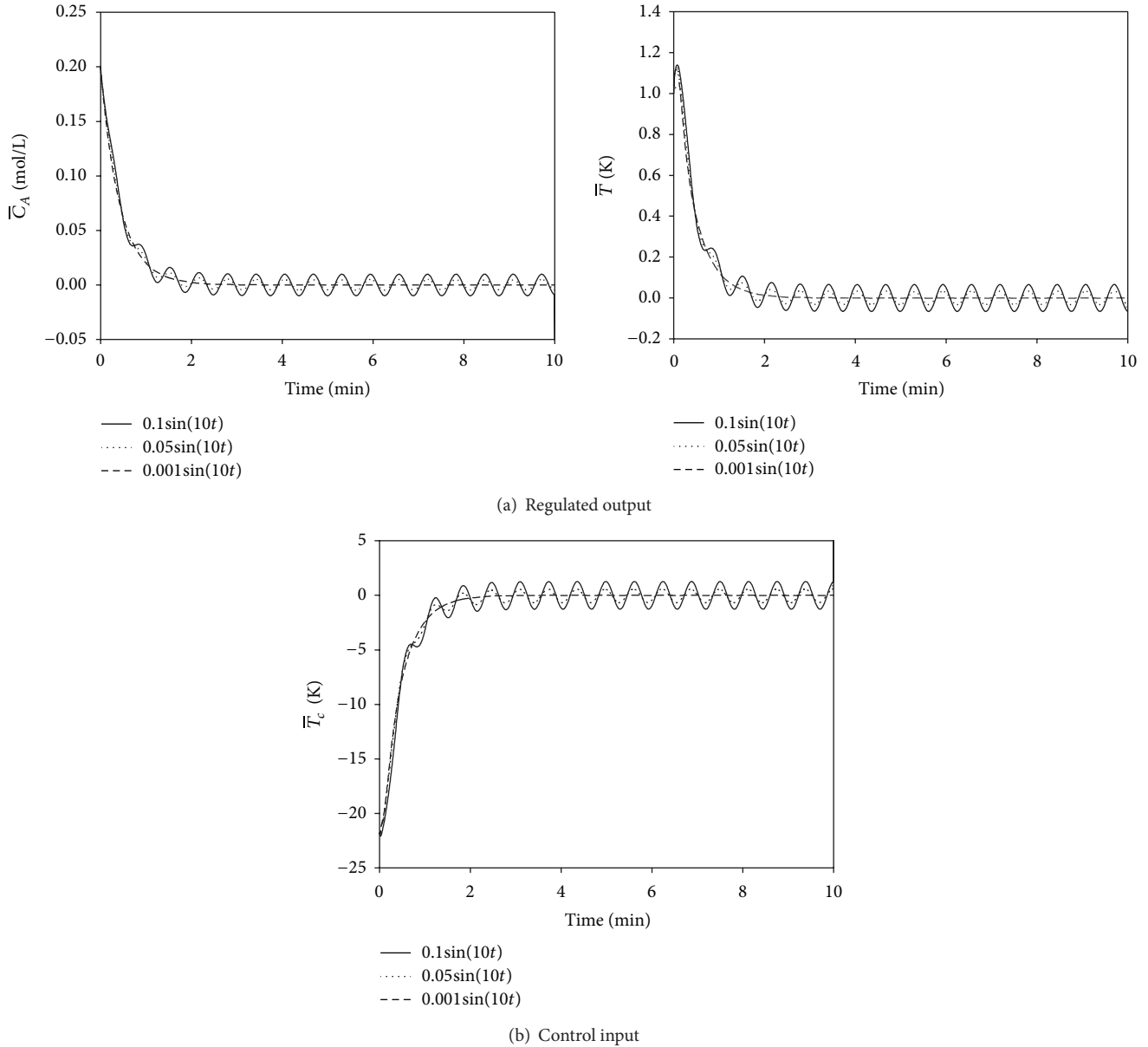
(a) Regulated output

(b) Control input

Figure 2: The closed-loop responses in Example 1 (a) regulated output and (b) control input.

can be observed from the figure that all state trajectories are restricted to lie in a sequence of positively invariant sets. The average time for Algorithm 6 to compute a real-time control law is 0.001 s.

5. Conclusions

In this paper, we have presented an offline MPC algorithm for constrained LPV systems. The main contribution is to develop an offline MPC algorithm for LPV systems that can deal with both persistent disturbance and time-varying scheduling parameter. The norm-bounding technique is used to derive an offline MPC algorithm based on the parameter-dependent state feedback control law and the parameter-dependent Lyapunov functions. Most of the optimization

problems are solved offline so the algorithm is applicable to fast systems. At each sampling time, a parameter-dependent state feedback control law is calculated by linear interpolation between the precomputed state feedback control laws. The controller design is illustrated with two examples.
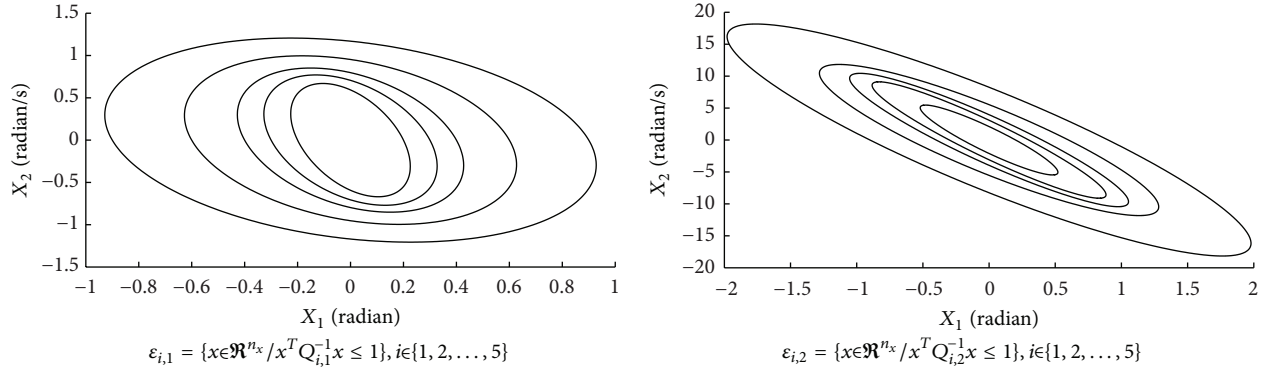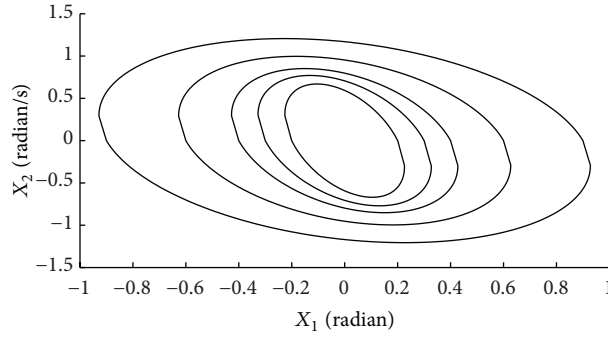
Appendices

A. Proof of Proposition 3

Lemma A.1 will be used in the proof.

Lemma A.1 (see [23]). *Suppose $M > 0$ is a symmetric matrix, while a and b are vectors with appropriate dimensions. Then,*

$$\varepsilon_{i,1} = \{x \in \mathfrak{R}^{n_x} / x^T Q_{i,1}^{-1} x \leq 1\}, i \in \{1, 2, \ldots, 5\}$$

$$\varepsilon_{i,2} = \{x \in \mathfrak{R}^{n_x} / x^T Q_{i,2}^{-1} x \leq 1\}, i \in \{1, 2, \ldots, 5\}$$

(a) Two sequences of ellipsoids corresponding to A_j

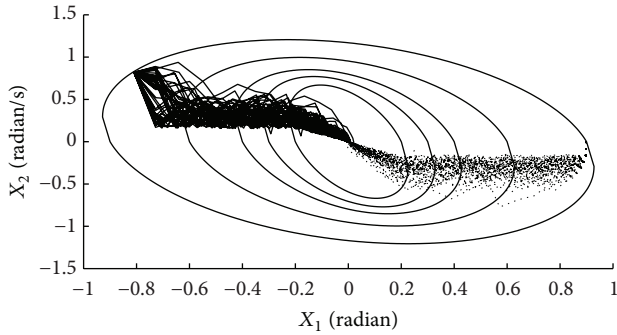(b) A sequence of positively invariant sets

FIGURE 3: The ellipsoids computed offline in Example 2 (a) two sequences of ellipsoids corresponding to A_j and (b) a sequence of positively invariant sets.

FIGURE 4: The state trajectories of the closed-loop system in Example 2.

for any scalar $\delta > 0$, $(a + b)^T M(a + b) \leq (1 + \delta)a^T Ma + (1 + (1/\delta))b^T Mb$.

The norm-bounding technique [23] is used to derive an offline MPC algorithm based on the parameter-dependent state feedback control law and the parameter-dependent Lyapunov functions.

By substituting $x(k + i + 1 \mid k) = [A(p(k + i)) + BK(p(k + i))]x(k + i \mid k) + D(\alpha(k + i))v(k + i)$ into (A.1)

$$\|x(k + i + 1 \mid k)\|_{P(p(k+i+1))}^2 \leq \gamma \tag{A.1}$$

TABLE 1: The operating parameters of nonlinear CSTR in Example 1.

Parameter	Value	Unit
q	100	L/min
T_f	350	K
C_{AF}	1	mol/L
V	100	L
ρ	1,000	g/L
C_p	0.239	J/(g K)
ΔH	-5×10^4	J/mol
E_a/R	8,750	K
k_o	7.2×10^{10}	min^{-1}
UA	5×10^4	J/(min K)
$C_{A,eq}$	0.5	mol/L
T_{eq}	350	K
$T_{c,eq}$	300	K
D_C	0.01	mol/(L min)
D_T	0.1	K/min

TABLE 2: The numerical burdens in Example 1.

Algorithm	CPU time (s)	
	Offline	Online (per step)
Algorithm 6	20.043	0.001

and applying Lemma A.1, for any $\delta_1 > 0$, we see that (A.1) is satisfied if

$$(1 + \delta_1)$$
$$\times \left\| \left[A \left(p \left(k+i \right) \right) + BK \left(p \left(k+i \right) \right) \right] x \left(k+i \mid k \right) \right\|^2_{P\left(p\left(k+i+1 \right) \right)/\gamma}$$
$$+ \left(1 + \frac{1}{\delta_1} \right) \left\| D(\alpha(k+i))v(k+i) \right\|^2_{P\left(p\left(k+i+1 \right) \right)/\gamma} \leq 1. \tag{A.2}$$

Consider the term $\left\| \left[A(p(k+i)) + BK(p(k+i)) \right] x(k+i \mid k) \right\|^2_{P(p(k+i+1))/\gamma}$ in (A.2); let $\theta \| x(k+i \mid k) \|^2_{P(p(k+i))/\gamma}$

be the maximum value of this term, where $0 < \theta < 1$ is a prespecified scalar:

$$\left\| \left[A \left(p \left(k+i \right) \right) + BK \left(p \left(k+i \right) \right) \right] x \left(k+i \mid k \right) \right\|^2_{P\left(p\left(k+i+1 \right) \right)/\gamma}$$
$$\leq \theta \| x \left(k+i \mid k \right) \|^2_{P\left(p\left(k+i \right) \right)/\gamma}. \tag{A.3}$$

Applying the Schur complement leads to

$$\begin{bmatrix} \theta \dfrac{P\left(p \left(k+i \right) \right)}{\gamma} & * \\ \dfrac{P\left(p \left(k+i+1 \right) \right)}{\gamma} \left[A \left(p \left(k+i \right) \right) + BK \left(p \left(k+i \right) \right) \right] & \dfrac{P\left(p \left(k+i+1 \right) \right)}{\gamma} \end{bmatrix} \geq 0. \tag{A.4}$$

From the convexity of the polytopic description, with $P_j = \gamma Q_j^{-1}$, $P_l = \gamma Q_l^{-1}$ and $K_j = Y_j G_j^{-1}$, (A.4) is satisfied if

$$\begin{bmatrix} \theta Q_j^{-1} & * \\ Q_l^{-1} \left[A_j + BY_j G_j^{-1} \right] & Q_l^{-1} \end{bmatrix} \geq 0, \quad j, l \in \{1, 2, \ldots, L\}. \tag{A.5}$$

Multiplying (A.5) from the left by $\mathrm{diag}\{G_j^T, Q_l^T\}$ and from the right by $\mathrm{diag}\{G_j, Q_l\}$ and applying $G_j^T Q_j^{-1} G_j \geq G_j + G_j^T - Q_j$, we obtain LMIs (A.6)

$$\begin{bmatrix} \theta \left(G_j + G_j^T - Q_j \right) & * \\ A_j G_j + BY_j & Q_l \end{bmatrix} \geq 0, \quad j, l \in \{1, 2, \ldots, L\}. \tag{A.6}$$

Consider the term $\| D(\alpha(k+i))v(k+i) \|^2_{P(p(k+i+1))/\gamma}$ in (A.2); let ξ be the maximum value of this term:

$$\| D(\alpha(k+i))v(k+i) \|^2_{P(p(k+i+1))/\gamma} \leq \xi. \tag{A.7}$$

Applying the Schur complement leads to

$$\begin{bmatrix} \xi & * \\ \dfrac{P\left(p \left(k+i+1 \right) \right)}{\gamma} D \left(\alpha \left(k+i \right) \right) v \left(k+i \right) & \dfrac{P\left(p \left(k+i+1 \right) \right)}{\gamma} \end{bmatrix}$$
$$\geq 0. \tag{A.8}$$

From the convexity of the polytopic description, with $P_l = \gamma Q_l^{-1}$, (A.8) is satisfied if

$$\begin{bmatrix} \xi & * \\ Q_l^{-1} D_t v_s & Q_l^{-1} \end{bmatrix} \geq 0, \quad l \in \{1, 2, \ldots, L\}, \; t \in \{1, 2, \ldots, n_D\},$$
$$s \in \{1, 2, \ldots, m_v\}. \tag{A.9}$$

Multiplying (A.9) from the left by $\mathrm{diag}\{I, Q_l\}$ and from the right by $\mathrm{diag}\{I, Q_l\}$, we obtain LMIs (A.10):

$$\begin{bmatrix} \xi & * \\ D_t v_s & Q_l \end{bmatrix} \geq 0, \quad t \in \{1, 2, \ldots, n_D\}, \; s \in \{1, 2, \ldots, m_v\},$$
$$l \in \{1, 2, \ldots, L\}. \tag{A.10}$$

From (A.3) and (A.7), with $\| x(k+i \mid k) \|^2_{P(p(k+i))/\gamma} \leq 1$, (A.2) is equivalent to

$$(1 + \delta_1) \theta + \left(1 + \frac{1}{\delta_1} \right) \xi \leq 1. \tag{A.11}$$

The maximum allowable value of ξ can be calculated by solving

$$\xi = \max_{\delta_1} \frac{1 - (1 + \delta_1) \theta}{(1 + (1/\delta_1))}. \tag{A.12}$$

From (A.12), ξ is obtained as

$$\xi = \left(1 - \theta^{1/2} \right)^2. \tag{A.13}$$

B. Proof of Proposition 5 (Input Constraint)

Defining ζ_h as the hth row of the n_u-dimensional identity matrix, we have

$$\max_{i \geq 0} |\zeta_h u \left(k+i \mid k \right)|^2 = \max_{i \geq 0} |\zeta_h K \left(p \left(k+i \right) \right) x \left(k+i \mid k \right)|^2. \tag{B.1}$$

Applying Cauchy-Schwarz inequality, with $\|x(k+i \mid k)\|^2_{P(p(k+i))/\gamma} \leq 1$, leads to

$$\max_{i \geq 0} |\zeta_h K (p(k+i)) x(k+i \mid k)|^2$$

$$\leq \left\| \zeta_h K(p(k+i)) \left(\frac{P(p(k+i))}{\gamma} \right)^{-1/2} \right\|^2. \tag{B.2}$$

The input constraint is satisfied if

$$\left\| \zeta_h K (p(k+i)) \left(\frac{P(p(k+i))}{\gamma} \right)^{-1/2} \right\|^2 \leq \bar{u}_h^2, \tag{B.3}$$

$$h \in \{1, 2, \ldots, n_u\}.$$

By applying the Schur complement, (B.3) is equivalent to

$$\begin{bmatrix} \chi & * \\ K(p(k+i))^T & \left(\frac{P(p(k+i))}{\gamma} \right) \end{bmatrix} \geq 0, \tag{B.4}$$

$$\chi_{hh} \leq \bar{u}_h^2, \quad h \in \{1, 2, \ldots, n_u\}.$$

From the convexity of the polytopic description, with $P_j = \gamma Q_j^{-1}$ and $K_j = Y_j G_j^{-1}$, (B.4) is satisfied if

$$\begin{bmatrix} \chi & * \\ (Y_j G_j^{-1})^T & Q_j^{-1} \end{bmatrix} \geq 0, \quad \chi_{hh} \leq \bar{u}_h^2, \tag{B.5}$$

$$h \in \{1, 2, \ldots, n_u\}, \quad j \in \{1, 2, \ldots, L\}.$$

Premultiplying by $\text{diag}\{I, G_j^T\}$, postmultiplying by $\text{diag}\{I, G_j\}$ and applying $G_j^T Q_j^{-1} G_j \geq G_j + G_j^T - Q_j$, we obtain LMIs (B.6)

$$\begin{bmatrix} \chi & * \\ Y_j^T & G_j + G_j^T - Q_j \end{bmatrix} \geq 0, \quad \chi_{hh} \leq \bar{u}_h^2, \tag{B.6}$$

$$h \in \{1, 2, \ldots, n_u\}, \quad j \in \{1, 2, \ldots, L\}.$$

C. Proof of Proposition 5 (Output Constraint)

Defining ζ_r as the rth row of the n_y-dimensional identity matrix, we have

$$\max_{i \geq 0} |\zeta_r y(k+i+1 \mid k)|^2$$

$$= \max_{i \geq 0} |\zeta_r C (p(k+i+1)) x(k+i+1 \mid k)|^2. \tag{C.1}$$

Substituting $x(k+i+1 \mid k) = [A(p(k+i)) + BK(p(k+i))]x(k+i \mid k) + D(\alpha(k+i))v(k+i)$ and applying Cauchy-Schwarz inequality and Lemma A.1, for any $\delta_2 > 0$, we see that the output constraint is satisfied if

$$(1 + \delta_2) \|\zeta_r C (p(k+i+1))$$
$$\times [A(p(k+i)) + BK(p(k+i))] x(k+i \mid k) \|^2$$
$$+ \left(1 + \frac{1}{\delta_2} \right) \|\zeta_r C(p(k+i+1))D(\alpha(k+i))v(k+i)\|^2 \leq \bar{y}_r^2. \tag{C.2}$$

Consider the term $\|\zeta_r C(p(k+i+1))[A(p(k+i)) + BK(p(k+i))]x(k+i \mid k)\|^2$ in (C.2); let Ξ_r be the maximum value of this term

$$\|\zeta_r C (p(k+i+1)) [A(p(k+i)) + BK(p(k+i))]$$
$$\times x(k+i \mid k) \|^2 \leq \Xi_r. \tag{C.3}$$

Since $\|x(k+i \mid k)\|^2_{P(p(k+i))/\gamma} \leq 1$, applying the Schur complement leads to

$$\begin{bmatrix} \Gamma & * \\ [A(p(k+i)) + BK(p(k+i))]^T C(p(k+i+1))^T & \left(\frac{P(p(k+i))}{\gamma} \right) \end{bmatrix} \geq 0, \tag{C.4}$$

$$\Gamma_{rr} \leq \Xi_r, \quad r \in \{1, 2, \ldots, n_y\}.$$

From the convexity of the polytopic description, with $P_j = \gamma Q_j^{-1}$ and $K_j = Y_j G_j^{-1}$, (C.4) is satisfied if

$$\begin{bmatrix} \Gamma & * \\ [A_j + BY_j G_j^{-1}]^T C_l^T & Q_j^{-1} \end{bmatrix} \geq 0, \quad \Gamma_{rr} \leq \Xi_r, \tag{C.5}$$

$$r \in \{1, 2, \ldots, n_y\}, \quad j, l \in \{1, 2, \ldots, L\}.$$

Premultiplying by $\text{diag}\{I, G_j^T\}$, postmultiplying by $\text{diag}\{I, G_j\}$, and applying $G_j^T Q_j^{-1} G_j \geq G_j + G_j^T - Q_j$, we obtain LMIs (C.6)

$$\begin{bmatrix} \Gamma & * \\ (A_j G_j + BY_j)^T C_l^T & G_j + G_j^T - Q_j \end{bmatrix} \geq 0, \quad \Gamma_{rr} \leq \Xi_r, \tag{C.6}$$

$$r \in \{1, 2, \ldots, n_y\}, \quad j, l \in \{1, 2, \ldots, L\}.$$

Consider the term $\|\zeta_r C(p(k+i+1))D(\alpha(k+i))v(k+i)\|^2$ in (C.2); let Φ_r be the maximum value of this term:

$$\|\zeta_r C(p(k+i+1))D(\alpha(k+i))v(k+i)\|^2 \le \Phi_r. \qquad \text{(C.7)}$$

By applying the Schur complement, we obtain

$$\begin{bmatrix} \Phi_r & * \\ \zeta_r C\left(p\left(k+i+1\right)\right)D\left(\alpha\left(k+i\right)\right)v\left(k+i\right) & 1 \end{bmatrix} \ge 0. \qquad \text{(C.8)}$$

From the convexity of the polytopic description, (C.8) is satisfied if

$$\begin{bmatrix} \Phi_r & * \\ C_{r,l} D_t v_s & 1 \end{bmatrix} \ge 0, \quad r \in \{1, 2, \dots, n_y\}, \ l \in \{1, 2, \dots, L\},$$

$$t \in \{1, 2, \dots, n_D\}, \quad s \in \{1, 2, \dots, m_v\}, \qquad \text{(C.9)}$$

where $C_{r,l}$ is the rth row of C_l. Thus, Φ_r can be calculated by solving

$$\min_{\Phi_r} \Phi_r \quad \text{s.t. (C.9)}. \qquad \text{(C.10)}$$

From (C.3) and (C.7), (C.2) is equivalent to

$$(1 + \delta_2) \Xi_r + \left(1 + \frac{1}{\delta_2}\right)\Phi_r \le \overline{y}_r^2. \qquad \text{(C.11)}$$

The maximum allowable value of Ξ_r can be calculated by solving

$$\Xi_r = \max_{\delta_2} \frac{\overline{y}_r^2 - (1 + (1/\delta_2))\Phi_r}{(1 + \delta_2)}. \qquad \text{(C.12)}$$

From (C.12), Ξ_r is obtained as

$$\Xi_r = \left(\overline{y}_r - \Phi_r^{1/2}\right)^2. \qquad \text{(C.13)}$$

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

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