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Research Article

Perturbation of m-Isometries by Nilpotent Operators

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We prove that if T is an m-isometry on a Hilbert space and Q an n-nilpotent operator commuting with T, then $T+Q$ is a $(2n+m-2)$-isometry. Moreover, we show that a similar result for (m,q)-isometries on Banach spaces is not true.

1. Introduction

The notion of m-isometric operators on Hilbert spaces was introduced by Agler [1]. See also [2–5]. Recently Sid Ahmed [6] has defined m-isometries on Banach spaces, Bayart [7] introduced (m,q)-isometries on Banach spaces, and (m,q)-isometries on metric spaces were considered in [8]. Moreover, Hoffman et al. [9] have studied the role of the second parameter q. Recall the main definitions.

A map $T : E \to E$ ($m \geq 1$ integer and $q > 0$ real), defined on a metric space E with distance d, is called an *(m,q)-isometry* if, for all $x, y \in E$,

$$\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} d\left(T^k x, T^k y\right)^q = 0. \quad (1)$$

We say that T is a *strict (m,q)-isometry* if either $m = 1$ or T is an (m,q)-isometry with $m > 1$ but is not an $(m-1,q)$-isometry. Note that $(1,q)$-isometries are isometries.

The above notion of an (m,q)-isometry can be adapted to Banach spaces in the following way: a bounded linear operator $T : X \to X$, where X is a Banach space with norm $\|\cdot\|$, is an (m,q)-isometry if and only if, for all $x \in X$,

$$\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \left\|T^k x\right\|^q = 0. \quad (2)$$

In the setting of Hilbert spaces, the case $q = 2$ can be expressed in a special way. Agler [1] gives the following definition: a linear bounded operator $T : H \to H$ acting on a Hilbert space H is an $(m,2)$-isometry if

$$\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} T^{*k} T^k = 0. \quad (3)$$

$(m,2)$-isometries on Hilbert spaces will be called for short m-isometries.

The paper is organized as follows. In the next section we collect some results about applications of arithmetic progressions to m-isometric operators.

In Section 3 we prove that, in the setting of Hilbert spaces, if T is an m-isometry, Q is an n-nilpotent operator, and they commute, and then $T+Q$ is a $(2n+m-2)$-isometry. This is a partial generalization of the following result obtained in [10, Theorem 2.2]: if T is an isometry and Q is a nilpotent operator of order n commuting with T, then $T+Q$ is a strict $(2n-1)$-isometry.

In the last section we give some examples of operators on Banach spaces which are of the form identity plus nilpotent, but they are not (m,q)-isometries, for any positive integer m and any positive real number q.

Notation. Throughout this paper H denotes a Hilbert space and $B(H)$ the algebra of all linear bounded operators on H. Given $T \in B(H)$, T^* denotes its adjoint. Moreover, $m \geq 1$ is an integer and $q > 0$ a real number.

2. Preliminaries: Arithmetic Progressions and (m,q)-Isometries

In this section we give some basic properties of m-isometries. We need some preliminaries about arithmetic progressions and their applications to m-isometries. In [11], some results about this topic are recollected.

Let G be a commutative group and denote its operation by $+$. Given a sequence $a = (a_n)_{n\geq 0}$ in G, the difference sequence $Da = (Da)_{n\geq 0}$ is defined by $(Da)_n := a_{n+1} - a_n$. The powers of D are defined recursively by $D^0 a := a$, $D^{k+1}a = D(D^k a)$. It is easy to show that

$$\left(D^k a\right)_n = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} a_{i+n}, \tag{4}$$

for all $k \geq 0$ and $n \geq 0$ integers.

A sequence a in a group G is called an *arithmetic progression* of order $h = 0, 1, 2\ldots$, if $D^{h+1}a = 0$. Equivalently,

$$\sum_{i=0}^{h+1} (-1)^{h+1-i} \binom{h+1}{i} a_{i+j} = 0 \tag{5}$$

for $j = 0, 1, 2, \ldots$. It is well known that the sequence a in G is an arithmetic progression of order h if and only if there exists a polynomial $p(n)$ in n, with coefficients in G and of degree less than or equal to h, such that $p(n) = a_n$, for every $n = 0, 1, 2\ldots$; that is, there are $\gamma_h, \gamma_{h-1}, \ldots, \gamma_1, \gamma_0 \in G$, which depend only on a, such that, for every $n = 0, 1, 2, \ldots$,

$$a_n = p(n) = \sum_{i=0}^{h} \gamma_i n^i. \tag{6}$$

We say that the sequence a is an *arithmetic progression of strict order* $h = 0, 1, 2\ldots$, if $h = 0$ or if it is of order $h > 0$ but is not of order $h-1$; that is, the polynomial p of (6) has degree h.

Moreover, a sequence a in a group G is an arithmetic progression of order h if and only if, for all $n \geq 0$,

$$a_n = \sum_{k=0}^{h} (-1)^{h-k} \frac{n(n-1)\cdots\overbrace{(n-k)}\cdots(n-h)}{k!\,(h-k)!} a_k; \tag{7}$$

that is,

$$a_n = \sum_{k=0}^{h} (-1)^{h-k} \binom{n}{k} \binom{n-k-1}{h-k} a_k. \tag{8}$$

Now we give a basic result about m-isometries.

Theorem 1. *Let H be a Hilbert space. An operator $T \in B(H)$ is a strict m-isometry if and only if there are $A_{m-1} \neq 0, A_{m-2}, \ldots, A_1, A_0$ in $B(H)$, which depend only on T, such that, for every $n = 0, 1, 2\ldots$,*

$$T^{*n}T^n = \sum_{i=0}^{m-1} A_i n^i; \tag{9}$$

that is, the sequence $\left(T^{*n}T^n\right)_{n\geq 0}$ is an arithmetic progression of strict order $m-1$ in $B(H)$.

Proof. If $T \in B(H)$ is a strict m-isometry, then it satisfies (3). Hence, for each integer $i \geq 0$,

$$\begin{aligned}\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} T^{*i}T^{*k}T^kT^i \\ = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} T^{*k+i}T^{k+i} = 0,\end{aligned} \tag{10}$$

but

$$\sum_{k=0}^{m-1} (-1)^{m-1-k} \binom{m-1}{k} T^{*k}T^k \neq 0. \tag{11}$$

By (5), the operator sequence $(T^{*n}T^n)_{n\geq 0}$ is an arithmetic progression of strict order $m-1$. Therefore, from (6) we obtain that there is a polynomial $p(n)$ of degree $m-1$ in n, with coefficients in $B(H)$ satisfying $p(n) = T^{*n}T^n$; that is, there are operators $A_{m-1} \neq 0, A_{m-2}, \ldots, A_1, A_0$ in $B(H)$, such that, for every $n = 0, 1, 2\ldots$,

$$T^{*n}T^n = A_{m-1}n^{m-1} + A_{m-2}n^{m-2} + \cdots + A_1 n + A_0. \tag{12}$$

Conversely, if $(T^{*n}T^n)_{n\geq 0}$ is an arithmetic progression of strict order $m-1$, then (10) and (11) hold. Taking $i = 0$ we obtain (3), so T is a strict m-isometry. □

Now we recall an elementary property of (m,q)-isometries on metric spaces which will be used in the next sections.

Proposition 2 (see [8, Proposition 3.11]). *Let E be a metric space and let $T : E \to E$ be an (m,q)-isometry. If T is an invertible strict (m,q)-isometry, then m is odd.*

3. m-Isometry Plus n-Nilpotent

Recall that an operator $Q \in B(H)$ is *nilpotent of order n* ($n \geq 1$ integer), or *n-nilpotent*, if $Q^n = 0$ and $Q^{n-1} \neq 0$.

In any finite dimensional Hilbert space H, strict m-isometries can be characterized in a very simple way: a linear operator $T \in B(H)$ is a strict m-isometry if and only if m is odd and $T = A + Q$, where A and Q are commuting operators on H and A is unitary and Q a nilpotent operator of order $(m+1)/2$ ([12, page 134] and [10, Theorem 2.7]).

It was proved in [10, Theorem 2.2] that if $A \in B(H)$ is an isometry and $Q \in B(H)$ is an n-nilpotent operator such that $TQ = QT$, then $T + Q$ is a strict $(2n-1)$-isometry. Now we obtain a partial generalization of this result: if $T \in B(H)$ is an m-isometry and $Q \in B(H)$ is an n-nilpotent operator commuting with T, then $T + Q$ is a $(2n+m-2)$-isometry. However, $T + Q$ is not necessarily a strict $(2n+m-2)$-isometry. For example, if T is an isometry and Q any n-nilpotent operator ($n > 1$) such that $TQ = QT$, then $T = T + Q + (-Q)$ is not a strict $(4n-3)$-isometry.

Theorem 3. *Let H be a Hilbert space. Let $T \in B(H)$ be an m-isometry and $Q \in B(H)$ an n-nilpotent operator ($n \geq 1$ integer) such that $TQ = QT$. Then $T + Q$ is $(2n+m-2)$-isometry.*

Proof. Fix an integer $k \geq 0$ and denote $h := \min\{k, n-1\}$. Then we have

$$\begin{aligned}(T+Q)^{*k}(T+Q)^k &= \left(\sum_{i=0}^{h}\binom{k}{i}Q^{*i}T^{*k-i}\right)\left(\sum_{j=0}^{h}\binom{k}{j}T^{k-j}Q^j\right)\\ &= \sum_{i,j=0}^{h}\binom{k}{i}\binom{k}{j}Q^{*i}T^{*k-i}T^{k-j}Q^j\\ &= \sum_{0\le i<j\le h}\binom{k}{i}\binom{k}{j}Q^{*i}T^{*j-i}T^{*k-j}T^{k-j}Q^j\\ &\quad+\sum_{0\le j\le i\le h}\binom{k}{i}\binom{k}{j}Q^{*i}T^{*k-i}T^{k-i}T^{i-j}Q^j.\end{aligned} \tag{13}$$

From (9) we obtain, for certain $A_{m-1}, \ldots, A_0 \in B(H)$,

$$\begin{aligned}(T+Q)^{*k}(T+Q)^k &= \sum_{0\le i<j\le h}\binom{k}{i}\binom{k}{j}Q^{*i}T^{*j-i}\left(\sum_{r=0}^{m-1}A_r(k-j)^r\right)Q^j\\ &\quad+\sum_{0\le j\le i\le h}\binom{k}{i}\binom{k}{j}Q^{*i}\left(\sum_{r=0}^{m-1}A_r(k-i)^r\right)T^{i-j}Q^j.\end{aligned} \tag{14}$$

Write

$$\begin{aligned}B_{r,i,j} &:= Q^{*i}T^{*j-i}A_rQ^j \in B(H),\\ C_{r,i,j} &:= Q^{*i}A_rT^{i-j}Q^j \in B(H),\\ q_{r,i,j} &:= \binom{k}{i}\binom{k}{j}(k-j)^r,\\ p_{r,i,j} &:= \binom{k}{i}\binom{k}{j}(k-i)^r.\end{aligned} \tag{15}$$

Note that $\binom{k}{i}$ and $\binom{k}{j}$ are real polynomials in k of degree less than or equal to $h \leq n-1$, and $(k-j)^r$ and $(k-i)^r$ have degree $r \leq m-1$. Hence $q_{r,i,j}$ and $p_{r,i,j}$ are real polynomials of degree less than or equal to $m-1+2(n-1) = 2n+m-3$. Consequently we can write

$$(T+Q)^{*k}(T+Q)^k = \sum_{r=0}^{m-1}\sum_{0\le i<j\le h}B_{r,i,j}q_{r,i,j} + \sum_{r=0}^{m-1}\sum_{0\le j\le i\le h}C_{r,i,j}p_{r,i,j}, \tag{16}$$

which is a polynomial in k, of degree less than or equal to $2n+m-3$ with coefficients in $B(H)$. By Theorem 1, the operator $T+Q$ is an $(2n+m-2)$-isometry. □

For isometries it is possible to say more [10, Theorem 2.2].

Theorem 4. *Let H be a Hilbert space. Let $T \in B(H)$ be an isometry and let $Q \in B(H)$ be an n-nilpotent operator ($n \geq 1$ integer) such that $TQ = QT$. Then $T+Q$ is a strict $(2n-1)$-isometry.*

Proof. By Theorem 3 we obtain that $T+Q$ is a $(2n-1)$-isometry; that is, $((T+Q)^{*k}(T+Q)^k)_{k\geq 0}$ is an arithmetic progression of order less than or equal to $2n-2$. Now we prove that it is an arithmetic progression of strict order $2n-2$, or equivalently the polynomial (9) has degree $2n-2$. Note that as T is an isometry we have $T^{*k}T^k = I$, for every positive integer k.

As in the proof of Theorem 3, for any integer $k \geq 0$, we have that

$$\begin{aligned}(T+Q)^{*k}(T+Q)^k &= \sum_{i,j=0}^{h}\binom{k}{i}\binom{k}{j}Q^{*i}T^{*k-i}T^{k-j}Q^j\\ &= \sum_{0\le i<j\le h}\binom{k}{i}\binom{k}{j}Q^{*i}T^{*j-i}Q^j\\ &\quad+\sum_{0\le j\le i\le h}\binom{k}{i}\binom{k}{j}Q^{*i}T^{i-j}Q^j,\end{aligned} \tag{17}$$

where $h := \min\{k, n-1\}$.

The coefficient of k^{2n-2} in the polynomial $(T+Q)^{*k}(T+Q)^k$ is

$$\left(\frac{1}{(n-1)!}\right)^2 Q^{*n-1}Q^{n-1}, \tag{18}$$

which is null if and only if $Q^{*n-1}Q^{n-1} = 0$, that is, if and only if $Q^{n-1} = 0$. Therefore, if Q is nilpotent of order n, then $(T+Q)^{*k}(T+Q)^k$ can be written as a polynomial in k, of degree $2n-2$ and coefficients in $B(H)$. Consequently $T+Q$ is a strict $(2n-1)$-isometry. □

Now we obtain the following corollary of Theorem 4.

Corollary 5. *Let H be a Hilbert space. Let $Q \in B(H)$ be an n-nilpotent operator ($n \geq 1$ integer). Then $I+Q$ is a strict $(2n-1)$-isometry.*

Recall that an operator $T \in B(H)$ is N-*supercyclic* ($N \geq 1$ integer) if there exists a subspace $F \subset H$ of dimension N such that its orbit $\{T^n x : n \geq 0, x \in F\}$ is dense in H. Moreover, T is called *supercyclic* if it is 1-supercyclic. See [13, 14].

Bayart [7, Theorem 3.3] proved that on an infinite dimensional Banach space an (m, q)-isometry is never N-supercyclic, for any $N \geq 1$. In the setting of Banach spaces, Yarmahmoodi et al. [15, Theorem 2.2] showed that any sum of an isometry and a commuting nilpotent operator is never supercyclic. For Hilbert space operators we extend the result [15, Theorem 2.2] to m-isometries plus commuting nilpotent operators.

Corollary 6. *Let H be an infinite dimensional Hilbert space. If $T \in B(H)$ is an m-isometry that commutes with a nilpotent operator Q, then $T+Q$ is never N-supercyclic for any N.*

4. Some Examples in the Setting of Banach Spaces

Theorem 4 is not true for finite-dimensional Banach spaces even for $m = 1$.

Denote $\ell_p^d := (\mathbb{C}^d, \|\cdot\|_p)$.

Example 1. Let $Q : \mathbb{C}^2 \to \mathbb{C}^2$ be defined by $Q(x, y) := (y, 0)$; hence Q is a 2-nilpotent operator. The following assertions hold:

(1) $I+Q$ is not a $(3, p)$-isometry on ℓ_p^2 for any $1 \le p < \infty$ and $p \ne 2$;

(2) $I + Q$ is not a $(3, p)$-isometry on ℓ_∞^2 for any $p > 0$;

(3) $I+Q$ is a strict $(2k+1, 2k)$-isometry on $(\mathbb{C}^2, \|\cdot\|_{2k})$ for any $k = 1, 2, 3, \ldots$.

Proof. For $(x, y) \in \mathbb{C}^2$ we have

$$\begin{aligned} (I+Q)(x,y) &= (x+y, y), \\ (I+Q)^2(x,y) &= (x+2y, y), \\ (I+Q)^3(x,y) &= (x+3y, y). \end{aligned} \tag{19}$$

Write

$$\begin{aligned} A(x,y;p,q) := {} & \left\|(I+Q)^3(x,y)\right\|_p^q \\ & - 3\left\|(I+Q)^2(x,y)\right\|_p^q \\ & + 3\left\|(I+Q)(x,y)\right\|_p^q - \left\|(x,y)\right\|_p^q. \end{aligned} \tag{20}$$

(1) We consider two cases: $1 < p < \infty$ and $p = 1$.

(a) Case $1 < p < \infty$. For $x = 0$, $y = 1$, and $q = p$, we have

$$\begin{aligned} A(0,1;p,p) &= 3^p + 1 - 3\cdot 2^p - 3 + 6 - 1 \\ &= 3^p - 3\cdot 2^p + 3. \end{aligned} \tag{21}$$

So $A(0, 1; p, p) = 0$ if and only if $3^{p-1} + 1 = 2^p$, which is true only when $p = 2$ or $p = 1$ since the function $f(t) = 3^{t-1} + 1 - 2^t$ is null only for $t = 1$ and $t = 2$.

Consequently $I + Q$ is not a $(3, p)$-isometry on ℓ_p^2 if $p \ne 2$ and $1 < p < \infty$.

(b) Case $p = 1$. In order to prove that $I+Q$ is not a $(3, 1)$-isometry on ℓ_1^2, we take the vector $(1, -1)$ and obtain that

$$\begin{aligned} A(1,-1;1,1) = {} & \left\|(I+Q)^3(1,-1)\right\|_1 \\ & - 3\left\|(I+Q)^2(1,-1)\right\|_1 + 3\left\|(I+Q)(1,-1)\right\|_1 \\ & - \left\|(1,-1)\right\|_1 \ne 0. \end{aligned} \tag{22}$$

(2) For $(x, y) \in \mathbb{C}^2$ we have

$$\begin{aligned} A(x,y;\infty,p) := {} & \left\|(I+Q)^3(x,y)\right\|_\infty^p \\ & - 3\left\|(I+Q)^2(x,y)\right\|_\infty^p + 3\left\|(I+Q)(x,y)\right\|_\infty^p \\ & - \left\|(x,y)\right\|_\infty^p \\ = {} & \max\{|x+3y|, |y|\}^p - 3\max\{|x+2y|, |y|\}^p \\ & + 3\max\{|x+y|, |y|\}^p - \max\{|x|, |y|\}^p. \end{aligned} \tag{23}$$

In particular, for $x := 1$ and $y := -1$,

$$A(1,-1;\infty,p) = 2^p - 1 \ne 0. \tag{24}$$

Therefore $I + Q$ is not a $(3, p)$-isometry on ℓ_∞^2 for any $p > 0$.

(3) First we prove by induction on k that $I + Q$ is a $(2k + 1, 2k)$-isometry on ℓ_{2k}^2 for any $k = 1, 2, 3 \ldots$. Note that, for $(x, y) \in \mathbb{C}^2$,

$$(I+Q)^s(x,y) = (x+sy, y), \quad (s = 0, 1, 2 \ldots). \tag{25}$$

By Corollary 5, the operator $I + Q$ is a strict $(3, 2)$-isometry on ℓ_2^2. Hence $I + Q$ is a strict $(2k + 1, 2k)$-isometry on ℓ_2^2 for all $k = 1, 2, 3 \ldots$ [9, Corollary 4.6]. Thus for $(x, y) \in \mathbb{C}^2$,

$$\sum_{s=0}^{2k+1} (-1)^{2k+1-s} \binom{2k+1}{s} \left(|x+sy|^2 + |y|^2\right)^k = 0. \tag{26}$$

Suppose that $I+Q$ is a $(2i-1, 2i-2)$-isometry on ℓ_{2i-2}^2 for every $i = 2, 3, \ldots, k$. Hence $I + Q$ is also a $(2k + 1, 2i - 2)$-isometry on ℓ_{2i-2}^2. Then, for $(x, y) \in \mathbb{C}^2$,

$$\sum_{s=0}^{2k+1} (-1)^{2k+1-s} \binom{2k+1}{s} \left(|x+sy|^{2i-2} + |y|^{2i-2}\right) = 0, \quad (2 \le i \le k). \tag{27}$$

Therefore,

$$\sum_{s=0}^{2k+1} (-1)^{2k+1-s} \binom{2k+1}{s} |x+sy|^{2i-2} = 0, \quad (2 \le i \le k). \tag{28}$$

Taking into account equality (28) we can write (26) in the following way:

$$\begin{aligned} 0 &= \sum_{s=0}^{2k+1} (-1)^{2k+1-s} \binom{2k+1}{s} \sum_{i=0}^{k} \binom{k}{i} |x+sy|^{2i} |y|^{2(k-i)} \\ &= \sum_{i=0}^{k-1} \binom{k}{i} |y|^{2(k-i)} \sum_{s=0}^{2k+1} (-1)^{2k+1-s} \binom{2k+1}{s} |x+sy|^{2i} \end{aligned}$$

$$+\sum_{s=0}^{2k+1}(-1)^{2k+1-s}\binom{2k+1}{s}|x+sy|^{2k}$$
$$=\sum_{s=0}^{2k+1}(-1)^{2k+1-s}\binom{2k+1}{s}\left(|x+sy|^{2k}+|y|^{2k}\right). \quad (29)$$

Therefore $I+Q$ is a $(2k+1,2k)$-isometry on ℓ^2_{2k}.

Now we prove that $I+Q$ is a strict $(2k+1,2k)$-isometry on ℓ^2_{2k}. Suppose on the contrary that $I+Q$ is a $(2k,2k)$-isometry on ℓ^2_{2k}. Then,

$$\sum_{s=0}^{2k-1}(-1)^{2k-1-s}\binom{2k-1}{s}\left(|x+sy|^{2k}+|y|^{2k}\right)=0 \quad (30)$$

for all $(x,y)\in\mathbb{C}^2$. So

$$\sum_{s=0}^{2k-1}(-1)^{2k-1-s}\binom{2k-1}{s}|x+sy|^{2k}=0 \quad (31)$$

for all $(x,y)\in\mathbb{C}^2$. In particular, for $y=1$ and $x=0,1,2,\ldots$, we have

$$\sum_{s=0}^{2k-1}(-1)^{2k-1-s}\binom{2k-1}{s}(x+s)^{2k}=0. \quad (32)$$

So $(s^{2k})_{s=0}^{\infty}$ is an arithmetic progression of order $2k-2$, which is a contradiction with (6). □

Remark 7. Notice that, in any Hilbert space of dimension n, there are strict m-isometries only for any $m\le 2n-1$. However, as the above example shows, there are strict $(2k+1,2k)$-isometries for any integer k in a Banach space of dimension 2.

The following example gives an operator of the form $I+Q$ with Q a nilpotent operator such that $I+Q$ is not an (m,q)-isometry for any integer m and any $q>0$.

Example 2. Let X be the Banach space of all real continuous functions f on $[0,1]$ such that $f(1)=0$ endowed with the supremun norm. Define $Q: X\to X$ by

$$(Qf)(t):=\begin{cases} f\left(t+\frac{1}{2}\right), & \text{if } 0\le t\le \frac{1}{2},\\ 0, & \text{if } \frac{1}{2}<t\le 1.\end{cases} \quad (33)$$

Then $Q\in B(X)$ is 2-nilpotent operator. Moreover, $I+Q$ is not an (m,q)-isometry for any $m=1,2,3,\ldots$ and any $q>0$.

Proof. It is clear that $I+Q$ is not an isometry since the function $f\in X$ given by

$$f(t):=\begin{cases} 1, & \text{if } 0\le t\le \frac{1}{2},\\ -2t+2, & \text{if } \frac{1}{2}<t\le 1\end{cases} \quad (34)$$

satisfies $\|f\|=1$ and $\|(I+Q)f\|=2$.

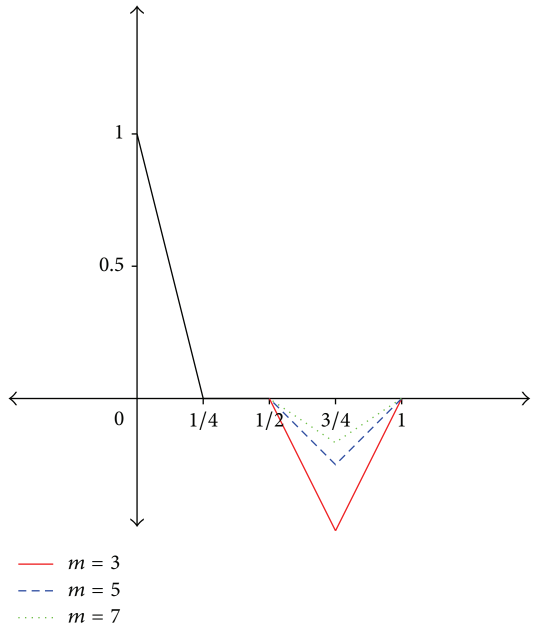

Figure 1: Graphics of functions f_3, f_5, and f_7.

For $m=2,3,4,\ldots$ consider the function $f_m\in X$ defined by

$$f_m(t):=\begin{cases} -4t+1, & \text{if } 0\le t\le \frac{1}{4},\\ 0, & \text{if } \frac{1}{4}<t\le \frac{1}{2},\\ \frac{-4}{m-1}t+\frac{2}{m-1}, & \text{if } \frac{1}{2}<t\le \frac{3}{4},\\ \frac{4}{m-1}t-\frac{4}{m-1}, & \text{if } \frac{3}{4}<t\le 1.\end{cases} \quad (35)$$

Note that $f_m(3/4)=1/(1-m)=\min_{0\le t\le 1}f_m(t)$ (Figure 1). □

Fix $q>0$. For $k=0,1,2,\ldots$, we have

$$\left\|(I+Q)^k f_m\right\|^q=\left\|(I+kQ)f_m\right\|^q = \sup_{0\le t\le 1}\left|f_m(t)+k(Qf_m)(t)\right|^q. \quad (36)$$

If $0\le k\le m-1$, then

$$\left\|(I+Q)^k f_m\right\|^q=\left|f_m(0)+kf_m\left(\frac{1}{2}\right)\right|^q=1, \quad (37)$$

since $k(1/(m-1))\le 1$. But as $m(1/(m-1))>1$ we obtain

$$\left\|(I+Q)^m f_m\right\|^q=\left|f_m\left(\frac{1}{4}\right)+mf_m\left(\frac{3}{4}\right)\right|^q = \left(\frac{m}{m-1}\right)^q>1. \quad (38)$$

Consequently,

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\left\|(I+Q)^{k}f_{m}\right\|^{q}$$
$$=\sum_{k=0}^{m-1}(-1)^{m-\ell}\binom{m}{k}+\left\|(I+Q)^{m}f_{m}\right\|^{q} \quad (39)$$
$$=-1+\left(\frac{m}{m-1}\right)^{q}\neq 0.$$

Therefore $I+Q$ is not an (m,q)-isometry for any $m=1,2,3\ldots$ and any $q>0$.

Disclosure

After submitting this paper for publication we received from Le and Gu et al. the papers [16, 17], in which they obtained (independently) Theorem 3. Their arguments are different from ours, using the Hereditary Functional Calculus.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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