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Research Article

Convergence Axioms on Dislocated Symmetric Spaces

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Dislocated symmetric spaces are introduced, and implications and nonimplications among various kinds of convergence axioms are derived.

1. Introduction

A metric space is a special kind of topological space. In a metric space, topological properties are characterized by means of sequences. Sequences are not sufficient in topological spaces for such purposes. It is natural to try to find classes intermediate between those of topological spaces and those of metric spaces in which members sequences play a predominant part in deciding their topological properties. A galaxy of mathematicians consisting of such luminaries as Frechet [1], Chittenden [2], Frink [3], Wilson [4], Niemytzki [5], and Aranđelović and Kečkić [6] have made important contributions in this area. The basic definition needed by most of these studies is that of a symmetric space. If X is a nonempty set, a function $: X \times X \rightarrow R^+$ is called a dislocated symmetric on X if $d(x, y) = 0$ implies that $x = y$ and $d(x, y) = d(y, x)$ for all $x, y \in X$. A dislocated symmetric (simply d-symmetric) on X is called symmetric on X if $d(x, x) = 0$ for all x in X. The names dislocated symmetric space and symmetric space have expected meanings. Obviously, a symmetric space that satisfies the triangle inequality is a metric space. Since the aim of our study is to find how sequential properties and topological properties influence each other, we collect various properties of sequences that have been shown in the literature to have a bearing on the problem under study. In what follows "d" denotes a dislocated distance on a nonempty set X. x_n, y_n, x, y, and so forth are elements of X and C_i for $1 \leq i \leq 5$ and W_i for $1 \leq i \leq 3$ indicate properties of sequences in (X, d). Consider

C_1: $\lim d(x_n, y_n) = 0 = \lim d(x_n, x) \Rightarrow \lim d(y_n, x) = 0$,

C_2: $\lim d(x_n, x) = 0 = \lim d(y_n, x) \Rightarrow \lim d(x_n, y_n) = 0$,

C_3: $\lim d(x_n, y_n) = 0 = \lim d(y_n, z_n) \Rightarrow \lim d(x_n, z_n) = 0$.

A space in which C_1 is satisfied is called coherent by Pitcher and Chittenden [7]. Niemytzki [5] proved that a coherent symmetric space (X, d) is metrizable, and in fact there is a metric ρ on X such that (X, d) and (X, ρ) have identical topologies and also that $\lim d(x_n, x) = 0$ if and only if $\lim \rho(x_n, x) = 0$.

Cho et al. [8] have introduced

C_4: $\lim d(x_n, x) = 0 \Rightarrow \lim d(x_n, y) = d(x, y)$ for all y in X,

C_5: $\lim d(x_n, x) = \lim d(x_n, y) = 0 \Rightarrow x = y$.

The following properties were introduced by Wilson [4]:

W_1: for each pair of distinct points a, b in X there corresponds a positive number $r = r(a, b)$ such that $r < \inf_{c \in X} d(a, c) + d(b, c)$,

W_2: for each $a \in X$, for each $k > 0$, there corresponds a positive number $r = r(a,k)$ such that if b is a point of X such that $d(a,b) \geq k$ and c is any point of X then $d(a,c) + d(c,b) \geq r$,

W_3: for each positive number k there is a positive number $r = r(k)$ such that $d(a,c) + d(c,b) \geq r$ for all c in X and all a, b in X with $d(a,b) \geq k$.

2. Implications among the Axioms

Proposition 1. *In a d-symmetric space (X,d), $C_3 \Rightarrow C_1 \Rightarrow C_5$, $C_3 \Rightarrow C_2$, and $C_4 \Rightarrow C_5$.*

Proof. Assume that C_3 holds in (X,d) and let $\lim d(x_n, y_n) = 0$ and $\lim d(x_n, x) = 0$. Put $z_n = x\ \forall n$ so that

$$\begin{aligned} \lim d(x_n, z_n) &= \lim d(x_n, x) = 0 \\ &= \lim d(x_n, y_n) = \lim d(y_n, x_n). \end{aligned} \tag{1}$$

By C_3, $\lim d(y_n, z_n) = 0$; that is, $\lim d(y_n, x) = 0$.

Hence

$$C_3 \Rightarrow C_1. \tag{2}$$

Assume that C_1 holds in (X,d) and let $\lim d(x_n, x) = 0$ and $\lim d(x_n, y) = 0$. Put $y_n = y\ \forall n$; then

$$\lim d(x_n, y_n) = \lim d(x_n, x) = 0. \tag{3}$$

By C_1, $\lim d(y_n, x) = 0$; that is, $\lim d(y, x) = 0$.

Consider $\lim d(x, y) = 0$; this implies that $x = y$. Hence C_5 holds. Thus

$$C_1 \Longrightarrow C_5. \tag{4}$$

Assume that C_3 holds and let $\lim d(x_n, x) = 0$ and $\lim d(y_n, x) = 0$.

Put $z_n = x\ \forall n$; then $\lim d(x_n, z_n) = \lim d(z_n, y_n) = 0$.

By C_3, $\lim d(x_n, y_n) = 0$. Hence

$$C_3 \Longrightarrow C_2. \tag{5}$$

Assume that C_4 holds and let $\lim d(x_n, x) = 0$ and $\lim d(x_n, y) = 0$.

By C_4, $\lim d(x_n, y) = d(x, y)$. Hence $d(x, y) = 0$. Hence $x = y$. □

The following proposition explains the relationship between Wilson's axioms [4] W_1, W_2, and W_3 and the C_i's.

Proposition 2. *Let (X,d) be a d-symmetric space; then*

(i) $W_1 \Leftrightarrow C_5$, (ii) $W_2 \Leftrightarrow C_1$, and (iii) $W_3 \Leftrightarrow C_3$.

Proof. (i) Assume W_1. Suppose $\lim d(a, x_n) = \lim d(b, x_n) = 0$ but $a \neq b$.

Then

$$\lim \{d(a, x_n) + d(b, x_n)\} = 0 \quad \text{but } a \neq b. \tag{6}$$

By

$$W_1\ \exists\, r > 0 \ni \forall x, \quad d(a, x) + d(b, x) \geq r, \tag{7}$$

equations (6) and (7) are contradictory. Hence $a = b$. Thus $W_1 \Rightarrow C_5$.

Suppose that W_1 fails. Then there exist $a \neq b$ in X such that for every n there corresponds x_n in X such that $d(a, x_n) + d(b, x_n) < 1/n$:

$$\Longrightarrow \lim d(a, x_n) = \lim d(b, x_n) = 0 \quad \text{but } a \neq b. \tag{8}$$

Thus if W_1 fails then C_5 fails. That is, $C_5 \Rightarrow W_1$. Hence $W_1 \Leftrightarrow C_5$.

(ii) Assume W_2. Then for each $a \in X$ and each $k > 0$ there corresponds $r > 0$ such that, for all $b \in X$ with $d(a,b) \geq k$ and $\forall x \in X$, $d(a,x) + d(b,x) \geq r$.

Suppose that C_1 fails. There exist $a \in X$, $\{b_n\}$, and $\{c_n\}$ in X such that $\lim d(a, b_n) = \lim d(b_n, c_n) = 0$ but $\lim d(a, c_n) \neq 0$.

Since $\lim d(a, c_n) \neq 0$ there exists $k > 0$ and a subsequence (c_{n_k}) such that

$$d(a, c_{n_k}) > k \quad \forall n_k. \tag{9}$$

Since

$$d(a, c_{n_k}) > k, \qquad d(a, b_{n_k}) + d(b_{n_k}, c_{n_k}) \geq r, \tag{10}$$

this implies that $\lim\{d(a, b_n) + d(b_n, c_n)\} \neq 0$, a contradiction.

Conversely assume that W_2 fails. Then there exist $a \in X$ and $k > 0$ such that $\forall n > 0\ \exists\, b_n \in X$ and $c_n \in X$ such that

$$d(a, b_n) \geq k \quad \text{but } d(a, c_n) + d(b_n, c_n) < \frac{1}{n}. \tag{11}$$

This implies that $\lim d(a, c_n) = \lim d(b_n, c_n) = 0$ but $\lim d(a, b_n) \neq 0$.

Hence C_1 fails.

(iii) Assume W_3. Suppose that C_3 fails. Then there exist sequences $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ in X such that $\lim d(a_n, b_n) = \lim d(b_n, c_n) = 0$ but $\lim d(a_n, c_n) \neq 0$.

Since W_3 holds, $\forall k > 0$ there corresponds $r > 0$ such that for all a, b with

$$d(a, b) \geq k, \qquad d(a, c) + d(b, c) \geq r \quad \forall c. \tag{12}$$

Since $\lim d(a_n, c_n) \neq 0$ there exists a positive number ϵ and a subsequence of positive integers $\{n_k\}$ such that $d(a_{n_k}, c_{n_k}) > \epsilon$. Choose r_1 corresponding to ϵ so that

$$d(a_{n_k}, b_{n_k}) + d(b_{n_k}, c_{n_k}) \geq r_1. \tag{13}$$

Thus

$$\lim \{d(a_{n_k}, b_{n_k}) + d(b_{n_k}, c_{n_k})\} \neq 0. \tag{14}$$

This contradicts the assumption that $\lim d(a_n, b_n) = \lim d(b_n, c_n) = 0$.

Hence

$$W_3 \Longrightarrow C_3. \tag{15}$$

Assume that W_3 fails.

Then there exists $k > 0$ such that, $\forall$ positive integer n, there exist a_n, b_n, and c_n with

$$d(a_n, b_n) \geq k \quad \text{but } d(a_n, c_n) + d(b_n, c_n) < \frac{1}{n}. \tag{16}$$

Hence

$$\lim d(a_n, b_n) \neq 0 \quad \text{but } \lim d(a_n, c_n) = \lim d(c_n, b_n) = 0. \tag{17}$$

Hence C_3 fails.
Hence

$$C_3 \Longrightarrow W_3. \tag{18}$$

This completes the proof of the proposition. □

We introduce the following.

Axiom C. Every convergent sequence satisfies Cauchy criterion. That is, if (x_n) is a sequence in X, $x \in X$ and $\lim d(x_n, x) = 0$; then given $\epsilon > 0$ $\exists$ $N(\epsilon) \in \mathbf{N}$ such that $d(x_n, x_m) < \epsilon$ whenever $m, n \geq N(\epsilon)$ we have the following.

Proposition 3. *In a d-symmetric space (X, d), $C_1 \Rightarrow C \Rightarrow C_2$.*

Proof. For $C_1 \Rightarrow C$, suppose that a sequence (x_n) in (X, d) is convergent to x but does not satisfy Cauchy criterion. Then $\exists r > 0$ such that for every positive integer k there correspond integers m_k, n_k such that

$$m_{k+1} > n_{k+1} > m_k > n_k, \quad d\left(x_{m_k}, x_{n_k}\right) > \gamma \quad \forall k. \tag{19}$$

Let

$$y_k = x_{m_k}, \quad z_k = x_{n_k} \quad \forall k. \tag{20}$$

Then

$$\lim d\left(y_k, x\right) = 0, \qquad \lim d\left(z_k, x\right) = 0. \tag{21}$$

But $\lim d(y_k, z_k) \neq 0$; this contradicts C_1. □

Proof. For $C \Rightarrow C_2$, suppose that $\lim d(x_n, x) = \lim d(y_n, x) = 0$.

Let (z_n) be the sequence defined by $z_{2n-1} = x_n$ and $z_{2n} = y_n$. Then $\lim d(z_n, x) = 0$. Hence (z_n) satisfies Cauchy criterion.

Given $\epsilon > 0$ $\exists N(\epsilon) \in \mathbf{N}$ such that $d(z_n, z_m) < \epsilon$ for $m, n \geq N(\epsilon)$:

$$\Rightarrow d(z_{2n-1}, z_{2n}) < \epsilon \quad \text{for } n \geq N(\epsilon),$$

$$\Rightarrow \lim d(x_n, y_n) < \epsilon \quad \text{for } n \geq N(\epsilon),$$

$$\Rightarrow \lim d(x_n, y_n) = 0.$$

□

3. Examples for Nonimplications

Example 4. A d-symmetric space in which the triangular inequality fails and C_1 through C_5 hold.

Let $X = [0, 1]$. Define d on $X \times X$ as follows:

$$d(x, y) = \begin{cases} x + y & \text{if } x \neq y, \\ 1 & \text{if } x = y \neq 0, \\ 0 & \text{if } x = y = 0. \end{cases} \tag{22}$$

Clearly d is a d-symmetric space. d does not satisfy the triangular inequality since $d(0.1, 0.2) + d(0.2, 0.1) = 0.6 < 1 = d(0.1, 0.1)$.

We show that C_1 through C_5 holds. We first show that $\lim d(x_n, x) = 0$ iff $x = 0$ and $\lim x_n = 0$ in R.

If $x \neq 0$ then $\lim d(x_n, x) = x_n + x \geq x > 0$. Hence $\lim d(x_n, x) \geq x > 0$.

If $x = 0$ then $\lim d(x_n, 0) = 0$ or x_n. Hence $\lim d(x_n, x) = 0 \Leftrightarrow \lim x_n = 0$ in R.

Now we show that $\lim d(x_n, y_n) = 0$ if and only if $\lim x_n = \lim y_n = 0$ in R.

Consider $\lim d(x_n, y_n) = 0 \Rightarrow d(x_n, y_n) < 1/2$ for large n:

$$\Rightarrow d(x_n, y_n) = x_n + y_n \text{ or } 0 \text{ for large } n,$$

$\Rightarrow$ either $x_n = y_n = 0$ or $d(x_n, y_n) = x_n + y_n$ for large n,

$$\Rightarrow \lim x_n = \lim y_n = 0 \text{ in } R.$$

Conversely if $\lim x_n = \lim y_n = 0$ in R then $\lim d(x_n, y_n) = 0$ or $x_n + y_n$ for large n.

Hence $\lim d(x_n, y_n) = 0$.

Verification of validity of C_1 through C_5 is done as follows.

C_1: let $\lim d(x_n, y_n) = 0$ and $\lim d(x_n, x) = 0$; then $\lim x_n = \lim y_n = 0$ in R and $x = 0$.

Hence $d(y_n, x) = d(y_n, 0) = y_n$ or 0. This implies that $\lim d(y_n, x) = 0$.

C_2: let $d(x_n, x) = d(y_n, x) = 0$. Then $x = 0$ and $\lim x_n = \lim y_n = 0$ in R.

Hence $\lim d(y_n, x_n) = 0$.

C_3: let $d(x_n, y_n) = d(y_n, z_n) = 0$; then $\lim x_n = \lim y_n = \lim z_n = 0$ in R.

Hence $\lim d(x_n, z_n) = 0$.

C_4: let $\lim d(x_n, x) = 0$. Then $x = 0$ and $\lim x_n = 0$.

If $y = 0$, $0 \leq d(x_n, y) \leq x_n$. Hence $\lim d(x_n, y) = 0 = d(x, y)$.

If $y \neq 0$, $d(x_n, y) = x_n + y$. Hence $\lim d(x_n, y) = y = 0 + y = d(x, y)$.

C_5: let $\lim d(x_n, x) = 0$ and $\lim d(x_n, y) = 0$.

Then $x = 0$, $y = 0$ and $\lim x_n = 0$. Hence $x = y$.

Example 5. A d-symmetric space (X, d) in which C_1 [hence C_5] holds while C_j does not hold for $j = 2, 3, 4$.

Let $X = [0, \infty)$. Define d on $X \times X$ as follows:

$$d(x, y) = \begin{cases} x + y & \text{if } x \neq 0 \neq y, \\ \dfrac{1}{x} & \text{if } x \neq 0 = y, \\ \dfrac{1}{y} & \text{if } x = 0 \neq y, \\ 0 & \text{if } x = 0 = y. \end{cases} \quad (23)$$

Clearly (X, d) is a d-symmetric space. We show that C_1, C_5 hold.

Let $\lim d(x_n, x) = 0 = \lim d(x_n, y_n)$.

If $x \neq 0$, $d(x_n, x) > x$ if $x_n \neq 0$.

$$= \frac{1}{x} \text{ if } x_n = 0. \quad (24)$$

This implies that

$$\lim d(x_n, x) \geq \min\left\{x, \frac{1}{x}\right\} > 0. \quad (25)$$

Thus $\lim d(x_n, x) = 0 \Rightarrow x = 0$ and (x_n) can be split into two subsequences $({x_n}^{(1)}), ({x_n}^{(2)})$, where $({x_n}^{(1)}) = 0\ \forall n$, $({x_n}^{(2)}) \neq 0$ for every n and if $({x_n}^{(2)})$ is infinite subsequence $\lim({x_n}^{(2)}) = \infty$. We consider the case where both $({x_n}^{(1)})$ and $({x_n}^{(2)})$ are infinite sequences as when one is a finite sequence the same proof works with minor modifications. Consider

$$\begin{aligned} \lim d(x_n, y_n) = 0 \Longrightarrow \lim d\left({x_n}^{(1)}, {y_n}^{(1)}\right) \\ = \lim d\left({x_n}^{(2)}, {y_n}^{(2)}\right) = 0. \end{aligned} \quad (26)$$

If we show that ${y_n}^{(2)}$ cannot be positive for infinitely many n, it will follow that $\lim d({x_n}^{(2)}, {y_n}^{(2)}) = \lim d({x_n}^{(2)}, 0) = 0$ so that $\lim d(0, y_n) = 0$. Hence C_1 holds.

If ${y_n}^{(2)} \neq 0$ for infinitely many n, say $\{{y_{n_k}}^{(2)}\}$ is the infinite subsequence of $\{{y_n}^{(2)}\}$ with ${y_{n_k}}^{(2)} \neq 0\ \forall n_k$, then $d({x_{n_k}}^{(2)}, {y_{n_k}}^{(2)}) = {x_{n_k}}^{(2)} + {y_{n_k}}^{(2)} > {x_{n_k}}^{(2)}$ so that $\lim d({x_{n_k}}^{(2)}, {y_{n_k}}^{(2)}) \geq \lim {x_{n_k}}^{(2)} \geq \infty$ contradicting the assumption that $\lim d(x_n, y_n) = 0$. Thus C_1 holds. Since $C_1 \Rightarrow C_5$, C_5 holds.

C_2 does not hold since $d(n, 0) = 1/n$ while $d(n, n) = 2n\ \forall n$ so that $\lim d(n, n) \neq 0$.

C_3 does not hold since $\lim d(n, 0) = \lim d(0, n)$ while $\lim d(n, n) = \infty$.

C_4 does not hold since $\lim d(n, 0) = 0$ but $\lim d(n, 2) = \infty$ while $d(0, 2) = 1/2$.

Example 6. A d-symmetric space (X, d) in which C_2 holds but C_1, C_3, C_4, and C_5 fail.

Let $X = [0, 1] \cup \{2\}$. Define d on $X \times X$ as follows:

$$d(x, y) = \begin{cases} x + y & \text{if } 0 \leq x \leq 1,\ 0 \leq y \leq 1 \\ x & \text{if } 0 \leq x \leq 1,\ y = 2 \\ y & \text{if } x = 2,\ 0 \leq y \leq 1 \\ 1 & \text{if } \begin{cases} x = 2,\ y \in \{0, 2\} \\ \text{or} \\ x \in \{0, 2\},\ y = 2. \end{cases} \end{cases} \quad (27)$$

Clearly (X, d) is a d-symmetric space.

We first show that if $\{x_n\}$ in X converges to x in (X, d) then $x \in \{0, 2\}$.

Suppose that $x \neq 0$ and $x \neq 2$; then $x \in (0, 1]$:

$$\Rightarrow \lim d(x_n, x) = 0 = x_n + x \text{ or } x,$$
$$\Rightarrow \lim d(x_n, x) \geq x > 0,$$
$$\Rightarrow \lim d(x_n, x) \neq 0.$$

Hence if $\lim d(x_n, x) = 0$ then $x \in \{0, 2\}$.

C_1 fails: $x_n = 1/n$, $y_n = 2$, and $x = 0$;

$$d(x_n, y_n) = \frac{1}{n}, \qquad d(x_n, x) = \frac{1}{n}, \qquad d(y_n, x) = 1$$
$$\Longrightarrow \lim d(x_n, y_n) = 0 = d(x_n, x) \quad \text{but } \lim d(y_n, x) \neq 0. \quad (28)$$

C_2 holds: suppose that $\lim d(x_n, x) = \lim d(y_n, x) = 0$; then $x \in \{0, 2\}$.

Case 1. If $x = 2$, $\lim d(x_n, x) \rightarrow 0 \Rightarrow d(x_n, x) = x_n$ eventually and $\lim x_n = 0$ in R. Hence $\exists N \in \mathbf{N} \ni x_n < 1$ and $y_n < 1$ for $n \geq N$.

Here $d(x_n, y_n) = x_n + y_n$. This implies that $\lim d(x_n, y_n) = 0$.

Case 2. If $x = 0$,

$$d(x_n, 0) = \begin{cases} 1 & \text{if } x_n = 2 \text{ or } 0, \\ x_n & \text{if } 0 \leq x_n \leq 1. \end{cases} \quad (29)$$

If $\lim d(x_n, 0) = 0$, $d(x_n, 0) = x_n$ eventually and $\lim x_n = 0$ in R.

Similarly $d(y_n, 0) = y_n$ eventually and $\lim y_n = 0$ in R. As in Case 1 it follows that

$$\lim d(x_n, y_n) = \lim(x_n + y_n) = 0. \quad (30)$$

Thus C_2 holds.

C_3 fails since $C_3 \Rightarrow C_1$.

C_5 fails: let $x_n = 1/n$, $x = 0$, and $y = 2$

$$\lim d(x_n, 0) = \lim\left(\frac{1}{n}\right) = 0 = \lim d(x_n, 2) \quad (31)$$

C_4 fails since $C_4 \Rightarrow C_5$.

Example 7. A d-symmetric space (X, d) in which C_4 holds but C_1 fails.

Let $X = N \cup \{0\}$. Define d on $X \times X$ as follows:

$$d(m,n) = d(n,m) \quad \forall m,n \in X,$$
$$d(0,n) = \begin{cases} \frac{1}{n} & \text{if } n \text{ is odd,} \\ 1 & \text{if } n \text{ is even,} \end{cases}$$
$$d(0,0) = 0,$$
$$d(m,n) = \begin{cases} \left|\frac{1}{m} - \frac{1}{n}\right| & \text{if } m+n \text{ is even} \\ & \text{or } m+n \text{ is odd and } |m-n| = 1, \\ 1 & \text{if } m+n \text{ is odd and } |m-n| > 2. \end{cases} \tag{32}$$

If $\{x_n\}$ in X and $\lim d(x_n, 0) = 0$ then x_n is eventually odd.

If $x \neq 0, d(x_n, x)$ cannot be 1 so $x_n + x$ is even or odd and $|x_n - x| = 1$.

But in this case $d(x_n, x) = |1/x_n - 1/x|$ so that $d(x_n, x) \neq 0$.

Thus $d(x_n, x) = 0 \Leftrightarrow x = 0$ and x_n is eventually odd.

If m is a fixed even integer and x_n is odd, $x_n + m$ is odd and eventually >2.

So

$$\lim d(x_n, m) = 1 = d(0,m). \tag{33}$$

If m is a fixed odd integer and x_n is odd, $x_n + m$ is even.

So $d(x_n, m) = |1/m - 1/x_n|$ so that $\lim d(x_n, 0) = 0 \Rightarrow \lim d(x_n, m) = d(0, m)$.

If $m{=}0$ and x_n is odd eventually

$$d(x_n, 0) = \frac{1}{n} \quad \text{so } \lim d(x_n, m) = \lim \frac{1}{n} = 0 = d(0,m). \tag{34}$$

If $m = 0$ and $x_n = 0$ eventually

$$d(x_n, 0) = \frac{1}{n} \quad \text{so } \lim d(x_n, m) = \lim \frac{1}{n} = 0 = d(0,m). \tag{35}$$

Hence C_4 holds in (X, d).

C_1 does not hold: let $x_n = 2n - 1$ and $y_n = 2n$:

$$d(x_n, 0) = \frac{1}{2n-1}, \qquad d(x_n, y_n) = \frac{1}{2n-1} - \frac{1}{2n}, \qquad d(y_n, 0) = 1. \tag{36}$$

Hence $d(x_n, 0) = d(x_n, y_n) = 0$ and $d(y_n, 0) \neq 0$.

Example 8. A d-symmetric space (X, d) in which C_3 holds but C_4 does not hold.

Let $X = [0,1] \cup \{2\}$. Define d on $X \times X$ as follows:

$$d(x,y) = \begin{cases} x+y & \text{if } 0 \le x \neq y \le 1, \\ 1 & \text{if } x = y \neq 0 \text{ or } x = y = 2 \\ & \text{or } x \in (0,1] \text{ and } y = 2, \\ 2 & \text{if } x = 0 \ \& \ y = 2 \text{ or } x = 2 \text{ and } y = 0, \\ 0 & \text{if } x = y = 0. \end{cases} \tag{37}$$

Clearly (X, d) is a d-symmetric space which is not a symmetric space.

We first show that if $\{x_n\}$ converges to x in (X, d) then $x \in \{0, 2\}$.

Suppose that $0 \neq x \neq 2$; then $x \in (0, 1]$:

$$\Longrightarrow d(x, x_n) = \begin{cases} x + x_n & \text{if } 0 < x \neq x_n \le 1, \\ 1 & \text{if } x = x_n \neq 0 \\ & \text{or } x_n = 2 \text{ or } x \in (0,1] \\ & \text{and } x_n = 2. \end{cases} \tag{38}$$

Since $\lim d(x, x_n) = 0 \ \exists N \ni d(x, x_n) < 1$ for $n \ge N$

$\Rightarrow d(x, x_n) = x + x_n \ge x$ for $n \ge N$,

$\Rightarrow \lim d(x, x_n) \neq 0$, a contradiction.

We now show that $\lim d(x_n, y_n) = 0$ if and only if $\lim x_n = \lim y_n = 0$. Consider

$$\begin{aligned} &\lim d(x_n, y_n) \\ &= 0 \Longrightarrow \exists N \in \mathbf{N} \ni d(x_n, y_n) < 1 \quad \text{for } n \ge N \\ &\Longrightarrow \lim d(x_n, y_n) = x_n + y_n \quad \text{or} \quad 0 \quad \text{for } n \ge N \\ &\Longrightarrow \text{either } x_n = y_n = 0 \quad \text{or} \quad d(x_n, y_n) = x_n + y_n \\ &\qquad\qquad \text{for } n \ge N \\ &\Longrightarrow \lim x_n = \lim y_n = 0. \end{aligned} \tag{39}$$

Conversely if $\lim x_n = \lim y_n = 0$ then $\exists N \in \mathbf{N} \ni x_n < 1, y_n < 1$ for $n \ge N \Rightarrow \lim d(x_n, y_n) = 0$ or $x_n + y_n$ for large n.

Hence $d(x_n, y_n) = 0$.

Thus $d(x_n, y_n) = 0$ if and only if $\lim x_n = \lim y_n = 0$.

As a consequence we have

$$\lim d(x_n, y_n) = 0 = \lim d(y_n, z_n) \Longrightarrow \lim d(x_n, z_n) = 0. \tag{40}$$

Hence C_3 holds in (X, d).

C_4 fails: $x_n = 1/(n+1)$ for $n \ge 1$:

$$d(x_n, 0) = \frac{1}{n+1} \Longrightarrow \lim d(x_n, 0) = 0,$$
$$d(x_n, 2) = 1 \quad \forall n \Longrightarrow \lim d(x_n, 2) = 1 \quad \text{but } d(0,2) = 2. \tag{41}$$

Example 9. A d-symmetric space (X, d) in which C_4 holds but C_2, C_3 fail to hold.

Let $X = N \cup \{0, \infty\}$. Define d on $X \times X$ as follows:

$$d(m,\infty) = d(\infty, m) = 1 \quad \text{if } m \in X,$$
$$d(m,0) = d(0,m) = \frac{1}{m} \quad \text{if } m \in N, \tag{42}$$
$$d(0,0) = 0.$$

If $m, n \in N$,

$$d(m,n) = \begin{cases} \left|\frac{1}{m} - \frac{1}{n}\right| & \text{if } |m-n| \geq 2, \\ 1 & \text{if } |m-n| \leq 1. \end{cases} \tag{43}$$

Clearly (X, d) is a d-symmetric space which is not a symmetric space.

We show that if $\lim d(x_n, x) = 0$ then $x = 0$ and $\{x_n\}$ consists of two subsequences $\{y_n\}$ and $\{z_n\}$, one of which may possibly be finite, where $y_n = 0\ \forall n$ and $0 \neq z_n \in N\ \forall n$ and $\lim(1/z_n) = 0$ (in case $\{z_n\}$ is an infinite sequence).

To prove this we first note that $\lim d(x_n, x) = 0 \Rightarrow x \neq \infty$ and $x_n \neq \infty$ eventually.

If $x \in N$, $d(x_n, x) = 1/x$ or 1 or $|1/x_n - 1/x|$.

Hence $\lim d(x_n, x) = 0 \Rightarrow x \notin N$; hence $x = 0$.

Further $d(x_n, 0) = 0$ or $1/x_n$. Consequently $\{x_n\}$ may be split into two sequences $\{y_n\}$ and $\{z_n\}$ as described above.

We show that C_4 holds. Assume that $\lim d(x_n, x) = 0$. Then $x = 0$.

Let $m \in N$ and $y_n = 0\ \forall n$. Then $d(y_n, m) = d(o, m) = 1/m$.

So $\lim d(y_n, m) = d(o, m)$.

If $z_n \neq 0\ \forall n$, and $\lim(1/z_n) = 0$ the $d(z_n, m) = |1/z_n - 1/m|$ for $n > m$ so that $\lim d(z_n, m) = 1/m = d(0, m)$.

Thus if $m \in N$ and $\lim d(x_n, x) = 0$ then $\lim d(x_n, m) = d(x, m)$.

Clearly this holds when $m = \infty$ or $m = 0$ as well.

Hence C_4 holds.

C_2 does not hold: let $x_n = x$, $y_n = n + 1$, and $x = 0$:

$$d(x_n, x) = d(n, 0) = \frac{1}{n}, \quad \text{hence } \lim d(x_n, x) = 0,$$
$$d(y_n, x) = d(n+1, 0) = \frac{1}{n+1}, \quad \text{hence } \lim d(y_n, x) = 0,$$
$$\lim d(x_n, y_n) = \lim d(n, n+1) \Longrightarrow \lim d(x_n, y_n) \neq 0. \tag{44}$$

C_3 does not hold:

$$x_n = n, \qquad y_n = n + 2, \qquad z_n = x_n,$$
$$d(x_n, y_n) = d(n, n+2) = \left|\frac{1}{n+2} - \frac{1}{n}\right| = \frac{1}{n} - \frac{1}{n+2},$$
$$d(y_n, z_n) = d(x_n, y_n) = \frac{1}{n} - \frac{1}{n+2},$$
$$\lim d(x_n, z_n) = \lim d(n, n) = 1,$$
$$\lim d(x_n, y_n) = \lim d(y_n, z_n) = 0 \quad \text{but } \lim d(x_n, z_n) = 1. \tag{45}$$

C_5 holds since $C_4 \Rightarrow C_5$.

Remarks. From this example we can conclude that

(1) C_5 does not imply C_2 as otherwise, since $C_4 \Rightarrow C_5$ it would follow that $C_4 \Rightarrow C_2$ which does not hold as is evident from the above example,

(2) in a d-symmetric space, convergent sequences are necessarily Cauchy sequences.

Example 10. A d-symmetric space (X, d) in which C_4 holds but C_2, C_3 fail to hold.

Let $X = N \cup \{0\}$. Define d on $X \times X$ as follows:

$$d(x, y) = d(y, x) = 1 \quad \text{for every } x, y \in X,$$
$$d(2m, 0) = 1,$$
$$d(2m-1, 0) = \frac{1}{2m-1} \quad \forall m,$$
$$d(0, 0) = 0,$$
$$d(m, n) = \begin{cases} \frac{1}{m} + \frac{1}{n} & \text{if } m+n \text{ is even or } |m-n| = 1, \\ 1 & \text{if } m+n \text{ is odd and } |m-n| > 2. \end{cases} \tag{46}$$

Clearly (X, d) is a d-symmetric space.

We first characterize all convergent sequences in (X, d).

Suppose that $\lim d(x_n, x) = 0$. We show that $x = 0$.

If x is odd and x_n is even $d(x_n, x) = 1$ if $x_n > x + 2$.

So $\lim d(x_n, x) \neq 0$. Thus x_n is even for at most finitely many n.

We may thus assume that x_n is odd $\forall n$.

The $d(x_n, x) = 1/x_n + 1/x$ so that $d(x_n, x) \geq 1/x > 0$.

Hence x cannot be odd. Now suppose that $x > 0$ and x is even.

Then $d(x_n, x) = 1$ if $x_n = 0$ if x_n is odd and $|x_n - x| > 2$while $d(x_n, x) = 1/x_n + 1/x$ if $x_n + x$ is even or $|x_n - x| = 1$. In all cases $\lim d(x_n, x) \neq 0$.

Hence the only possibility is $x = 0$.

We now show that the following are equivalent.

(a) $\lim d(x_n, x) = 0$ in R,

(b) there exists a positive integer N such that x_n is positive and even, only if $n < N$.

Assumption (b): x_n is odd or zero if $n \geq N$ so that $\lim d(x_n, 0) = \lim(1/x_n) = 0$.

Hence (b)$\Rightarrow$(a).

Assumption (a): since $d(2m, 0) = 1$ for $m \in N$, it follows that at most finitely many terms of $\{x_n\}$ can be even. This proves (b). Thus $\lim d(x_n, x) = 0 \Leftrightarrow x = 0$ and $\exists N \in \mathbf{N} \ni x_n$ is "0" or odd for $n \geq N$.

Consequently C_5 holds.

C_1 does not hold: let $x_n = 2n + 1$, $y_n = 2n$ and $x = 0$;

$$\lim d(x_n, x) = \lim \frac{1}{2n+1} = 0,$$
$$\lim d(x_n, y_n) = \lim \frac{1}{2n+1} + \frac{1}{2n} = 0. \tag{47}$$

But $\lim d(y_n, x) = 1$ since $\lim d(2n, 0) = 1\ \forall n$.

C_2 holds: assume that $\lim d(x_n, x) = 0 = \lim d(y_n, x)$.

Then $x = 0$ and then there exists N such that x_n is "0" or odd and $y_n = 0$ or odd for $n \geq N$ and $\lim(1/x_n) = \lim(1/y_n) = 0$.

If $x_n = y_n = 0$, $d(x_n, y_n) = 0$.

If $x_n = 0$, y_n is odd, $d(x_n, y_n) = 1/y_n$.

If $y_n = 0$, x_n is odd, $d(x_n, y_n) = 1/x_n$.

If x_n is odd and y_n is odd, $d(x_n, y_n) = 1/x_n + 1/y_n$.

Consequently $\lim d(x_n, y_n) = 0$.

C_3 does not hold: let $x_n = 0$, $y_n = 2n + 1$, and $z_n = 2n$:

$$d(x_n, y_n) = \frac{1}{2n+1}, \qquad d(y_n, z_n) = \frac{1}{2n+1} + \frac{1}{2n}, \qquad d(x_n, z_n) = 1 \tag{48}$$

so that $\lim d(x_n, y_n) = \lim d(y_n, z_n) = 0$ but $\lim d(x_n, z_n) = 1$.

C_4 does not hold: let $x_n = 2n + 1$, $x = 0$, and $y = 3$:

$$\lim d(x_n, 0) = \lim \frac{1}{2n+1} = 0, \qquad \lim d(x_n, 3) = 1, \qquad \lim d(0, 3) = \frac{1}{3}. \tag{49}$$

Example 11. The following example shows that there exist symmetric spaces in which C does not hold.

Let $X = \{0, 1/2, 1/3, 1/4, \ldots\}$.

Define $d(x, x) = 0$, $d(x, y) = d(y, x)$

$$d\left(\frac{1}{n}, 0\right) = d\left(0, \frac{1}{n}\right) = \frac{1}{n} \quad \forall n \text{ in } N, \qquad d\left(\frac{1}{n}, \frac{1}{m}\right) = 1 \quad \forall n, m \text{ in } N. \tag{50}$$

Then (X, d) is a symmetric space; $\{1/n\}$ converges to 0 but is not a Cauchy sequence.

Disclosure

Professor I. Ramabhadra Sarma is a retired professor from Acharya Nagarjuna University.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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