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Research Article

An Inventory Model under Trapezoidal Type Demand, Weibull-Distributed Deterioration, and Partial Backlogging

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This paper studies an inventory model for Weibull-distributed deterioration items with trapezoidal type demand rate, in which shortages are allowed and partially backlogging depends on the waiting time for the next replenishment. The inventory models starting with no shortage is are to be discussed, and an optimal inventory replenishment policy of the model is proposed. Finally, numerical examples are provided to illustrate the theoretical results, and a sensitivity analysis of the major parameters with respect to the optimal solution is also carried out.

1. Introduction

The effect of deteriorating for items cannot be disregarded in many inventory systems and it is a general phenomenon in real life. Deterioration is defined as any process that decreases the usefulness or the value of the original item, such as decay or physical depletion. For example, fruits, vegetables, or foodstuffs are subject to spoilage directly while being kept in store, and electronic products, radioactive substances, and photographic film deteriorate through a gradual loss of potential or utility with the passage of time.

Due to the variability in economic circumstances, the basic assumptions of the EOQ model should be constantly modified according to the studied inventory model. In recent years, many researchers have studied kinds of EOQ models for deteriorating items. Ghare and Schrader [1] established the classical no-shortage inventory model with a constant rate of decay. Wu et al. [2] studied an inventory model with a Weibull-distributed deteriorating rate for items and assumed the demand rate with a continuous function of time. Wee [3] developed an inventory model with quantity discount, pricing, and partial backordering when the product in stock deteriorates with time. Related literature also includes Skouri and Papachristos [4], Wee [5], and Dye et al. [6].

practically, the demand rate of deterioration items is impossible to increase continuously all the time. Hill [7] proposed an inventory model with ramp type demand rate. Mandal and Pal [8] extended the inventory model with ramp type demand for deterioration items and allowed shortage. Wu [9] considered an inventory model with Weibull distribution deterioration and ramp type demand rate in which shortages are allowed and the backlogging rate is dependent on waiting time. Giri et al. [10] extended the ramp type demand inventory model with a more generalized Weibull deterioration distribution. Manna and Chaudhuri [11] developed an inventory model for time-dependent deteriorating items with ramp type demand rate. Skouri et al. [12] considered an inventory model with general ramp type demand rate, partial backlogging, and Weibull deterioration rate. Hung [13] extended their inventory model from ramp type demand rate and Weibull deterioration rate to arbitrary demand rate and arbitrary deterioration rate. Kumar et al. [14] studied fuzzy EOQ models with ramp type demand rate, partial backlogging, and time-dependent deterioration rat. Cheng et al. [15] considered an inventory model for time-dependent deteriorating items with trapezoidal type demand rate and partial backlogging. Uthayakumar and Rameswari [16] studied an inventory model for defective items with trapezoidal type demand rate to determine the optimal product reliability. Tan and Weng [17] considered a discrete-in-time inventory model for deteriorating items with partially backlogged. Ahmed et al. [18] proposed a method for finding the economic order quantity for an inventory model with ramp type demand rate, partial backlogging, and general deterioration rate. Lin [19] explored the inventory model with a general demand rate in which both the Weibull-distributed deterioration and partial backlogging are considered.

In the above mentioned research, one of assumptions was considered: the ramp type demand rate, partial backlogging, and Weibull-distributed deterioration rate. However, for fashionable commodities, high-tech products, and other short life cycle products, the demand rate should increase with the time up to certain point at first stage then reach a stabilized period and finally the demand rate decrease to zero and the products retreat from market in their product life cycle, that is, the demand rate with continuous trapezoidal function of time. On the other hand, in many real situations, customers encountering shortages will respond differently. Some customers are willing to wait until the next replenishment, while others may be impatient and go elsewhere as waiting time increases; that is, the willingness for a customer to wait for backlogging is diminishing with the length of the waiting time. In this paper, we consider an inventory model with Weibull-distributed deterioration items, trapezoidal type demand rate, and time-dependent partial backlogging. By analyzing the inventory model, a useful inventory replenishment policy is proposed. Finally, numerical examples are provided to illustrate the theoretical results, and a sensitivity analysis of the optimal solution with respect to major parameters is also carried out.

The rest of the paper is organized as follows. Section 2 describes the notation and assumptions used throughout this paper. Section 3 analyzes the inventory model, and some numerical examples to illustrate the solution procedure are provided. Sensitivity analysis of the major parameters is also carried out in Section 4, and the final Section concludes this paper.

2. Notations and Assumptions

The fundamental notations and assumptions used in inventory model and considered in this paper are given as below.

(i) $I(t)$ the level of inventory at time t, $0 \le t \le T$.

(ii) T the fixed length of each ordering cycle.

(iii) t_1 the time when the inventory level reaches zero for the inventory model.

(iv) t_1^* the optimal point.

(v) S the maximum inventory level for each ordering cycle.

(vi) Q^* the optimal ordering quantity.

(vii) A_0 the fixed cost per order.

(viii) c_1 the cost of each deteriorated item.

(ix) c_2 the inventory holding cost per unit per unit of time.

(x) c_3 the shortage cost per unit per unit of time.

(xi) c_4 the lost sales cost per unit.

(xii) $C_i(t_1)$ $i = 1, 2, 3$, the average total cost per unit time under different conditions, respectively.

(xiii) $TC(t_1)$ the average total cost per unit time.

(xiv) The demand rate, $D(t)$, which is positive and consecutive, is assumed to be a trapezoidal type function of time; that is,

$$D(t) = \begin{cases} f(t), & t \le \mu_1; \\ D_0, & \mu_1 < t < \mu_2; \\ g(t), & \mu_2 \le t < T, \end{cases} \tag{1}$$

where μ_1 is time point changing from the increasing demand function $f(t)$ to constant demand D_0, and μ_2 is time point changing from the constant demand D_0 to the decreasing demand function $g(t)$.

(xv) The replenishment rate is infinite; that is, replenishment is instantaneous.

(xvi) The deterioration rate of the item is defined as Weibull (α, β); that is the deterioration rate is $\theta(t) = \alpha\beta t^{\beta-1}$ $(\alpha > 0, \beta > 0, t > 0)$.

(xvii) Shortages are allowed and they adopt the notation used in Abad [20], where the unsatisfied demand is backlogged and the fraction of shortages backordered is $e^{-\delta t}$, where t is the waiting time up to the next replenishment. We also assume that $te^{-\delta t}$ is an increasing function, which had appeared in Skouri et al. [12].

(xviii) The time horizon of the inventory model is finite.

3. Model Formulation

In this section, we consider an inventory model starting with no shortage. The behavior of the model during a given cycle is depicted in Figure 1. Replenishment occurs at time $t = 0$ and the inventory level attains its maximum. From $t = 0$ to t_1, the inventory level reduces due to demand and deterioration. At t_1, the inventory level achieves zero, then shortage is allowed to occur during the time interval (t_1, T), and all of the demand during the shortage period (t_1, T) is partially backlogged. According to the notations and assumptions mentioned above, the behavior of the model at any time can be described by the following differential equations:

$$\frac{dI(t)}{dt} = \begin{cases} -\theta(t) I(t) - D(t), & 0 < t < t_1; \\ -e^{-\delta(T-t)} D(t), & t_1 < t < T, \end{cases} \tag{2}$$

with boundary conditions $I(0) = S$, $I(t_1) = 0$.

In the following, we consider three possible cases based on the values of t_1, μ_1, and μ_2. These three cases are shown.

Case 1 $(0 < t_1 \le \mu_1)$. Due to the deteriorating and trapezoidal type demand rate, the inventory level gradually diminishes during the time interval $[0, t_1]$ and ultimately falls to zero at time t_1. Thus, from (2), we have

$$\frac{dI(t)}{dt} = \begin{cases} -\alpha\beta t^{\beta-1} I(t) - f(t), & 0 < t < t_1; \\ -e^{-\delta(T-t)} f(t), & t_1 < t < \mu_1; \\ -e^{-\delta(T-t)} D_0, & \mu_1 < t < \mu_2; \\ -e^{-\delta(T-t)} g(t), & \mu_2 < t < T. \end{cases} \tag{3}$$

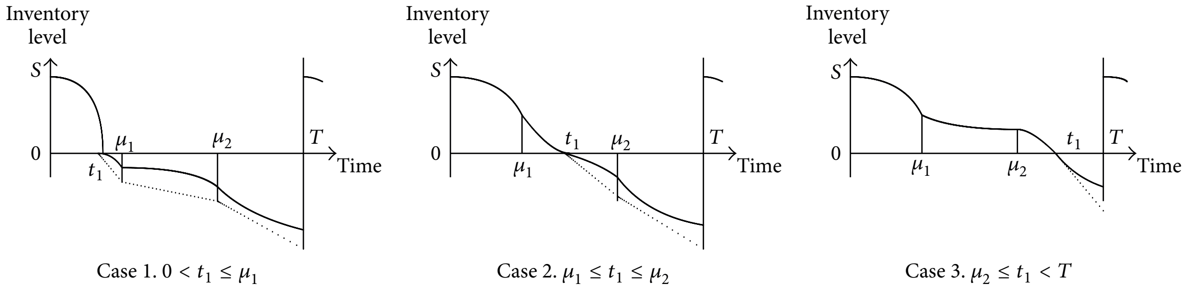

FIGURE 1: Graphical representation of inventory level over the cycle.

By using the boundary condition $I(t_1) = 0$, the solutions of (3) are given by

$$I(t) = \begin{cases} \int_t^{t_1} f(x) e^{\alpha(x^\beta - t^\beta)} dx, & 0 < t < t_1; \\ -\int_{t_1}^{t} e^{\delta(x-T)} f(x)\, dx, & t_1 < t < \mu_1; \\ \frac{D_0}{\delta}\left(e^{\delta(\mu_1 - T)} - e^{\delta(t-T)}\right) \\ \quad -\int_{t_1}^{\mu_1} e^{\delta(x-T)} f(x)\, dx, & \mu_1 < t < \mu_2; \\ -\int_{\mu_2}^{t} g(x) e^{\delta(x-T)} dx \\ \quad +\frac{D_0}{\delta}\left(e^{\delta(\mu_1 - T)} - e^{\delta(\mu_2 - T)}\right) \\ \quad -\int_{t_1}^{\mu_1} e^{\delta(x-T)} f(x)\, dx, & \mu_2 < t < T. \end{cases} \tag{4}$$

The maximum inventory level per cycle is

$$S = I(0) = \int_0^{t_1} f(x) e^{\alpha x^\beta} dx. \tag{5}$$

Then, the total number of deteriorated items D_T in the interval $[0, t_1]$ is

$$D_T = S - \int_0^{t_1} D(t)\, dt = \int_0^{t_1} f(x)\left(e^{\alpha x^\beta} - 1\right) dx. \tag{6}$$

The total number of inventory H_T carried during the interval $[0, t_1]$ is

$$H_T = \int_0^{t_1} I(t)\, dt = \int_0^{t_1} \int_t^{t_1} f(x) e^{\alpha(x^\beta - t^\beta)} dx\, dt. \tag{7}$$

The total shortage quantity B_T during the interval $[t_1, T]$ is

$$\begin{aligned} B_T &= -\int_{t_1}^{T} I(t)\, dt \\ &= \int_{t_1}^{\mu_1} \left[\int_{t_1}^{t} e^{\delta(x-T)} f(x)\, dx\right] dt \\ &\quad - \int_{\mu_1}^{\mu_2} \left[\frac{D_0}{\delta}\left(e^{\delta(\mu_1 - T)} - e^{\delta(t-T)}\right) - \int_{t_1}^{\mu_1} e^{\delta(x-T)} f(x)\, dx\right] dt \\ &\quad + \int_{\mu_2}^{T} \left[\int_{\mu_2}^{t} e^{\delta(x-T)} g(x)\, dx - \frac{D_0}{\delta}\left(e^{\delta(\mu_1 - T)} - e^{\delta(\mu_2 - T)}\right) + \int_{t_1}^{\mu_1} e^{\delta(x-T)} f(x)\, dx\right] dt \\ &= \int_{t_1}^{\mu_1} (T - t) e^{\delta(t-T)} f(t)\, dt \\ &\quad + \int_{\mu_2}^{T} (T - t) e^{\delta(t-T)} g(t)\, dt \\ &\quad + \frac{D_0}{\delta^2}\left(e^{\delta(\mu_2 - T)} - e^{\delta(\mu_1 - T)}\right) \\ &\quad + \frac{D_0}{\delta}\left[(T - \mu_2) e^{\delta(\mu_2 - T)} - (T - \mu_1) e^{\delta(\mu_1 - T)}\right]. \end{aligned} \tag{8}$$

The total of lost sales L_T during the interval $[t_1, T]$ is

$$\begin{aligned} L_T &= \int_{t_1}^{\mu_1} \left(1 - e^{\delta(t-T)}\right) f(t)\, dt + \int_{\mu_1}^{\mu_2} \left(1 - e^{\delta(t-T)}\right) D_0 dt \\ &\quad + \int_{\mu_2}^{T} \left(1 - e^{\delta(t-T)}\right) g(t)\, dt. \end{aligned} \tag{9}$$

Therefore, the average total cost per unit time under the condition $t_1 \le \mu_1$ can be given by

$$\begin{aligned} C_1(t_1) &= \frac{1}{T}\left[A_0 + c_1 D_T + c_2 H_T + c_3 B_T + c_4 L_T\right] \\ &= \frac{1}{T}\left\{A_0 + c_1\int_0^{t_1} f(x)\left(e^{\alpha x^\beta}-1\right)dx \right. \\ &\quad + c_2\int_0^{t_1}\int_t^{t_1} f(x)\,e^{\alpha(x^\beta - t^\beta)}dx\,dt \\ &\quad + c_4\left[\int_{t_1}^{\mu_1}\left(1-e^{\delta(t-T)}\right)f(t)\,dt \right. \\ &\qquad + \int_{\mu_1}^{\mu_2}\left(1-e^{\delta(t-T)}\right)D_0 dt \\ &\qquad \left. + \int_{\mu_2}^{T}\left(1-e^{\delta(t-T)}\right)g(t)\,dt\right] \\ &\quad + c_3\left[\int_{t_1}^{\mu_1} e^{\delta(t-T)}(T-t)\,f(t)\,dt \right. \\ &\qquad + \int_{\mu_2}^{T} e^{\delta(t-T)}(T-t)\,g(t)\,dt \\ &\qquad + \frac{D_0}{\delta^2}\left(e^{\delta(\mu_2-T)} - e^{\delta(\mu_1-T)}\right) \\ &\qquad + \frac{D_0}{\delta}\left((T-\mu_2)\,e^{\delta(\mu_2-T)} \right. \\ &\qquad \left.\left.\left. - (T-\mu_1)\,e^{\delta(\mu_1-T)}\right)\right]\right\}. \end{aligned} \tag{10}$$

Case 2 ($\mu_1 \le t_1 \le \mu_2$). The differential equations governing the inventory model can be expressed as follows:

$$\frac{dI(t)}{dt} = \begin{cases} -\alpha\beta t^{\beta-1} I(t) - f(t), & 0 < t < \mu_1; \\ -\alpha\beta t^{\beta-1} I(t) - D_0, & \mu_1 < t < t_1; \\ -e^{-\delta(T-t)} D_0, & t_1 < t < \mu_2; \\ -e^{-\delta(T-t)} g(t), & \mu_2 < t < T. \end{cases} \tag{11}$$

Solving the differential equation (11) with $I(t_1) = 0$, we have

$$I(t) = \begin{cases} \int_t^{\mu_1} f(x)\,e^{\alpha(x^\beta-t^\beta)}dx \\ \quad + D_0\int_{\mu_1}^{t_1} e^{\alpha(x^\beta - t^\beta)}dx, & 0 < t < \mu_1; \\ D_0\int_t^{t_1} e^{\alpha(x^\beta-t^\beta)}dx, & \mu_1 < t < t_1; \\ \frac{D_0}{\delta}\left(e^{\delta(t_1-T)} - e^{\delta(t-T)}\right), & t_1 < t < \mu_2; \\ -\int_{\mu_2}^{t} e^{\delta(x-T)} g(x)\,dx \\ \quad + \frac{D_0}{\delta}\left(e^{\delta(t_1-T)} - e^{\delta(\mu_2-T)}\right), & \mu_2 < t < T. \end{cases} \tag{12}$$

The beginning inventory level can be computed as

$$S = I(0) = \int_0^{\mu_1} f(x)\,e^{\alpha x^\beta}dx + D_0\int_{\mu_1}^{t_1} e^{\alpha x^\beta}dx. \tag{13}$$

The total number of items which perish in the interval $[0, t_1]$ is

$$D_T = \int_0^{\mu_1} f(x)\left(e^{\alpha x^\beta}-1\right)dx + D_0\int_{\mu_1}^{t_1}\left(e^{\alpha x^\beta}-1\right)dx. \tag{14}$$

The total number of inventory carried during the interval $[0, t_1]$ is

$$H_T = \int_0^{\mu_1}\int_t^{\mu_1} e^{\alpha(x^\beta-t^\beta)} f(x)\,dx\,dt + D_0\int_{\mu_1}^{t_1}\int_t^{t_1} e^{\alpha(x^\beta-t^\beta)}dx\,dt. \tag{15}$$

The total shortage quantity during the interval $[t_1, T]$ is

$$\begin{aligned} B_T = &\int_{\mu_2}^{T}(T-t)\,e^{\delta(t-T)} g(t)\,dt \\ &+ \frac{D_0}{\delta}\left[e^{\delta(t_1-T)}\left(T - t_1 + \frac{1}{\delta}\right)\right. \\ &\left. - e^{-\delta(\mu_2-T)}\left(T - \mu_2 + \frac{1}{\delta}\right)\right]. \end{aligned} \tag{16}$$

The total of lost sales during the interval $[t_1, T]$ is

$$L_T = D_0\int_{t_1}^{\mu_2}\left(1-e^{\delta(t-T)}\right)dt + \int_{\mu_2}^{T}\left(1-e^{\delta(t-T)}\right)g(t)\,dt. \tag{17}$$

Therefore, the average total cost per unit time under the condition $\mu_1 \le t_1 \le \mu_2$ can be given by

$$\begin{aligned} C_2(t_1) &= \frac{1}{T}\left[A_0 + c_1 D_T + c_2 H_T + c_3 B_T + c_4 L_T\right] \\ &= \frac{1}{T}\left\{A_0 + c_1\left[\int_0^{\mu_1} f(x)\left(e^{\alpha x^\beta}-1\right)dx \right.\right. \\ &\qquad \left. + D_0\int_{\mu_1}^{t_1}\left(e^{\alpha x^\beta}-1\right)dx\right] \\ &\quad + c_2\left[\int_0^{\mu_1}\int_t^{\mu_1} e^{\alpha(x^\beta-t^\beta)} f(x)\,dx\,dt \right. \\ &\qquad \left. + D_0\int_0^{t_1}\int_t^{t_1} e^{\alpha(x^\beta-t^\beta)}dx\,dt\right] \end{aligned}$$

$$+ c_3 \left[\int_{\mu_2}^{T} (T - t) e^{\delta(t-T)} g(t)\, dt + \frac{D_0}{\delta} \left(e^{\delta(t_1-T)} \left(T - t_1 + \frac{1}{\delta} \right) - e^{\delta(\mu_2-T)} \left(T - \mu_2 + \frac{1}{\delta} \right) \right) \right] + c_4 \left[D_0 \int_{t_1}^{\mu_2} \left(1 - e^{\delta(t-T)} \right) dt + \int_{\mu_2}^{T} \left(1 - e^{\delta(t-T)} \right) g(t)\, dt \right] \Bigg\}. \quad (18)$$

Case 3 ($\mu_2 \le t_1 < T$). The differential equations governing the inventory model can be expressed as follows:

$$\frac{dI(t)}{dt} = \begin{cases} -\alpha\beta t^{\beta-1} I(t) - f(t), & 0 < t < \mu_1; \\ -\alpha\beta t^{\beta-1} I(t) - D_0, & \mu_1 < t < \mu_2; \\ -\alpha\beta t^{\beta-1} I(t) - g(t), & \mu_2 < t < t_1; \\ -e^{-\delta(T-t)} g(t), & t_1 < t < T. \end{cases} \quad (19)$$

Solving the differential equation (19) with $I(t_1) = 0$, we have

$$I(t) = \begin{cases} \int_t^{\mu_1} e^{\alpha(x^\beta - t^\beta)} f(x)\, dx + D_0 \int_{\mu_1}^{\mu_2} e^{\alpha(x^\beta - t^\beta)} dx + \int_{\mu_2}^{t_1} e^{\alpha(x^\beta - t^\beta)} g(x)\, dx, & 0 < t < \mu_1; \\ D_0 \int_t^{\mu_2} e^{\alpha(x^\beta - t^\beta)} dx + \int_{\mu_2}^{t_1} e^{\alpha(x^\beta - t^\beta)} g(x)\, dx, & \mu_1 < t < \mu_2; \\ \int_t^{t_1} e^{\alpha(x^\beta - t^\beta)} g(x)\, dx, & \mu_2 < t < t_1; \\ -\int_{t_1}^{t} e^{\delta(x-T)} g(x)\, dx, & t_1 < t < T. \end{cases} \quad (20)$$

The beginning inventory level can be computed as

$$S = I(0) = \int_0^{\mu_1} e^{\alpha x^\beta} f(x)\, dx + D_0 \int_{\mu_1}^{\mu_2} e^{\alpha x^\beta} dx + \int_{\mu_2}^{t_1} e^{\alpha x^\beta} g(x)\, dx. \quad (21)$$

The total number of items which perish in the interval $[0, t_1]$ is

$$D_T = \int_0^{\mu_1} \left(e^{\alpha x^\beta} - 1 \right) f(x)\, dx + D_0 \int_{\mu_1}^{\mu_2} \left(e^{\alpha x^\beta} - 1 \right) dx + \int_{\mu_2}^{t_1} \left(e^{\alpha x^\beta} - 1 \right) g(x)\, dx. \quad (22)$$

The total number of inventory carried during the interval $[0, t_1]$ is

$$H_T = \int_0^{\mu_1} \int_t^{\mu_1} e^{\alpha(x^\beta - t^\beta)} f(x)\, dx\, dt + D_0 \int_0^{\mu_1} \int_{\mu_1}^{\mu_2} e^{\alpha(x^\beta - t^\beta)} dx\, dt + D_0 \int_{\mu_1}^{\mu_2} \int_t^{\mu_2} e^{\alpha(x^\beta - t^\beta)} dx\, dt + \int_0^{\mu_2} \int_{\mu_2}^{t_1} e^{\alpha(x^\beta - t^\beta)} g(x)\, dx\, dt + \int_{\mu_2}^{t_1} \int_t^{t_1} e^{\alpha(x^\beta - t^\beta)} g(x)\, dx\, dt. \quad (23)$$

The total shortage quantity during the interval $[t_1, T]$ is

$$B_T = \int_{t_1}^{T} (T - t) e^{-\delta(T-t)} g(t)\, dt. \quad (24)$$

The total of lost sales during the interval $[t_1, T]$ is

$$L_T = \int_{t_1}^{T} \left(1 - e^{-\delta(T-t)} \right) g(t)\, dt. \quad (25)$$

Therefore, the average total cost per unit time under the condition $\mu_2 \le t_1 \le T$ can be given by

$$C_3(t_1) = \frac{1}{T} \left[A_0 + c_1 D_T + c_2 H_T + c_3 B_T + c_4 L_T \right] = \frac{1}{T} \Bigg\{ A_0 + c_1 \left[\int_0^{\mu_1} \left(e^{\alpha x^\beta} - 1 \right) f(x)\, dx + D_0 \int_{\mu_1}^{\mu_2} \left(e^{\alpha x^\beta} - 1 \right) dx + \int_{\mu_2}^{t_1} \left(e^{\alpha x^\beta} - 1 \right) g(x)\, dx \right]$$

$$
\begin{aligned}
&+c_2\left[\int_0^{\mu_1}\int_t^{\mu_1} e^{\alpha(x^\beta-t^\beta)} f(x)\,dx\,dt\right.\\
&\quad+D_0\int_0^{\mu_1}\int_{\mu_1}^{\mu_2} e^{\alpha(x^\beta-t^\beta)}dx\,dt\\
&\quad+D_0\int_{\mu_1}^{\mu_2}\int_t^{\mu_2} e^{\alpha(x^\beta-t^\beta)}dx\,dt\\
&\quad+\int_0^{\mu_2}\int_{\mu_2}^{t_1} e^{\alpha(x^\beta-t^\beta)} g(x)\,dx\,dt\\
&\quad\left.+\int_{\mu_2}^{t_1}\int_t^{t_1} e^{\alpha(x^\beta-t^\beta)} g(x)\,dx\,dt\right]\\
&+c_3\left[\int_{t_1}^{T}(T-t)\,e^{\delta(t-T)} g(t)\,dt\right]\\
&+c_4\left[\int_{t_1}^{T}\left(1-e^{\delta(t-T)}\right) g(t)\,dt\right]\Bigg\}.
\end{aligned}
\tag{26}
$$

From the above analysis, we obtain that the total average cost of the model over the time interval $[0,T]$ is

$$
TC(t_1)=\begin{cases} C_1(t_1), & 0<t_1\le\mu_1;\\ C_2(t_1), & \mu_1<t_1\le\mu_2;\\ C_3(t_1), & \mu_2<t_1<T,\end{cases}
\tag{27}
$$

where $C_1(t_1)$, $C_2(t_1)$, and $C_3(t_1)$ are obtained from (10), (18), and (26), respectively.

In the following, we will provide the results which ensure the existence of a unique t_1, say t_1^*, so as to minimize the total average cost for the model system starting with no shortages.

If $0<t_1\le\mu_1$, taking the first-order derivative of $C_1(t_1)$ with respect to t_1, we obtain

$$
\frac{dC_1(t_1)}{dt_1}=\frac{f(t_1)}{T}h(t_1),
\tag{28}
$$

where

$$
\begin{aligned}
h(t_1)&=c_1\left(e^{\alpha t_1^\beta}-1\right)\\
&\quad+c_2\int_0^{t_1} e^{\alpha(t_1^\beta-t^\beta)}dt+c_3(t_1-T)\,e^{\delta(t_1-T)}\\
&\quad+c_4\left(e^{\delta(t_1-T)}-1\right),
\end{aligned}
\tag{29}
$$

then we can obtain $h(0)<0$ and $h(T)>0$. By using the assumption ($te^{-\delta t}$ is an increasing function, where t is the waiting time up to the next replenishment), we have

$$
\begin{aligned}
\frac{dh(t_1)}{dt_1}&=\alpha\beta t_1^{\beta-1}\left(c_1e^{\alpha t_1^\beta}+c_2\int_0^{t_1} e^{\alpha(t_1^\beta-t^\beta)}dt\right)\\
&\quad+\left[c_3\left(\delta(t_1-T)+1\right)+c_4\delta\right]e^{\delta(t_1-T)}+c_2>0,
\end{aligned}
\tag{30}
$$

which implies that $h(t_1)$ is a strictly monotone increasing function. Therefore, the equation

$$
\begin{aligned}
h(t_1)&=c_1\left(e^{\alpha t_1^\beta}-1\right)+c_2\int_0^{t_1} e^{\alpha(t_1^\beta-t^\beta)}dt\\
&\quad+c_3(t_1-T)\,e^{\delta(t_1-T)}+c_4\left(e^{\delta(t_1-T)}-1\right)\\
&=0
\end{aligned}
\tag{31}
$$

has a unique root $t_1^*\in(0,T)$ obtained by using Mathematica 9.0. Further, t_1^* is the only zero-point of $dC_1(t_1)/dt_1=0$ since $f(t_1)>0$.

If $0<t_1^*\le\mu_1$, for this t_1^*, we have

$$
\left.\frac{d^2C_1(t_1)}{dt_1^2}\right|_{t_1=t_1^*}=f(t_1^*)\frac{dh(t_1^*)}{Tdt_1^*}>0,
\tag{32}
$$

which means that the total average cost $C_1(t_1)$ can obtain its minimum value at t_1^*.

The optimal value of the order level, $S=I(0)$, is

$$
S^*=\int_0^{t_1^*} f(x)\,e^{\alpha x^\beta}dx,
\tag{33}
$$

and the optimal order quantity Q^* is

$$
\begin{aligned}
Q^*&=S^*+\int_{t_1^*}^{\mu_1} e^{\delta(t-T)} f(t)\,dt\\
&\quad+D_0\int_{\mu_1}^{\mu_2} e^{\delta(t-T)}dt+\int_{\mu_2}^{T} e^{\delta(t-T)} g(t)\,dt.
\end{aligned}
\tag{34}
$$

If $t_1^*\ge\mu_1$, then the optimal value of $C_1(t_1)$ is obtained at $t_1=\mu_1$.

If $\mu_1<t_1\le\mu_2$, taking the first-order and second-order derivative of $C_2(t_1)$ with respect to t_1, respectively, we obtain

$$
\frac{dC_2(t_1)}{dt_1}=\frac{D_0}{T}h(t_1).
\tag{35}
$$

If $\mu_1<t_1^*\le\mu_2$, for this t_1^*, we have

$$
\left.\frac{d^2C_2(t_1)}{dt_1^2}\right|_{t_1=t_1^*}=D_0\frac{dh(t_1^*)}{Tdt_1^*}>0,
\tag{36}
$$

where the function $h(t_1)$ is given by (31), and (36) implies that $C_2(t_1)$ is a strictly convex function in t_1 and obtained its minimum value at t_1^*. Therefore, the equation $h(t_1)=0$ has a unique root t_1^* in $(0,T)$.

The optimal value of the order level, $S=I(0)$, is

$$
S^*=\int_0^{\mu_1} f(x)\,e^{\alpha x^\beta}dx+D_0\int_{\mu_1}^{t_1^*} e^{\alpha x^\beta}dx,
\tag{37}
$$

and the optimal order quantity Q^* is

$$
Q^*=S^*+\int_{t_1^*}^{\mu_2} e^{\delta(t-T)}D_0\,dt+\int_{\mu_2}^{T} e^{\delta(t-T)} g(t)\,dt.
\tag{38}
$$

If $t_1^* \le \mu_1$, then the optimal value of $C_2(t_1)$ is obtained at $t_1^* = \mu_1$, and if $t_1^* \ge \mu_2$, then the optimal value of $C_2(t_1)$ is obtained at $t_1^* = \mu_2$.

If $\mu_2 < t_1 \le T$, taking the first-order and second-order derivative of $C_3(t_1)$ with respect to t_1, respectively, we obtain

$$\frac{dC_3(t_1)}{dt_1} = \frac{g(t_1)}{T} h(t_1). \quad (39)$$

If $\mu_2 < t_1^* \le T$, for this t_1^*, we have

$$\left.\frac{d^2 C_3(t_1)}{dt_1^2}\right|_{t_1 = t_1^*} = g(t_1^*) \frac{dh(t_1^*)}{T dt_1^*} > 0. \quad (40)$$

The function $h(t_1)$ is given by (31), and (40) implies that $C_3(t_1)$ can obtain its minimum value at t_1^*.

The optimal value of the order level, $S = I(0)$, is

$$S^* = \int_0^{\mu_1} e^{\alpha x^\beta} f(x)\, dx + D_0 \int_{\mu_1}^{\mu_2} e^{\alpha x^\beta} dx + \int_{\mu_2}^{t_1^*} e^{\alpha x^\beta} g(x)\, dx, \quad (41)$$

and the optimal order quantity Q^* is

$$Q^* = S^* + \int_{t_1^*}^{T} e^{\delta(t-T)} g(t)\, dt. \quad (42)$$

If $t_1^* \le \mu_2$, then the optimal value of $C_3(t_1)$ is obtained at $t_1^* = \mu_2$.

The above analysis shows that the three average cost functions $C_1(t_1)$, $C_2(t_2)$, and $C_3(t_1)$ can obtain their minimum value at $t_1^* \in (0, T)$ which is determined by (31). Therefore, based on the results analyzed above, this paper derives a procedure to locate the optimal replenishment policy starting with no shortage for the three cases. The procedure is as follows.

Step 1. Solve t_1^* from (31).

Step 2. Compare t_1^* to μ_1 and μ_2, respectively.

Step 2.1. If $t_1^* \in (0, \mu_1]$, then the optimal total average cost and the optimal order quantity can be obtained by (10) and (34), respectively.

Step 2.2. If $t_1^* \in (\mu_1, \mu_2]$, then the optimal total average cost and the optimal order quantity can be obtained by (18) and (38), respectively.

Step 2.3. If $t_1^* \in (\mu_2, T]$, then the optimal total average cost and the optimal order quantity can be obtained by (26) and (42), respectively.

Remark 1. In such considered inventory model starting with no shortage, if t_1 satisfies $\mu_1 < t_1 \le T < \mu_2$, the considered inventory model reduces to that of Skouri et al. [12].

4. Numerical Example

In order to demonstrate the above procedure which can be applied to obtain the optimal solution of the model, this paper presents several examples for the model, respectively. Examples are based on piecewise demand rate, such as $f(t) = a_1 + b_1 t$ and $g(t) = a_2 e^{-b_2 t}$.

Example 1. The parameter values are given as follows: $T = 12$ weeks, $\mu_1 = 4$ weeks, $\mu_2 = 8$ weeks, $\alpha = 0.005$, $\beta = 2$, $\delta = 0.04$, $a_1 = 30$ unit, $b_1 = 5$ unit, $a_2 = 100$ unit, $A_0 = \$500$, $c_1 = \$2$, $c_2 = \$3$, $c_3 = \$12$, and $c_4 = \$8$.

The model starting with no shortage; by solving the equation $h(t_1) = 0$, we have $t_1^* = 8.7622$. From (42) and (26), we obtain $Q^* = 582.5217$ and $TC(t_1^*) = 793.9986$, respectively.

Example 2. The parameter values are given as follows: $T = 12$ weeks, $\mu_1 = 4$ weeks, $\mu_2 = 8$ weeks, $\alpha = 0.005$, $\beta = 2$, $\delta = 0.02$, $a_1 = 30$ unit, $b_1 = 5$ unit, $a_2 = 100$ unit, $A_0 = \$500$, $c_1 = \$5$, $c_2 = \$10$, $c_3 = \$12$, and $c_4 = \$8$.

The model starting with no shortage, by solving the equation $h(t_1) = 0$, we have $t_1^* = 5.8330$. From (38) and (18), we obtain $Q^* = 528.4725$ and $TC(t_1^*) = 1601.4013$, respectively.

Example 3. The parameter values are given as follows: $T = 12$ weeks, $\mu_1 = 4$ weeks, $\mu_2 = 6$ weeks, $\alpha = 0.005$, $\beta = 1.6$, $\delta = 0.2$, $a_1 = 30$ unit, $b_1 = 5$ unit, $a_2 = 100$ unit, $A_0 = \$500$, $c_1 = \$5$, $c_2 = \$10$, $c_3 = \$12$, and $c_4 = \$8$.

The model starting with no shortage, solving the equation $h(t_1) = 0$, the optimal value of t_1 is $t_1^* = 2.3235$. The optimal ordering quantity is $Q^* = 272.6678$, and the minimum cost $TC(t_1^*) = 1258.82$.

In order to clearly indicate the effects of parameters such as δ, α, β, c_1, c_2, c_3, and c_4 on the optimal on-hand inventory S^*, the optimal ordering quantity Q^*, and the optimal total cost $TC(t_1^*)$, respectively, the paper will study the sensitivity of the optimal solution to changes in the value of different parameter associated with the studied inventory model. The sensitivity analysis is performed on the base of Example 1, and the results are shown in Table 1–7.

By studying the results of Table 1, it is found that the shortage time t_1^*, inventory level S^*, order quantity Q^*, and the total average cost $TC(t_1^*)$ gradually decrease as the shortage parameter δ increases for the model, respectively. We also find that the percentage increase of δ from 14.3% to 100% causes $TC(t_1^*)$ to decrease from 0.45% to 0.34%, Q^* decrease from 0.75% to 0.52%, t_1^* decrease from 0.78% to 0.53%, and S^* decrease from 1.02% to 0.69%. It is also observed that the value of t_1^*, S^*, Q^*, and $TC(t_1^*)$ all are lowly sensitive to the changes of δ for the considered inventory model.

By studying the results of Table 2, it is found that S^*, Q^*, and $TC(t_1^*)$ coordinates to the deterioration parameter α; the shortage time t_1^* decreases as α increases for the model. It is also found that the percentage increase of α from 16.7% to 100% causes $TC(t_1^*)$ to decrease by 2.066%–2.595%, Q^* to increase by 1.597%–2.44%, the shortage time t_1^* to decrease by 1.513%–1.455%, and S^* to increase by 0.917%–1.651%. It also observes that the value of t_1^*, S^*, Q^*, and $TC(t_1^*)$ all are moderately sensitive to the changes of α for the considered inventory model.

Table 1: The sensitivity of δ for the models in Example 1.

δ	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08
t_1^*	8.9664	8.9187	8.8689	8.8167	8.7622	8.7049	8.6449	8.5817	8.5152
S^*	472.9889	469.7083	466.2823	462.7001	458.9498	455.0167	450.8910	446.5519	441.9829
Q^*	595.6569	592.5872	589.3822	586.0308	582.5217	578.8412	574.9771	570.9112	566.6265
$TC(t_1^*)$	805.6323	802.8699	800.0139	797.0588	793.9986	790.8268	787.5365	784.1200	780.5690

Table 2: The sensitivity of α for the models in Example 1.

α	0	0.001	0.002	0.003	0.004	0.005	0.006	0.007
t_1^*	9.4545	9.3115	9.1702	9.0312	8.8950	8.7622	8.6328	8.5072
S^*	428.3286	435.4002	442.0048	448.1317	453.7785	458.9498	463.6555	467.9088
Q^*	523.8866	536.6685	548.9558	560.7091	571.9025	582.5217	592.5618	602.0259
$TC(t_1^*)$	704.6045	722.8867	741.0107	758.9297	776.6035	793.9986	811.0872	827.8474

Table 3: The sensitivity of β for the models in Example 1.

β	1.4	1.6	1.8	2.0	2.2	2.4	2.6	2.8
t_1^*	9.2876	9.1810	9.0142	8.7622	8.4047	7.9422	7.4073	6.8496
S^*	442.9058	447.5894	453.1802	458.9076	462.8345	461.6422	453.4686	438.8904
Q^*	545.1340	554.1055	566.4485	582.4772	601.2274	774.6099	766.4364	751.8582
$TC(t_1^*)$	745.7703	764.3043	791.7624	790.7114	829.3653	588.7417	473.1282	391.1266

Table 4: The sensitivity of c_1 for the models in Example 1.

c_1	0	0.4	1	1.6	2	2.4	2.6	3	3.6
t_1^*	8.8225	8.8103	8.7921	8.7741	8.7622	8.7503	8.7443	8.7326	8.7150
S^*	463.0980	462.2595	461.0101	459.7707	458.9498	458.1333	457.7266	456.9164	455.7090
Q^*	584.1909	583.8529	583.3499	582.8514	582.5217	582.1940	582.5217	581.7061	581.2226
$TC(t_1^*)$	783.5429	785.6518	788.7984	791.9251	793.9986	796.0633	797.0925	799.1443	802.2061

Table 5: The sensitivity of c_2 for the model in Example 1.

c_2	0	0.4	0.8	1.2	1.8	2.4	3	3.4	3.8
t_1^*	11.8343	11.267	10.7698	10.326	9.7377	9.2216	8.7622	8.4819	8.2196
S^*	673.7217	633.0852	597.9589	566.9228	526.1187	490.5382	458.9498	439.6922	421.6606
Q^*	679.6039	659.4483	642.7261	628.4787	610.4744	595.4175	582.5217	574.8648	567.8284
$TC(t_1^*)$	67.2679	193.9216	308.8975	413.9636	555.8150	681.6692	793.9986	862.3253	925.9516

Table 6: The sensitivity of c_3 for the models in Example 1.

c_3	10.4	10.6	10.8	11	11.2	11.6	12	12.4	12.8
t_1^*	8.4424	8.4859	8.5284	8.5698	8.6102	8.6879	8.7622	8.8329	8.9005
S^*	436.9771	439.9713	442.889	445.7332	448.5071	453.854	458.9498	463.8114	468.4587
Q^*	573.7973	574.9747	576.1255	577.2506	578.3509	580.4809	582.5217	584.4787	586.3587
$TC(t_1^*)$	763.8057	767.9203	771.9299	775.8387	779.6505	786.9976	793.9986	800.6779	807.0579

Table 7: The sensitivity of c_4 for the models in Example 1.

c_4	0	2	4	6	8	10	12	14	16
t_1^*	8.6989	8.7150	8.7309	8.7466	8.7622	8.7775	8.7928	8.8078	8.82272
S^*	454.6069	455.7103	456.8017	457.8815	458.9498	460.0048	461.0527	462.0877	463.1120
Q^*	580.7818	581.2231	581.6601	582.0929	582.5217	582.9455	583.367	583.7837	584.1966
$TC(t_1^*)$	788.1626	789.6445	791.1109	792.5621	793.9986	795.4203	796.8276	798.2207	799.5999

By studying the results of Table 3, it is found that S^*, Q^*, and $TC(t_1^*)$ coordinate to the deterioration parameter β, while the shortage time t_1^* decreases as β increases for the model. It is also found that the increase of β from 1.4 to 2.2 causes S^* to increase, while the increase of β from 2.4 to 2.8 causes S^* to decrease, Q^* to increase by 2.44%–1.597%, $TC(t_1^*)$ to increase by 2.595%–2.066%, and the shortage time t_1^* to decrease by 1.513%–1.455%. It is also observed that the value of t_1^*, S^*, Q^*, and $TC(t_1^*)$ all are moderately sensitive to the changes of β for the considered inventory model.

By studying the results of Table 4, it is found that $TC(t_1^*)$ coordinate to c_1, while the shortage time t_1^*, S^*, and Q^* decrease as c_1 increases for the model. It is also found that c_1 increases from 8.3% to 150%, $TC(t_1^*)$ decreases by 0.269%–0.383%, Q^* decreases by 0.083%–0.058%, t_1^* decreases by 0.264%–0.181%, and S^* decreases by 0.203%–0.138%, respectively. It is also observed that the values of t_1^*, S^*, Q^*, and $TC(t_1^*)$ all are lowly sensitive to the changes of c_1 for the considered inventory model.

By studying the results of Table 5, it is found that $TC(t_1^*)$ coordinates to c_2, while S^*, Q^*,and t_1^* decrease as c_1 increases for the model. It is also found that c_2 increases by 100%, $TC(t_1^*)$ decreases by 0.269%–0.383%, Q^* decreases by 0.083%–0.058%, t_1^* decreases by 0.264%–0.181%, and S^* decreases by 0.203%–0.138%.

By studying the results of Table 6, it is found that t_1^*, S^*, Q^*, and $TC(t_1^*)$ coordinate to c_3. It is also observed that the value of t_1^*, S^*, Q^*, and $TC(t_1^*)$ all are lowly sensitive to the changes of c_3 for the inventory models; that is, c_3 increases from 1.9% to 3.2%, the change of all the parameters is no more than 1%.

By studying the results of Table 7, it is found that t_1^*, S^*, Q^*, and $TC(t_1^*)$ coordinate to c_4. It is also observed that the value of t_1^*, S^*, Q^*, and $TC(t_1^*)$ all are lowly sensitive to the changes of c_4 for the inventory models; that is, c_4 increases from 14.3% to 100%, the change of all the parameters is no more than 1%.

5. Conclusion

An inventory model starting without shortage for Weibull-distributed deterioration with trapezoidal type demand rate and partial backlogging is considered in this paper. The optimal replenishment policy for the inventory model is proposed, and numerical examples are provided to illustrate the theoretical results. A sensitivity analysis of the optimal solution with respect to major parameters is also carried out. From Table 1–7, it can be found that the shortage time point t_1^*, order quantity Q^*, and the total average cost $TC(t_1^*)$ are moderately sensitive to the changes of α and β and lowly sensitive to the changes of δ, $c_i\,(i = 1, 2, 3, 4)$, respectively. The paper provides an interesting topic for further study, such that the joint influence from some of these parameters may be investigated to show the effects; the model starting with shortage will be studied and other types of models for deteriorating items in supply chain situation are also to be studied in the future.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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