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Research Article

The Local Stability of Solutions for a Nonlinear Equation

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The approach of Kruzkov's device of doubling the variables is applied to establish the local stability of strong solutions for a nonlinear partial differential equation in the space $L^1(R)$ by assuming that the initial value only lies in the space $L^1(R) \cap L^\infty(R)$.

1. Introduction and Main Results

Coclite and Karlsen [1] studied the well-posedness for the nonlinear equation

$$\begin{aligned} v_t - v_{txx} + 4h'(v)v_x \\ = h'''(v)v_x^3 + 3h''(v)v_xv_{xx} + h'(v)v_{xxx}, \end{aligned} \tag{1}$$

where function $h(v)$ satisfies

$$|h'(v)| \le c|v|, \qquad |h(v)| \le c|v|^2, \tag{2}$$

or

$$|h'(v)| \le c, \qquad |h(v)| \le c|v|, \tag{3}$$

where c is a positive constant. The existence of entropy weak solutions and several other dynamic properties for (1) are investigated in [1].

Consider the following partial differential equation:

$$\begin{aligned} v_t - v_{txx} + mh'(v)v_x \\ = h'''(v)v_x^3 + 3h''(v)v_xv_{xx} + h'(v)v_{xxx}, \end{aligned} \tag{4}$$

where m is a positive constant and $h(v) \in C^3$. If $m = 4$ and $h(v) = v^2/2$, (4) becomes the Degasperis-Procesi equation (see [2–12]). If $h(v) = v^3$, we note that $h(v)$ does not satisfy conditions (2) and (3). Recently, Wu [13] established the local well-posedness of strong solutions for (4) in the Sobolev space $C([0,T);H^s(R)) \cap C^1([0,T);H^{s-1}(R))$ provided that $h(v) = v^3$ and the initial value lies in the space $H^s(R)$ with $s > 3/2$. The objective of this work is to study (4) in the case $h(v) = v^3$. The local stability of strong solutions for this case is established in the space $L^1(R)$. We think that this stability result is a new conclusion for (4).

When $h(v) = v^3$, the Cauchy problem of (4) takes the form

$$\begin{gathered} v_t - v_{txx} + 3mv^2v_x = 6v_x^3 + 18vv_xv_{xx} + 3v^2v_{xxx}, \\ v(0,x) = v_0(x), \end{gathered} \tag{5}$$

which is equivalent to

$$\begin{gathered} v_t + 3v^2v_x + (m-1)\left(1-\partial_x^2\right)^{-1}\partial_x\left(v^3\right) = 0, \\ v(0,x) = v_0(x), \end{gathered} \tag{6}$$

where the operator $\Lambda^{-2}g = (1-\partial_x^2)^{-1}g = (1/2)\int_{-\infty}^{\infty} e^{-|x-y|} g(y)dy$ for any $g \in L^\infty(R)$ or $g \in L^{p_0}(R)$ with a parameter p_0 satisfying $1 \le p_0 < \infty$.

Using the approach of Kruzkov's device of doubling the variables in [14], we obtain the following result.

Theorem 1. *Let $v_1(t,x)$ and $v_2(t,x)$ be two strong solutions of problem (5) or (6) with initial data $v_1(0,x) = v_{10}(x) \in L^1(R) \cap L^\infty(R)$ and $v_2(0,x) = v_{20}(x) \in L^1(R) \cap L^\infty(R)$. Then, for an arbitrary $T > 0$,*

$$\int_{-\infty}^{\infty} \left|v_1(t,x) - v_2(t,x)\right| dx \le e^{ct} \int_{-\infty}^{\infty} \left|v_{10}(x) - v_{20}(x)\right| dx, \qquad t \in [0,T] \tag{7}$$

holds, where c is a constant depending on $\|v_{10}\|_{L^1(R)}$, $\|v_{10}\|_{L^\infty(R)}$, $\|v_{20}\|_{L^1(R)}$, $\|v_{20}\|_{L^\infty(R)}$, and T.

This paper is organized as follows. Several lemmas are given in Section 2 and the proof of Theorem 1 is completed in Section 3.

2. Several Lemmas

Lemma 1 (see [13]). *Assume that initial value $v_0 = v(0,x) \in L^2(R)$. Then the solution of problem (5) with $m > 0$ satisfies*

$$\int_R K_1 K\, dx = \int_R \frac{1+\xi^2}{m+\xi^2} \left|\widehat{v}(\xi)\right|^2 d\xi = \int_R \frac{1+\xi^2}{m+\xi^2} \left|\widehat{v}_0(\xi)\right|^2 d\xi, \tag{8}$$

where $K_1 = v - \partial^2_{xx} v$ and $K = (m - \partial^2_{xx})^{-1} v$. Moreover, there exist two constants $c_1 > 0$ and $c_2 > 0$ depending only on m such that

$$c_1 \|v_0\|_{L^2(R)} \le c_1 \|v\|_{L^2(R)} \le c_2 \|v_0\|_{L^2(R)}. \tag{9}$$

In fact, if $v \in L^1(R) \cap L^\infty(R)$, we know that $v \in L^2(R)$.

Lemma 2 (see [13]). *Assume that $v_0 \in L^2(R)$. Then*

$$\|v(t,x)\|_{L^\infty} \le \|v_0\|_{L^\infty} e^{ct}, \quad \forall t \in [0,\infty), \tag{10}$$

where $c > 0$ is a constant independent of t.

Lemma 3. *Let $J_v = (m-1)(1-\partial_x^2)^{-1}\partial_x(v^3)$ and $v_0 \in L^1(R) \cap L^\infty(R)$. Consider that*

$$\|J_v\|_{L^\infty(R)} \le |m-1| \, \|v_0\|^3_{L^\infty(R)} e^{ct}, \quad t \in [0,\infty) \tag{11}$$

holds, where $c > 0$ is a constant independent of t.

Proof. Since

$$\begin{aligned} \left|\Lambda^{-2}\partial_x\left(v^3\right)\right| &= \left|\frac{1}{2}\partial_x \int_{-\infty}^{\infty} e^{-|x-y|} v^3(t,y)\,dy\right| \\ &= \left|-\frac{1}{2}e^{-x}\int_{-\infty}^{x} e^{y} v^3(t,y)\,dy \right. \\ &\quad \left. +\frac{1}{2}e^{x}\int_{x}^{\infty} e^{-y} v^3(t,y)\,dy\right| \\ &\le \frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-y|}\left|v^3(t,y)\right| dy, \end{aligned} \tag{12}$$

we complete the proof by using Lemma 2 and $\int_{-\infty}^{\infty} e^{|x-y|} dy = 2$. □

Let $\zeta_T = [0,T] \times R$ for an arbitrary $T > 0$. $C_0^\infty(\zeta_T)$ is the space of all infinitely differentiable functions with compact support in ζ_T. We define $\theta(\omega)$ as a function which is infinitely differentiable on $(-\infty,+\infty)$ such that $\theta(\omega) \ge 0$, $\theta(\omega) = 0$ for $|\omega| \ge 1$, and $\int_{-\infty}^{\infty} \theta(\omega) d\omega = 1$. For any number $k > 0$, we let $\theta_k(\omega) = (\theta(k^{-1}\omega)/k)$. Then we have that $\theta_k(\omega)$ is a function in $C^\infty(-\infty,\infty)$ and

$$\begin{gathered} \theta_k(\omega) \ge 0, \qquad \theta_k(\omega) = 0 \quad \text{if } |\omega| \ge k, \\ \left|\theta_k(\omega)\right| \le \frac{c}{k}, \qquad \int_{-\infty}^{\infty} \theta_k(\omega)\,d\omega = 1, \end{gathered} \tag{13}$$

where c is a positive constant. Assume that the function $u(x)$ is locally integrable in $(-\infty,\infty)$. We define the approximation of function $u(x)$ as

$$u^k(x) = \frac{1}{k}\int_{-\infty}^{\infty} \theta\left(\frac{x-y}{k}\right) u(y)\,dy, \quad k > 0. \tag{14}$$

We know that a point x_0 is defined as a Lebesgue point of function $u(x)$ if

$$\lim_{k\to 0} \frac{1}{k}\int_{|x-x_0|\le k} \left|u(x) - u(x_0)\right| dx = 0. \tag{15}$$

For the Lebesgue point x_0 of the function $u(x)$, we get $\lim_{k\to 0} u^k(x_0) = u(x_0)$. Since the measure for the set of points which are not the Lebesgue points of $u(x)$ is zero, we have $u^k(x) \to u(x)$ as $k \to 0$ almost everywhere.

For any $N > 0$, let $R_0 > \sup_{t\in[0,\infty)} \|v\|^2_{L^\infty(R)} < \infty$. Let $\sqsupset$ represent the cone $\{(t,x) : |x| \le N - R_0 t, 0 < t < T_0 = \min(T, NR_0^{-1})\}$. We let S_τ represent the cross section of the cone $\sqsupset$ by the plane $t = \tau$, $\tau \in [0,T_0]$. Let $K_r = \{x : |x| \le r\}$, where $r > 0$.

Lemma 4 (see [14]). *Let the function $u(t,x)$ be bounded and measurable in cylinder $\Omega = [0,T] \times K_r$. If $\delta \in (0, \min[r,T])$ and $h \in (0,\delta)$, then the function*

$$U_k = \frac{1}{k^2}\iiiint_{|(t-\tau)/2|\le k,\, \delta\le(t+\tau)/2\le T-\delta,\, |(x-y)/2|\le k,\, |(x+y)/2|\le r-\delta} \left|u(t,x) - u(\tau,y)\right| dx\,dt\,dy\,d\tau \tag{16}$$

satisfies $\lim_{k\to 0} U_k = 0$.

Lemma 5 (see [14]). *If the function $|\partial F(u)/\partial u|$ is bounded, then the function $H(u,v) = \operatorname{sign}(u-v)(F(u)-F(v))$ satisfies the Lipschitz condition in u and v, respectively.*

Lemma 6. *Let v be the strong solution of problem (6) and $\phi(t,x) \in C_0^\infty(\zeta_T)$. Then*

$$\begin{aligned} \iint_{\zeta_T} \Big\{ &|v-\alpha|\,\phi_t + \operatorname{sign}(v-k)\left(v^3 - \alpha^3\right)\phi_x \\ &- \operatorname{sign}(v-\alpha)\, J_v(t,x)\,\phi \Big\}\, dx\,dt = 0, \end{aligned} \tag{17}$$

where α is an arbitrary constant.

Proof. Let $\Phi(v)$ be an arbitrary twice smooth function on the line $-\infty < v < \infty$. We multiply the first equation of problem (6) by the function $\Phi'(v)\phi(t,x)$, where $\phi(t,x) \in C_0^\infty(\zeta_T)$. Integrating over ζ_T and transferring the derivatives with respect to t and x to the test function ϕ, for any constant α, we obtain

$$\iint_{\zeta_T} \left\{\Phi(v)\phi_t + \left[\int_\alpha^v \Phi'(z)\,3z^2 dz\right]\phi_x - \Phi'(v) J_v(t,x)\phi\right\} dx\,dt = 0, \tag{18}$$

in which we have used $\int_{-\infty}^{\infty}[\int_\alpha^v \Phi'(z)3z^2dz]\phi_x dx = -\int_{-\infty}^{\infty}[\phi\Phi'(v)3v^2 v_x]dx$.

Integration by parts yields

$$\int_{-\infty}^{\infty}\left[\int_\alpha^v \Phi'(z)\,3z^2 dz\right]\phi_x dx = \int_{-\infty}^{\infty}\left[\left(v^3-\alpha^3\right)\Phi'(v) - \int_\alpha^v \left(z^3-\alpha^3\right)\Phi''(z)\,dz\right]\phi_x dx. \tag{19}$$

Let $\Phi^k(v)$ be an approximation of the function $|v-\alpha|$ and set $\Phi(v) = \Phi^k(v)$. Using the properties of the $\operatorname{sign}(v-\alpha)$, (18), and (19) and sending $k \to 0$, we have

$$\iint_{\zeta_T} \left\{|v-\alpha|\,\phi_t + \operatorname{sign}(v-\alpha)\left[v^3-\alpha^3\right]\phi_x - \operatorname{sign}(v-\alpha) J_v(t,x)\phi\right\} dx\,dt = 0, \tag{20}$$

which completes the proof. □

We note that the proof of (17) can also be found in [14].

Lemma 7. *Assume that $v_1(t,x)$ and $v_2(t,x)$ are two strong solutions of problem (6) associated with the initial data $v_{10} = v_1(0,x) \in L^1(R)\cap L^\infty(R)$ and $v_{20} = v_2(0,x) \in L^1(R)\cap L^\infty(R)$. For any $\phi \in C_0^\infty(\zeta_T)$,*

$$\left|\int_{-\infty}^{\infty} \operatorname{sign}(v_1-v_2)\left[J_{v_1}(t,x) - J_{v_2}(t,x)\right]\phi\,dx\right| \le c\int_{-\infty}^{\infty}|v_1-v_2|\,dx \tag{21}$$

holds, where c depends on $\|v_{10}\|_{L^1(R)}$, $\|v_{10}\|_{L^\infty(R)}$, $\|v_{20}\|_{L^1(R)}$, $\|v_{20}\|_{L^\infty(R)}$, T, and ϕ.

Proof. We obtain

$$\Lambda^{-2}\left[v_1^3(t,x) - v_2^3(t,x)\right] = \frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-y|}\left[v_1^3(t,y) - v_2^3(t,y)\right]dy, \tag{22}$$

$$\begin{aligned}\left|\Lambda^{-2}\partial_x\left[v_1^3(t,x) - v_2^3(t,x)\right]\right| &= \left|-\frac{1}{2}e^{-x}\int_{-\infty}^{x} e^{y}\left[v_1^3(t,y) - v_2^3(t,y)\right]dy + \frac{1}{2}e^{x}\int_{x}^{\infty} e^{-y}\left[v_1^3(t,y) - v_2^3(t,y)\right]dy\right| \\ &\le c\int_{-\infty}^{\infty} e^{-|x-y|}\left|v_1^3(t,y) - v_2^3(t,y)\right|dy \\ &\le c\int_{-\infty}^{\infty} e^{-|x-y|}\left|v_1(t,y) - v_2(t,y)\right|dy,\end{aligned} \tag{23}$$

in which we have used Lemma 2. Using (23) and the Fubini theorem completes the proof. □

3. Proof of Theorem 1

Here we state that the techniques used in this paper to establish the local stability of solutions for problem (6) come from the methods of Kruzkov's device of doubling the variables presented in Kruzkov's paper [14].

Proof of Theorem 1. For an arbitrary $T > 0$, set $\zeta_T = [0,T]\times R$. Let $f(t,x) \in C_0^\infty(\zeta_T)$. We assume that $f(t,x) = 0$ outside the cylinder

$$\Theta = \{(t,x)\} = [\delta, T-2\delta]\times H_{r-2\delta}, \quad 0 < 2\delta \le \min(T,r). \tag{24}$$

We define

$$h = f\left(\frac{t+\tau}{2}, \frac{x+y}{2}\right)\theta_k\left(\frac{t-\tau}{2}\right)\theta_k\left(\frac{x-y}{2}\right) = f(\cdots)\,\lambda_k(*), \tag{25}$$

where $(\cdots) = ((t+\tau)/2, (x+y)/2)$ and $(*) = ((t-\tau)/2, (x-y)/2)$. The function $\theta_k(\omega)$ is defined in (13). Note that

$$h_t + h_\tau = f_t(\cdots)\,\lambda_k(*), \qquad h_x + h_y = f_x(\cdots)\,\lambda_k(*). \tag{26}$$

Taking $v = v_1(t,x)$ and $\alpha = v_2(\tau,y)$ and assuming that $f(t,x) = 0$ outside the cylinder Θ, from Lemma 6, we have

$$\begin{aligned}\iiiint_{\zeta_T\times\zeta_T} \Big\{&\left|v_1(t,x) - v_2(\tau,y)\right| h_t \\ &+ \operatorname{sign}\left(v_1(t,x) - v_2(\tau,y)\right) \times \left(v_1^3(t,x) - v_2^3(\tau,y)\right)h_x \\ &- \operatorname{sign}\left(v_1(t,x) - v_2(\tau,y)\right) \times J_{v_1}(t,x)\,h\Big\}\,dx\,dt\,dy\,d\tau = 0.\end{aligned} \tag{27}$$

Similarly, it has

$$\iiiint_{\zeta_T\times\zeta_T}\Big\{\left|v_2(\tau,y)-v_1(t,x)\right|g_\tau + \operatorname{sign}\left(v_2(\tau,y)-v_1(t,x)\right)\times\left(v_2^3(\tau,y)-v_1^3(t,x)\right)h_y - \operatorname{sign}\left(v_2(\tau,y)-v_1(t,x)\right)\times J_{v_2}(\tau,y)\,h\Big\}\,dx\,dt\,dy\,d\tau=0, \tag{28}$$

from which we obtain

$$\begin{aligned}0\le{}&\iiiint_{\zeta_T\times\zeta_T}\left|v_1(t,x)-v_2(\tau,y)\right|\left(h_t+h_\tau\right)dx\,dt\,dy\,d\tau\\&+\iiiint_{\zeta_T\times\zeta_T}\operatorname{sign}\left(v_1(t,x)-v_2(\tau,y)\right)\times\left(v_1^3(t,x)-v_2^3(\tau,y)\right)\times\left[h_x+h_y\right]dx\,dt\,dy\,d\tau\\&+\left|\iiiint_{\zeta_T\times\zeta_T}\operatorname{sign}\left(v_1(t,x)-v_2(\tau,y)\right)\times\left(J_{v_1}(t,x)-J_{v_2}(\tau,y)\right)h\,dx\,dt\,dy\,d\tau\right|\\={}&\iiiint_{\zeta_T\times\zeta_T}\left(I_1+I_2\right)dx\,dt\,dy\,d\tau+\left|\iiiint_{\zeta_T\times\zeta_T}I_3\,dx\,dt\,dy\,d\tau\right|.\end{aligned}\tag{29}$$

We claim that

$$\begin{aligned}0\le{}&\iint_{\zeta_T}\left|v_1(t,x)-v_2(t,x)\right|f_t+\operatorname{sign}\left(v_1(t,x)-v_2(t,x)\right)\times\left(v_1^3(t,x)-v_2^3(t,x)\right)f_x\,dx\,dt\\&+\left|\iint_{\zeta_T}\operatorname{sign}\left(v_1(t,x)-v_2(t,x)\right)\times\left[J_{v_1}(t,x)-J_{v_2}(t,x)\right]f\,dx\,dt\right|.\end{aligned}\tag{30}$$

We note that the first two terms in the integrand of (29) can be represented in the form

$$H_k=H\left(t,x,\tau,y,v_1(t,x),v_2(\tau,y)\right)\lambda_k(*). \tag{31}$$

From Lemma 5, we know that H_k satisfies the Lipschitz condition in v_1 and v_2, respectively. By the choice of h, we have $H_k=0$ outside the region as follows:

$$\begin{aligned}\{(t,x;\tau,y)\}={}&\left\{\delta\le\frac{t+\tau}{2}\le T-2\delta,\ \frac{|t-\tau|}{2}\le k,\ \frac{|x+y|}{2}\le r-2\delta,\ \frac{|x-y|}{2}\le k\right\},\\\iiiint_{\zeta_T\times\zeta_T}H_k\,dx\,dt\,dy\,d\tau={}&\iiiint_{\zeta_T\times\zeta_T}\left[H\left(t,x,\tau,y,v_1(t,x),v_2(\tau,y)\right)-H\left(t,x,t,x,v_1(t,x),v_2(t,x)\right)\right]\times\lambda_k(*)\,dx\,dt\,dy\,d\tau\\&+\iiiint_{\zeta_T\times\zeta_T}H\left(t,x,t,x,v_1(t,x),v_2(t,x)\right)\times\lambda_k(*)\,dx\,dt\,dy\,d\tau\\={}&B_1(k)+B_2.\end{aligned}\tag{32}$$

Considering the estimate $|\lambda(*)|\le c/k^2$ and the expression of function H_k, we have

$$\left|B_1(k)\right|\le c\left[k+\frac{1}{k^2}\iiiint_{|(t-\tau)/2|\le k,\delta\le(t+\tau)/2\le T-\delta,|(x-y)/2|\le k,|(x+y)/2|\le r-\delta}\left|v_2(t,x)-v_2(\tau,y)\right|dx\,dt\,dy\,d\tau\right], \tag{33}$$

where the constant c does not depend on k. Using Lemma 4, we obtain $B_1(k)\to 0$ as $k\to 0$. The integral B_2 does not depend on k. In fact, substituting $t=\alpha_1$, $(t-\tau)/2=\beta$, $x=\eta$, $(x-y)/2=\xi$ and noting that

$$\int_{-k}^{k}\int_{-\infty}^{\infty}\lambda_k(\beta,\xi)\,d\xi\,d\beta=1, \tag{34}$$

we have

$$\begin{aligned}B_2={}&2^2\iint_{\zeta_T}H_k\left(\alpha_1,\eta,\alpha_1,\eta,v_1(\alpha_1,\eta),v_2(\alpha_1,\eta)\right)\times\left\{\int_{-k}^{k}\int_{-\infty}^{\infty}\lambda_k(\beta,\xi)\,d\xi\,d\beta\right\}d\eta\,d\alpha_1\\={}&4\iint_{\zeta_T}H\left(t,x,t,x,v_1(t,x),v_2(t,x)\right)dx\,dt.\end{aligned}\tag{35}$$

Hence

$$\lim_{k\to 0}\iiiint_{\zeta_T\times\zeta_T} H_k\,dx\,dt\,dy\,d\tau = 4\iint_{\zeta_T} H(t,x,t,x,v_1(t,x),v_2(t,x))\,dx\,dt. \tag{36}$$

Since

$$I_3 = \operatorname{sign}(v_1(t,x)-v_2(\tau,y)) \times \left(J_{v_1}(t,x)-J_{v_2}(\tau,y)\right) f\lambda_k(*), \tag{37}$$

$$\iiiint_{\zeta_T\times\zeta_T} I_3\,dx\,dt\,dy\,d\tau = \iiiint_{\zeta_T\times\zeta_T}[I_3(t,x,\tau,y)-I_3(t,x,t,x)]\,dx\,dt\,dy\,d\tau + \iiiint_{\zeta_T\times\zeta_T} I_3(t,x,t,x)\,dx\,dt\,dy\,d\tau = C_1(h)+C_2, \tag{38}$$

we obtain

$$\left|C_1(k)\right| \le c\left(k+\frac{1}{k^2}\iiiint_{|(t-\tau)/2|\le k,\delta\le(t+\tau)/2\le T-\delta,\left|(x-y)/2\right|\le k,\left|(x+y)/2\right|\le r-\delta}\left|J_{v_2}(t,x)-J_{v_2}(\tau,y)\right|dx\,dt\,dy\,d\tau\right). \tag{39}$$

By Lemmas 3 and 4, we have $C_1(h)\to 0$ as $k\to 0$. Using (34), we have

$$\begin{aligned}C_2 &= 2^2\iint_{\zeta_T} I_3(\alpha_1,\eta,\alpha_1,\eta,v_1(\alpha_1,\eta),v_2(\alpha_1,\eta)) \times\left\{\int_{-h}^{h}\int_{-\infty}^{\infty}\lambda_h(\beta,\xi)\,d\xi\,d\beta\right\}d\eta\,d\alpha_1\\ &= 4\iint_{\zeta_T} I_3(t,x,t,x,v_1(t,x),v_2(t,x))\,dx\,dt\\ &= 4\iint_{\zeta_T}\operatorname{sign}(v_1(t,x)-v_2(t,x)) \times\left(J_{v_1}(t,x)-J_{v_2}(t,x)\right)f(t,x)\,dx\,dt.\end{aligned} \tag{40}$$

From (36) and (38)–(40), we prove that inequality (30) holds.

Let

$$B(t)=\int_{-\infty}^{\infty}\left|v_1(t,x)-v_2(t,x)\right|dx. \tag{41}$$

We define

$$\rho_k(\sigma)=\int_{-\infty}^{\sigma}\theta_k(\omega)\,d\omega,\quad \left(\rho_k'(\sigma)=\theta_k(\sigma)\ge 0\right) \tag{42}$$

and choose the two numbers τ_1 and $\tau_2\in(0,T_0)$, $\tau_1<\tau_2$. In (30), we choose

$$f=[\rho_k(t-\tau_1)-\rho_k(t-\tau_2)]\,\chi(t,x),\quad k<\min(\tau_1,T_0-\tau_2), \tag{43}$$

where

$$\chi(t,x)=\chi_\varepsilon(t,x)=1-\rho_\varepsilon(|x|+R_0t-N+\varepsilon),\quad \varepsilon>0. \tag{44}$$

When ε is sufficiently small, we note that function $\chi(t,x)=0$ outside the cone $\sqsupset$ and $f(t,x)=0$ outside the set Θ. For $(t,x)\in\Theta$, we have

$$0=\chi_t+R_0\left|\chi_x\right|\ge\chi_t+R_0\chi_x. \tag{45}$$

Applying (30) and (42)–(45) and suitably choosing large R_0, we have the inequality

$$\begin{aligned}0\le&\iint_{\xi_{T_0}}\{[\theta_k(t-\tau_1)-\theta_k(t-\tau_2)] \times\chi_\varepsilon\left|v_1(t,x)-v_2(t,x)\right|\}\,dx\,dt\\ &+\left|\iint_{\xi_{T_0}}[\rho_k(t-\tau_1)-\rho_k(t-\tau_2)]\left[J_{v_1}(t,x)-J_{v_2}(t,x)\right]\times E(t,x)\,\chi(t,x)\,dx\,dt\right|,\end{aligned} \tag{46}$$

where $E(t,x)=\operatorname{sign}[v_1(t,x)-v_2(t,x)]$.

From (46), we obtain

$$\begin{aligned}0\le&\iint_{\xi_{T_0}}\{[\theta_k(t-\tau_1)-\theta_k(t-\tau_2)] \times\chi_\varepsilon\left|v_1(t,x)-v_2(t,x)\right|\}\,dx\,dt\\ &+\int_0^{T_0}(\rho_k(t-\tau_1)-\rho_k(t-\tau_2)) \times\left|\int_{-\infty}^{\infty}\left[J_{v_1}(t,x)-J_{v_2}(t,x)\right]E(t,x)\,\chi(t,x)\,dx\right|dt.\end{aligned} \tag{47}$$

Using Lemma 7, we have

$$\begin{aligned}0\le&\iint_{\xi_{T_0}}\{[\theta_k(t-\tau_1)-\theta_k(t-\tau_2)] \times\chi_\varepsilon\left|v_1(t,x)-v_2(t,x)\right|\}\,dx\,dt\\ &+c\int_0^{T_0}(\rho_k(t-\tau_1)-\rho_k(t-\tau_2)) \times\int_{-\infty}^{\infty}\left|v_1-v_2\right|dx\,dt,\end{aligned} \tag{48}$$

where $c>0$ is a constant as described in (21).

Letting $\varepsilon \to 0$ in (48) and sending $N \to \infty$, we have

$$\begin{aligned} 0 \le & \iint_{\xi_{T_0}} \{[\theta_k(t-\tau_1) - \theta_k(t-\tau_2)] \\ & \times |v_1(t,x) - v_2(t,x)|\}\, dx\, dt \\ & + c\int_0^{T_0} (\rho_k(t-\tau_1) - \rho_k(t-\tau_2)) \\ & \times \int_{-\infty}^{\infty} |v_1 - v_2|\, dx\, dt. \end{aligned} \tag{49}$$

By the properties of the function $\theta_k(\omega)$ for $k \le \min(\tau_1, T_0 - \tau_1)$, we have

$$\begin{aligned} & \left| \int_0^{T_0} \theta_k(t-\tau_1) B(t)\, dt - B(\tau_1) \right| \\ & = \left| \int_0^{T_0} \theta_k(t-\tau_1)[B(t) - B(\tau_1)]\, dt \right| \\ & \le c\frac{1}{k} \int_{\tau_1-k}^{\tau_1+k} |B(t) - B(\tau_1)|\, dt \longrightarrow 0 \quad \text{as } k \longrightarrow 0, \end{aligned} \tag{50}$$

where c is independent of h.

Set

$$P(\tau_1) = \int_0^{T_0} \rho_k(t-\tau_1) B(t)\, dt = \int_0^{T_0} \int_{-\infty}^{t-\tau_1} \theta_k(\sigma)\, d\sigma B(t)\, dt. \tag{51}$$

Using the similar proof of (50), we get

$$P'(\tau_1) = -\int_0^{T_0} \theta_k(t-\tau_1) B(t)\, dt \longrightarrow -B(\tau_1) \quad \text{as } k \longrightarrow 0, \tag{52}$$

from which we obtain

$$P(\tau_1) \longrightarrow P(0) - \int_0^{\tau_1} B(\sigma)\, d\sigma \quad \text{as } k \longrightarrow 0. \tag{53}$$

Similarly, we have

$$P(\tau_2) \longrightarrow P(0) - \int_0^{\tau_2} B(\sigma)\, d\sigma \quad \text{as } k \longrightarrow 0. \tag{54}$$

Then, we get

$$P(\tau_1) - P(\tau_2) \longrightarrow \int_{\tau_1}^{\tau_2} B(\sigma)\, d\sigma \quad \text{as } k \longrightarrow 0. \tag{55}$$

Letting $\tau_1 \to 0$ and $\tau_2 \to t$, from (49), (50), and (55), for any $t \in [0, T_0]$, we have

$$\begin{aligned} \int_{-\infty}^{\infty} |v_1(t,x) - v_2(t,x)|\, dx \le & \int_{-\infty}^{\infty} |v_1(0,x) - v_2(0,x)|\, dx \\ & + c\int_0^t \int_{-\infty}^{\infty} |v_1 - v_2|\, dx, \end{aligned} \tag{56}$$

where c depends on $\|v_{10}\|_{L^1(R)}$, $\|v_{10}\|_{L^\infty(R)}$, $\|v_{20}\|_{L^1(R)}$, $\|v_{20}\|_{L^\infty(R)}$, and T. Using the Gronwall inequality and (56) completes the proof of Theorem 1. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] G. M. Coclite and K. H. Karlsen, "On the well-posedness of the Degasperis-Procesi equation," *Journal of Functional Analysis*, vol. 233, no. 1, pp. 60–91, 2006.

[2] A. Constantin and D. Lannes, "The hydrodynamical relevance of the Camassa-Holm and Degasperis-Procesi equations," *Archive for Rational Mechanics and Analysis*, vol. 192, no. 1, pp. 165–186, 2009.

[3] J. Escher, Y. Liu, and Z. Y. Yin, "Global weak solutions and blow-up structure for the Degasperis-Procesi equation," *Journal of Functional Analysis*, vol. 241, no. 2, pp. 457–485, 2006.

[4] Y. Fu, Y. Liu, and C. Z. Qu, "On the blow-up structure for the generalized periodic Camassa-Holm and Degasperis-Procesi equations," *Journal of Functional Analysis*, vol. 262, no. 7, pp. 3125–3158, 2012.

[5] D. Henry, "Infinite propagation speed for the Degasperis-Procesi equation," *Journal of Mathematical Analysis and Applications*, vol. 311, no. 2, pp. 755–759, 2005.

[6] G. L. Gui, Y. Liu, and L. X. Tian, "Global existence and blow-up phenomena for the peakon *b*-family of equations," *Indiana University Mathematics Journal*, vol. 57, no. 3, pp. 1209–1234, 2008.

[7] Y. Liu and Z. Y. Yin, "Global existence and blow-up phenomena for the Degasperis-Procesi equation," *Communications in Mathematical Physics*, vol. 267, no. 3, pp. 801–820, 2006.

[8] S. Y. Lai and Y. H. Wu, "A model containing both the Camassa-Holm and Degasperis-Procesi equations," *Journal of Mathematical Analysis and Applications*, vol. 374, no. 2, pp. 458–469, 2011.

[9] S. Lai and A. Y. Wang, "The well-posedness of solutions for a generalized shallow water wave equation," *Abstract and Applied Analysis*, vol. 2012, Article ID 872187, 15 pages, 2012.

[10] S. Y. Lai, "The global weak solution for a generalized Camassa-Holm equation," *Abstract and Applied Analysis*, vol. 2013, Article ID 838302, 6 pages, 2013.

[11] Z. W. Lin and Y. Liu, "Stability of peakons for the Degasperis-Procesi equation," *Communications on Pure and Applied Mathematics*, vol. 62, no. 1, pp. 125–146, 2009.

[12] O. G. Mustafa, "A note on the Degasperis-Procesi equation," *Journal of Nonlinear Mathematical Physics*, vol. 12, no. 1, pp. 10–14, 2005.

[13] M. Wu, "The local strong solutions and global weak solutions for a nonlinear equation," *Abstract and Applied Analysis*, vol. 2013, Article ID 619068, 5 pages, 2013.

[14] S. N. Kruzkov, "First order quasi-linear equations in several independent variables," *Mathematics of the USSR-Sbornik*, vol. 10, pp. 217–243, 1970.