

Research Article

Global Dynamics of a Discretized Heroin Epidemic Model with Time Delay

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We derive a discretized heroin epidemic model with delay by applying a nonstandard finite difference scheme. We obtain positivity of the solution and existence of the unique endemic equilibrium. We show that heroin-using free equilibrium is globally asymptotically stable when the basic reproduction number $R_0 < 1$, and the heroin-using is permanent when the basic reproduction number $R_0 > 1$.

1. Introduction

As we all know, the use of heroin and other drugs in Europe and more specifically in Ireland and the resulting prevalence are well documented [1–3]. It shows that the use of heroin is very popular and causes many preventable deaths. Heroin is so soluble in the fat cells that it crosses the blood-brain barrier within 15–20 seconds, rapidly achieving a high level syndrome in the brain and central nervous system which causes both the “rush” experience by users and the toxicity. Heroin-related deaths are associated with the use of alcohol or other drugs [4]. Treatment of heroin users is a huge burden on the health system of any country.

We often study infectious diseases with mathematical and statistical techniques; see, for example, [5–11]; however, little has been done to apply this method to the heroin epidemics. In 1979, Mackintosh and Stewart [9] considered an exponential model which is simplified from infectious disease model of Kermack and McKendrick to illustrate how the heroin-using spreads in epidemic fashion. They arranged a numerical simulation to show how the dynamics of spread are influenced by parameters in the model. White and Comiskey [5] attempted to extend dynamic disease modeling to the drug-using career and formulated an ordinary differential equation. They divided the whole population into three classes, namely, susceptible, heroin users, and heroin users undergoing treatment. Their model allows a steady state

(constant) solution which represents an equilibrium between the number of susceptible, heroin users, and heroin users in treatment. Furthermore, this ODE model was revisited by Mulone and Straughan [12]; the authors proved that this equilibrium solution is stable both linearly and nonlinearly under the realistic condition in which relapse rate of those in treatment returning to untreated drug use is greater than the prevalence rate of susceptible becoming drug users. Recently, the study of the global properties and permanence of continuous heroin epidemic models attracted the researchers and have some very good results; see [13–16]. Specially, Samanta [15] considered a model with time-dependent coefficients and with different removal rates for three different classes, introduced some new threshold values R_* and R^* , and obtained the permanence of heroin-using career.

Motivated by Samanta [15] and Zhang and Teng [8], we alter a nonautonomous heroin epidemic model with time delay to an autonomous heroin epidemic model. For convenience, we replace U_1 and U_2 by U and V , respectively. Thus, we obtain the following continuous heroin epidemic model with a distributed time delay:

$$\begin{aligned} \dot{S}(t) = & \lambda - \beta_1(U) S(t) \int_0^h U(t-s) d\eta(s) \\ & - \mu_1 S(t) + \xi_1 U(t) + \xi_2 V(t), \end{aligned}$$

$$\begin{aligned} \dot{U}(t) &= \beta_1(U)S(t) \int_0^h U(t-s) d\eta(s) \\ &\quad + \beta_3U(t)V(t) - (\mu_2 + P + \xi_1)U(t), \\ \dot{V}(t) &= PU(t) - \beta_3U(t)V(t) - (\mu_3 + \xi_2)V(t), \end{aligned} \tag{1}$$

where $S(t)$, $U(t)$, and $V(t)$ represent the number of susceptible, heroin users not in treatment, and heroin users in treatment, respectively. We assume that the time taken to become heroin user is s . The function $\eta(s) : [0, h] \rightarrow [0, \infty)$ is nondecreasing and has bounded variation such that $\int_0^h \eta(s)ds = \eta(h) - \eta(0) = 1$.

For understanding more realistic phenomenon of heroin, a little complicated epidemic model is helpful. By applying Micken’s nonstandard discretization method [17] to continuous heroin epidemics model with time delay (1), we derive the following discretized heroin epidemic model with a distributed time delay:

$$\begin{aligned} S_{n+1} - S_n &= \lambda - \beta_1(U_n)S_{n+1} \sum_{k=0}^h U_{n-k}\eta_k \\ &\quad - \mu_1S_{n+1} + \xi_1U_{n+1} + \xi_2V_{n+1}, \\ U_{n+1} - U_n &= \beta_1(U_n)S_{n+1} \sum_{k=0}^h U_{n-k}\eta_k \\ &\quad + \beta_3U_{n+1}V_{n+1} - (\mu_2 + P + \xi_1)U_{n+1}, \\ V_{n+1} - V_n &= PU_{n+1} - \beta_3U_{n+1}V_{n+1} - (\mu_3 + \xi_2)V_{n+1}, \end{aligned} \tag{2}$$

where S_n is the susceptible class, U_n is the class of heroin users not in treatment, and V_n is the class of heroin users in treatment at n th step. Since the sufficient condition can be obtained, independently of the choice of a time step-size, we let the time step-size be one for the sake of simplicity. The nonnegative constants μ_1 , μ_2 , and μ_3 denote the death rate of the susceptible, heroin users not in treatment, and heroin users in treatment class, respectively. Throughout the paper, it is biologically natural to assume that $\mu_1 \leq \min\{\mu_2, \mu_3\}$. The constant $\lambda > 0$ denotes the recruitment rate of susceptible population from the general population. Constant $P > 0$ is the proportion of heroin users who enter the treatment class. The individuals in treatment who stop using heroin are susceptible at a constant rate $\xi_2 \geq 0$. Constant β_3 represents the transmission rate from heroin users in treatment to untreated heroin users. $\beta_1(U_n)$ is the probability per unit time and the transmission is used with the form $\beta_1(U_n)S_{n+1} \sum_{k=0}^h U_{n-k}\eta_k$, which includes various delays. By a natural biological meaning, we assume that $\beta_1(U)$ is a positive function and that there exists a constant $U_\beta > 0$ such that $\beta_1(U)$ is nondecreasing on the interval $[0, U_\beta]$. The integer $h \geq 0$ is the time delay. The sequence $\eta_k : -\infty < \eta_k < +\infty$ ($k = 0, 1, \dots, h$) is nondecreasing and has bounded.

The initial conditions of the system (2) are given by

$$\begin{aligned} S_n &= \psi_n^{(1)}, & U_n &= \psi_n^{(2)}, & V_n &= \psi_n^{(3)}, \\ & & & & & \text{for } n = -h, -h + 1, \dots, 0, \end{aligned} \tag{3}$$

where $\psi_n^{(i)} \geq 0$ ($n = -h, -h + 1, \dots, 0, i = 1, 2, 3$). Again, by biological meaning, we further assume that $\psi_0^{(i)} > 0$ for all $i = 1, 2, 3$.

The paper is organized as follows. In Section 2, we prove the positivity and boundedness of the solution of system (2). In Section 3, we deal with the global asymptotic stability of the heroin-using free equilibrium. In Section 4, we consider the permanence of the discrete epidemic model applying Wang’s technique. In the discretized epidemic model, sufficient condition for global asymptotic stability and permanence are the same as for the original continuous epidemic model. We give some numerical examples and conclusion in Sections 5 and 6.

2. Basic Properties

For system (2), the heroin-using free equilibrium is given by

$$E^0 = (S^0, 0, 0), \quad S^0 = \frac{\lambda}{\mu_1}. \tag{4}$$

Define a positive constant $A \equiv \sum_{k=0}^h \eta_k$. The stability of E^0 is studied by using the next generation method in [7]. The associated matrix F (of the new heroin-using terms) and the M-matrix V (of the remaining transfer terms) are given as follows, respectively:

$$F = \begin{pmatrix} \frac{\beta_1(0)\lambda A}{\mu_1} & 0 \\ \mu_1 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} \mu_2 + P + \xi_1 & 0 \\ -P & \mu_3 + \xi_2 \end{pmatrix}. \tag{5}$$

Clearly, F is nonnegative, V is a nonsingular M-matrix, and $V - F$ has Z sign pattern. The associated basic reproduction number, denoted by R_0 , is then given by $R_0 = \rho(FV^{-1})$, where ρ is the spectral radius of FV^{-1} . It follows that

$$R_0 = \frac{\beta_1(0)\lambda A}{\mu_1(\mu_2 + P + \xi_1)}. \tag{6}$$

Now, we will consider the positivity and boundedness of solution to system (2). For most continuous epidemic models, positivity of the solution is clear, but, for system (2), the positivity of the sequences S_n, U_n , and V_n holds in some condition.

Lemma 1. Let (S_n, U_n, V_n) be any solution of system (2) with initial condition (3); then (S_n, U_n, V_n) is positive for any $n \in N$ and $V_0 < P(1 + \mu_3 + \xi_2)/\beta_3$.

Proof. Let (S_n, U_n, V_n) be any solution of system (2) with initial condition (3). It is evident that system (2) is equivalent to the following iteration system:

$$\begin{aligned} U_{n+1} &= \frac{\beta_1(U_n)S_{n+1} \sum_{k=0}^h U_{n-k}\eta_k + U_n}{1 - \beta_3V_{n+1} + \mu_2 + P + \xi_1}, \\ V_{n+1} &= \frac{V_n + PU_{n+1}}{1 + \beta_3U_{n+1} + \mu_3 + \xi_2}, \\ S_{n+1} &= \frac{\lambda + S_n + \xi_1U_{n+1} + \xi_2V_{n+1}}{1 + \beta_1(U_n) \sum_{k=0}^h U_{n-k}\eta_k + \mu_1}. \end{aligned} \tag{7}$$

In the following, we will use the induction to prove the positivity of solution. When $n = 0$, we have

$$U_1 = \frac{\beta_1(U_0)S_1 \sum_{k=0}^h U_{-k}\eta_k + U_0}{1 - \beta_3V_1 + \mu_2 + P + \xi_2}, \tag{8}$$

$$V_1 = \frac{V_0 + PU_1}{1 + \beta_3U_1 + \mu_3 + \xi_2}, \tag{9}$$

$$S_1 = \frac{\lambda + S_0 + \xi_1U_1 + \xi_2V_1}{1 + \beta_1(U_0) \sum_{k=0}^h U_{-k}\eta_k + \mu_1}. \tag{10}$$

From (8)–(10), we see that, as long as U_1 is obtained, V_1 and S_1 will be obtained too.

If $U_1 > 0$, from (9), we directly obtain $V_1 > 0$ and, from (10), we further obtain that $S_1 > 0$. Furthermore, we also have $N_1 = S_1 + U_1 + V_1 > 0$.

Let $x = U_1$; then, from (8)–(10), we see that x satisfies the following equation:

$$\begin{aligned} \psi(x) &= x - \left(\left(1 + \beta_1(U_0) \sum_{k=0}^h U_{-k}\eta_k + \mu_1 \right) U_0 \right. \\ &\quad \left. + \beta_1 \sum_{k=0}^h U_{-k}\eta_k W_1(x) \right) \\ &\quad \times \left(\left(1 + \beta_1(U_0) \sum_{k=0}^h U_{-k}\eta_k + \mu_1 \right) W_2(x) \right)^{-1}, \end{aligned} \tag{11}$$

where

$$\begin{aligned} W_1(x) &= \lambda + S_0 + \xi_1x + \xi_2V_1, \\ W_2(x) &= 1 + \mu_2 + P + \xi_1 - \beta_3V_1, \\ V_1 &= \frac{V_0 + Px}{1 + \beta_3x + \mu_3 + \xi_2}. \end{aligned} \tag{12}$$

Substituting V_1 in $W_2(x)$ gets

$$W_2(x) = 1 + \mu_2 + \xi_1 + \frac{P(1 + \mu_3 + \xi_2) - \beta_3V_0}{1 + \mu_3 + \xi_2 + \beta_3x}. \tag{13}$$

Since $V_0 < P(1 + \mu_3 + \xi_2)/\beta_3$, we have

$$W_2(0) = 1 + \mu_2 + P + \xi_1 - \frac{\beta_3V_0}{1 + \mu_3 + \xi_2} > 0, \tag{14}$$

and because of $W_1(0) = \lambda + S_0 + \xi_2V_0/(1 + \mu_3 + \xi_2) > 0$, thus

$$\begin{aligned} \psi(0) &= - \left(\left(1 + \beta_1(U_0) \sum_{k=0}^h U_{-k}\eta_k + \mu_1 \right) U_0 \right. \\ &\quad \left. + \beta_1(U_0) \sum_{k=0}^h U_{-k}\eta_k W_1(0) \right) \\ &\quad \times \left(\left(1 + \beta_1(U_0) \sum_{k=0}^h U_{-k}\eta_k + \mu_1 \right) W_2(0) \right)^{-1} < 0. \end{aligned} \tag{15}$$

Substituting W_1 and W_2 in $\psi(x)$ yields

$$\psi(x) = \frac{ax^2 + bx - c}{\left(1 + \beta_1(U_0) \sum_{k=0}^h U_{-k}\eta_k + \mu_1 \right) (mx + n)}. \tag{16}$$

Here, constants a, b, c, m , and n are as follows:

$$a = (1 + \mu_2) \left(1 + \beta_1(U_0) \sum_{k=0}^h U_{-k}\eta_k + \mu_1 \right) + \xi_1(1 + \mu_1) > 0,$$

$$\begin{aligned} b &= (1 + \mu_3 + \xi_2)(1 + \mu_1)(1 + \mu_2 + P + \xi_1) \\ &\quad + (1 + \mu_1) \left(\beta_1(U_0) \sum_{k=0}^h U_{-k}\eta_k + \beta_3S_0 - \beta_3N_0 \right) \end{aligned}$$

$$\begin{aligned} &+ \beta_1(U_0) \sum_{k=0}^h U_{-k}\eta_k P(1 + \mu_3) \\ &\quad - \beta_3\beta_1(U_0) \sum_{k=0}^h U_{-k}\eta_k (\lambda + N_0), \end{aligned}$$

$$\begin{aligned} c &= \beta_1(U_0) \sum_{k=0}^h U_{-k}\eta_k [(1 + \mu_3 + \xi_2)(\lambda + N_0) - (1 + \mu_3)V_0] \\ &\quad - U_0(1 + \mu_1)\xi_1, \end{aligned}$$

$$m = \beta_3(1 + \mu_2 + \xi_1) > 0,$$

$$n = (1 + \mu_2 + P + \xi_1)(1 + \mu_3 + \xi_2) - \beta_3V_0 > 0. \tag{17}$$

We take the limit on both sides of the above equation:

$$\begin{aligned} & \lim_{x \rightarrow +\infty} \psi(x) \\ &= \lim_{x \rightarrow +\infty} \frac{ax^2 + bx - c}{(1 + \beta_1(U_0) \sum_{k=0}^h U_{-k} \eta_k + \mu_1)(mx + n)} = +\infty; \end{aligned} \tag{18}$$

this means that $\psi(x) = 0$ has at least one positive solution $\bar{x} \in (0, +\infty)$. So, we have $U_1 = \bar{x} > 0$. Therefore, the positivity of $S_1 > 0, U_1 > 0$, and $V_1 > 0$ is finally obtained. When $n = 2$, we have

$$\begin{aligned} U_2 &= \frac{\beta_1 S_2 \sum_{k=0}^h U_{1-k} \eta_k + U_1}{1 - \beta_3 V_2 + \mu_2 + P + \xi_1}, \\ V_2 &= \frac{V_1 + P U_2}{1 + \beta_3 U_2 + \mu_3 + \xi_2}, \\ S_2 &= \frac{\lambda + S_1 + \xi_1 U_2 + \xi_2 V_2}{1 + \beta_1 \sum_{k=0}^h U_{1-k} \eta_k + \mu_1}, \end{aligned} \tag{19}$$

a similar argument as in the above for U_1, V_1 , and S_1 ; we also can obtain that $U_2 > 0, V_2 > 0$, and $S_2 > 0$. Lastly, by using the induction, we can finally obtain that $S_n > 0, U_n > 0$, and $V_n > 0$, for all $n > 0$.

Now, we define the total population as $N_n = S_n + U_n + V_n$. Then, from system (2), we know that

$$N_{n+1} - N_n = \lambda - \mu_1 S_{n+1} - \mu_2 U_{n+1} - \mu_3 V_{n+1}. \tag{20}$$

Notice the assumption that $\mu_1 \leq \min(\mu_2, \mu_3)$; we obtain

$$\begin{aligned} N_n &\leq \frac{\lambda + N_{n-1}}{1 + \mu_1} \\ &\leq \frac{\lambda}{1 + \mu_1} \times \left\{ 1 + \frac{1}{1 + \mu_1} + \dots + \left(\frac{1}{1 + \mu_1} \right)^{n-1} \right\} \\ &\quad + \left(\frac{1}{1 + \mu_1} \right)^n N_0 \\ &= \frac{\lambda}{\mu_1} \left\{ 1 - \left(\frac{1}{1 + \mu_1} \right)^n \right\} + \left(\frac{1}{1 + \mu_1} \right)^n N_0 \\ &\leq \max \left\{ \frac{\lambda}{\mu_1}, N_0 \right\}. \end{aligned} \tag{21}$$

If $\lambda/\mu_1 \geq N_0$, it is easy to see that $N_n \leq \lambda/\mu_1 = S^0$, for all large n . If $\lambda/\mu_1 < N_0$, from right hand side of system (2), we obtain

$$N_1 \leq \frac{\lambda + N_0}{1 + \mu_1} < N_0. \tag{22}$$

Hence, we have $N_1 < N_0$ and there exists $i \in N$ such that $N_i \leq \lambda/\mu_1 = S^0$. Then, we may use this N_i as a starting value instead of N_0 . This argument leads to the following result. \square

Lemma 2. For any solution (S_n, U_n, V_n) of system (2), the total population $N_n = S_n + U_n + V_n$ satisfies

$$\limsup_{n \rightarrow +\infty} N_n \leq S^0 = \frac{\lambda}{\mu_1}; \tag{23}$$

thus (S_n, U_n, V_n) is ultimately bounded.

Let $\Omega = \{(S_n, U_n, V_n) : S_n, U_n, V_n \geq 0, S_n + U_n + V_n \leq \lambda/\mu_1\}$; then Ω is the positive invariant set to the solution of system (2).

In the following, we will examine the existence of endemic equilibrium for a special case of system (2).

Lemma 3. Assume that $\beta_1(U) = \beta_1 > 0$ is a constant. If $R_0 > 1$, system (2) admits a heroin-using equilibrium $E^* = (S^*, U^*, V^*)$ when $V < P/\beta_3$, where E^* satisfies following equality:

$$\begin{aligned} \lambda - A\beta_1 S^* U^* - \mu_1 S^* + \xi_1 U^* + \xi_2 V^* &= 0, \\ A\beta_1 S^* U^* + \beta_3 U^* V^* - (\mu_2 + P + \xi_1) U^* &= 0, \\ P U^* - \beta_3 U^* V^* - (\mu_3 + \xi_2) V^* &= 0. \end{aligned} \tag{24}$$

Proof. Consider the following equation:

$$\begin{aligned} \lambda - A\beta_1 S U - \mu_1 S + \xi_1 U + \xi_2 V &= 0, \\ A\beta_1 S U + \beta_3 U V - (\mu_2 + P + \xi_1) U &= 0, \\ P U - \beta_3 U V - (\mu_3 + \xi_2) V &= 0. \end{aligned} \tag{25}$$

From the first equation and the second equation of the system (25), we have

$$\lambda - \mu_1 S + \xi_2 V + \beta_3 U V - (\mu_2 + P) U = 0. \tag{26}$$

From the second equation and the third equation of the system (25), we obtain

$$A\beta_1 S U - (\mu_2 + \xi_1) U - (\mu_3 + \xi_2) V = 0; \tag{27}$$

thus,

$$U = \frac{(\mu_3 + \xi_2) V}{A\beta_1 S - \mu_2 + \xi_1}. \tag{28}$$

Since $U \neq 0$, from the second equation of the system, we have

$$S = \frac{\mu_2 + P + \xi_1 - \beta_3 V}{A\beta_1}. \tag{29}$$

Substituting S in (28), we obtain

$$U = \frac{(\mu_3 + \xi_2)V}{P - \beta_3 V}. \tag{30}$$

Substituting U and S in (26) yields a quadratic equation of V as follows:

$$F(V) = aV^2 + bV + c = 0, \tag{31}$$

where the coefficients are given by

$$\begin{aligned} a &= \mu_1 \beta_3^2 - A \beta_1 \beta_3 \mu_3, & c &= \mu_1 P (\mu_2 + P + \xi_1) - \lambda A \beta_1 P, \\ b &= \lambda A \beta_1 \beta_3 - \mu_1 \beta_3 (\mu_2 + \xi_1) + A \beta_1 \mu_3 (\mu_2 + P) + A \beta_1 \mu_2 \xi_2. \end{aligned} \tag{32}$$

Since $R_0 = \lambda A \beta_1 / \mu_1 (\mu_2 + P + \xi_1) > 1$, then it is easy to see that $c < 0$ and $b > 0$. According to Descartes' rule of signs, if $a \geq 0$, then $F(V) = 0$ has a positive solution; if $a < 0$, then $F(V) = 0$ has two positive solutions. From the expression of S and U , we note that $V < P/\beta_3$. Since

$$F(0) = c < 0, \quad F\left(\frac{P}{\beta_3}\right) = \frac{A \beta_1 P \mu_2 (\mu_3 + \xi_2)}{\beta_3} > 0. \tag{33}$$

This means that $F(V) = 0$ has a unique positive solution $V^* \in (0, P/\beta_3)$. Therefore, there exists a unique positive solution (S^*, U^*, V^*) of system (2). \square

For the local stability of the equilibria, we refer to Theorem 2 in [7] and have the following results.

Theorem 4. Assume that $\beta_1(U) = \beta_1$, β_1 is a positive constant. The heroin-using free equilibrium $E^0 = (S^0, 0, 0)$ of system (2) is locally asymptotically stable if $R_0 < 1$ and unstable if $R_0 > 1$.

3. Global Asymptotic Stability of the Heroin-Using Free Equilibrium

In this section, we still assume that $\beta_1(U) = \beta_1 > 0$, and obtain a sufficient condition for global asymptotic stability of the heroin-using free equilibrium E^0 of system (2).

Using a Lyapunov function similar to that in [11], we can easily prove the global asymptotic stability of the heroin-using free equilibrium E^0 .

Theorem 5. If $R_0 < 1$, the drug-using free equilibrium E^0 of system (2) is globally asymptotically stable.

Proof. Let us take the following Lyapunov function:

$$H_n = U_n + c_1 V_n + c_2 \sum_{k=0}^h \left(\sum_{l=n-k}^n U_l \right) \eta_k + \frac{c_3}{2} (S_{n+1} - S^0)^2, \tag{34}$$

where $c_i > 0$ ($i = 1, 2, 3$) are the constants to be defined later and $S^0 = \lambda/\mu_1$. Using system (2), the difference of H_n satisfies

$$\begin{aligned} \Delta H &= H_{n+1} - H_n \\ &= U_{n+1} - U_n + c_1 (V_{n+1} - V_n) \\ &\quad + c_2 \sum_{k=0}^h (U_{n+1} \eta_k - U_{n-k} \eta_k) \\ &\quad + \frac{c_3}{2} \left\{ (S_{n+1} - S^0)^2 - (S_n - S^0)^2 \right\}. \end{aligned} \tag{35}$$

From $S_n \leq N_n \leq S^0$, for all $n \geq 0$, we have

$$\begin{aligned} \Delta H &\leq \sum_{k=0}^h U_{n-k} \eta_k \left\{ \beta_1 S_{n+1} - c_2 - c_3 \beta_1 S_{n+1} (S_{n+1} - S^0) \right\} \\ &\quad + (1 - c_1) \beta_3 U_{n+1} V_{n+1} \\ &\quad + [c_2 A - \mu_2 - P - \xi_1 + c_1 P + c_3 \xi_1 (S_{n+1} - S^0)] U_{n+1} \\ &\quad - [c_1 (\mu_3 + \xi_2) - c_3 \xi_2 (S_{n+1} - S^0)] V_{n+1} \\ &\quad - c_3 \mu_1 (S_{n+1} - S^0)^2 \\ &\leq \sum_{k=0}^n U_{n-k} \eta_k \left\{ \beta_1 S_{n+1} - c_2 - c_3 \beta_1 S_{n+1} (S_{n+1} - S^0) \right\} \\ &\quad + (1 - c_1) \beta_3 U_{n+1} V_{n+1} \\ &\quad + [c_2 A - \mu_2 - P - \xi_1 + c_1 P] U_{n+1} - c_1 (\mu_3 + \xi_2) V_{n+1} \\ &\quad - c_3 \mu_1 (S_{n+1} - S^0)^2. \end{aligned} \tag{36}$$

Let us choose $c_i > 0$ ($i = 1, 2, 3$) such that these constants satisfy the following inequalities:

$$\beta_1 S_{n+1} - c_2 - c_3 \beta_1 S_{n+1} (S_{n+1} - S^0) < 0, \tag{37}$$

$$1 - c_1 < 0, \tag{38}$$

$$c_1 P + c_2 A < \mu_2 + P + \xi_1. \tag{39}$$

From (37), we have $c_3 \beta_1 S_{n+1}^2 + (c_3 \beta_1 S^0 - \beta_1) S_{n+1} + c_2 > 0$; since $S_{n+1} > 0$, then the following inequality is true:

$$\beta_1 (1 + c_3 S^0)^2 < 4c_2 c_3; \tag{40}$$

that is,

$$\beta_1 (S^0)^2 c_3^2 + (2\beta_1 S^0 - 4c_2) c_3 + \beta_1 < 0. \tag{41}$$

Since $R_0 < 1$, which implies that $\beta_1 A S^0 < \mu_2 + P + \xi_1$, we can choose $c_2 = \beta_1 S^0 + \epsilon$; here, ϵ ($0 < \epsilon < (\mu_2 + \xi_1 - A \beta_1 S^0)/A$) is a sufficiently small positive number such that $\beta_1 A S^0 + A \epsilon < \mu_2 + (1 - c_1)P + \xi_1$. Since $\beta_1 S^0 - 2c_2 < 0$ and $(\beta_1 S^0 - 2c_2)^2 > (\beta_1 S^0)^2$, we can choose $c_3 > 0$ to satisfy (41). We may further choose $c_1 > 1$ to satisfy (38). Therefore, ΔH is negative definite and is equal to zero if and only if $S_{n+1} = S^0$, $U_{n+1} = 0$, and $V_{n+1} = 0$. The proof is complete. \square

4. Permanence of System (2)

The system (2) is said to be permanent if there are positive constants m and M such that

$$m \leq \liminf_{n \rightarrow \infty} S_n \leq \limsup_{n \rightarrow \infty} S_n \leq M \tag{42}$$

hold for any sequence S_n of the system (2), and the same inequalities hold for U_n and V_n . For each class S_n, U_n , and V_n , m and M are independent of initial conditions.

Following the method used by Wang in [6], we will prove the permanence of system (2) for the general case; that is, assume that $\beta_1(U)$ is related to U .

Theorem 6. *If $R_0 > 1$, then system (2) is permanent for any initial condition (3).*

Proof. Firstly, from system (2) and Lemmas 1 and 2, for any $\epsilon_0 > 0$, there exists sufficiently large $n_0 > 0$ such that $U_n \leq \lambda/\mu_1 + \epsilon_0$ as $n \geq n_0 + h$. Then, we have

$$\begin{aligned} S_{n+1} &= \frac{\lambda + S_n + \xi_1 U_{n+1} + \xi_2 V_{n+1}}{1 + \beta_1(U_n) \sum_{k=0}^h U_{n-k} \eta_k + \mu_1} \\ &> \frac{\lambda}{1 + \beta_1(U_n) \sum_{k=0}^h U_{n-k} \eta_k}. \end{aligned} \tag{43}$$

Let $\beta_1^M(\epsilon_0) = \max_{U \in [0, \lambda/\mu_1 + \epsilon_0]} \beta_1(U)$. Thus, we have

$$\begin{aligned} S_{n+1} &\geq \frac{\lambda}{1 + \beta_1^M \sum_{k=0}^h U_{n-k} \eta_k} \\ &\geq \frac{\lambda}{1 + \beta_1^M (\lambda/\mu_1 + \epsilon_0) A}. \end{aligned} \tag{44}$$

Notice that ϵ_0 can be arbitrarily small. Then, we have

$$\liminf_{n \rightarrow +\infty} S_{n+1} \geq m_s = \frac{\lambda}{1 + \beta_1^M (A\lambda/\mu_1)}, \tag{45}$$

$$\beta_1^M = \max_{U \in [0, \lambda/\mu_1]} \beta_1(U).$$

Next, let us consider the positive sequences S_n and U_n of (2). According to these sequences, we define

$$H_n = U_n + \frac{\mu_2 + P + \xi_1}{A} \sum_{k=0}^h \left(\sum_{l=n-k}^n U_l \right) \eta_k. \tag{46}$$

Then, for $n \geq 0$, we obtain

$$\begin{aligned} \Delta H &= H_{n+1} - H_n \\ &= U_{n+1} - U_n + \frac{\mu_2 + P + \xi_1}{A} \sum_{k=0}^h (U_{n+1} \eta_k - U_{n-k} \eta_k) \\ &= \left(\beta_1(U_n) S_{n+1} - \frac{\mu_2 + P + \xi_1}{A} \right) \sum_{k=0}^h U_{n-k} \eta_k + \beta_3 U_{n+1} V_{n+1}. \end{aligned} \tag{47}$$

Since $R_0 = \beta_1(0)A\lambda/\mu_1(\mu_2 + P + \xi_1) > 1$, there exist $0 < \alpha < U_\beta$ and $\rho > 0$ such that

$$\begin{aligned} &\frac{A\beta_1(0)}{\mu_2 + P + \xi_1} \times \frac{\lambda}{\mu_1 + \alpha\beta_1(\alpha)A} \\ &\times \left\{ 1 - \left(\frac{1}{1 + \mu_1 + \alpha\beta_1(\alpha)A} \right)^{\rho h} \right\} > 1; \end{aligned} \tag{48}$$

note that

$$S^\Delta = \frac{\lambda}{\mu_1 + \alpha\beta_1(\alpha)A} \left\{ 1 - \left(\frac{1}{1 + \mu_1 + \alpha\beta_1(\alpha)A} \right)^{\rho h} \right\}. \tag{49}$$

We claim that it is impossible that $U_n \leq \alpha$ holds for all $n \geq n_1 \geq [\rho h]$. The function $[x]$ gives the smallest integer not less than x . Suppose the contrary, for $n \geq n_1 + h$. Consider

$$\begin{aligned} S_{n+1} &= \frac{\lambda + S_n + \xi_1 U_{n+1} + \xi_2 V_{n+1}}{1 + \mu_1 + \beta_1(U_n) \sum_{k=0}^h U_{n-k} \eta_k} > \frac{\lambda + S_n}{1 + \mu_1 + \alpha\beta_1(\alpha)A} \\ &> \frac{\lambda}{1 + \mu_1 + \alpha\beta_1(\alpha)A} \left\{ 1 + \frac{1}{1 + \mu_1 + \alpha\beta_1(\alpha)A} \right. \\ &\quad \left. + \left(\frac{1}{1 + \mu_1 + \alpha\beta_1(\alpha)A} \right)^{n-n_1-h-1} \right\} \\ &\quad + \left(\frac{1}{1 + \mu_1 + \alpha\beta_1(\alpha)A} \right)^{n-n_1-h} S_{n_1+h+1}. \end{aligned} \tag{50}$$

From Lemma 1, S_n satisfies

$$S_{n+1} > \frac{\lambda}{\mu_1 + \alpha\beta_1(\alpha)A} \left\{ 1 - \left(\frac{1}{1 + \mu_1 + \alpha\beta_1(\alpha)A} \right)^{n-n_1-h} \right\}, \tag{51}$$

and we have that, for $n \geq n_1 + h + [\rho h]$, we have

$$\begin{aligned} S_{n+1} &> \frac{\lambda}{\mu_1 + \alpha\beta_1(\alpha)A} \left\{ 1 - \left(\frac{1}{1 + \mu_1 + \alpha\beta_1(\alpha)A} \right)^{[\rho h]} \right\} \\ &\geq \frac{\lambda}{\mu_1 + \alpha\beta_1(\alpha)A} \left\{ 1 - \left(\frac{1}{1 + \mu_1 + \alpha\beta_1(\alpha)A} \right)^{\rho h} \right\} = S^\Delta. \end{aligned} \tag{52}$$

Hence, for $n \geq n_1 + h + [\rho h]$, we have

$$\begin{aligned} \Delta H &\geq \left(\beta_1(U_n) S_{n+1} - \frac{\mu_2 + P + \xi_1}{A} \right) \sum_{k=0}^h U_{n-k} \eta_k \\ &> \left(\beta_1(0) S^\Delta - \frac{\mu_2 + P + \xi_1}{A} \right) \sum_{k=0}^h U_{n-k} \eta_k \\ &= \frac{\mu_2 + P + \xi_1}{A} \left(\frac{A\beta_1(0)S^\Delta}{\mu_2 + P + \xi_1} - 1 \right) \sum_{k=0}^h U_{n-k} \eta_k. \end{aligned} \tag{53}$$

Let $\epsilon = \min_{\theta} \{U_{n_1 + [\rho h] + h + \theta}; \theta = -h, -h+1, \dots, 0\}$. Now, we will show that $U_n \geq \epsilon$ for all $n \geq n_1 + [\rho h] + h$. In fact, there is an integer $\bar{n} > 0$ such that

$$\begin{aligned} U_n &\geq \epsilon, & n_1 + [\rho h] + h \leq n \leq n_1 + [\rho h] + h + \bar{n}, \\ U_{n+1} &< \epsilon, & n = n_1 + [\rho h] + h + \bar{n}. \end{aligned} \tag{54}$$

However, for $n = n_1 + [\rho h] + h + \bar{n}$, we have

$$\begin{aligned} U_{n+1} - \epsilon &= \frac{\beta_1(U_n) S_{n+1} \sum_{k=0}^h U_{n-k} \eta_k + U_n}{1 + \mu_2 + P + \xi_1 - \beta_3 V_{n+1}} - \epsilon \\ &\geq \frac{\beta_1(U_n) S_{n+1} \sum_{k=0}^h U_{n-k} \eta_k + U_n}{1 + \mu_2 + P + \xi_1} \\ &\quad - \frac{1 + \mu_2 + P + \xi_1}{1 + \mu_2 + P + \xi_1} \epsilon \\ &\geq \frac{\beta_1(0) S^\Delta A - (\mu_2 + P + \xi_1)}{1 + \mu_1 + P + \xi_1} \epsilon \\ &= \frac{\mu_2 + P + \xi_1}{1 + \mu_2 + p + \xi_1} \left\{ \frac{\beta_1(0) S^\Delta A}{\mu_2 + P + \xi_1} - 1 \right\} \epsilon > 0. \end{aligned} \tag{55}$$

Which is a contradiction. Thus, $U_n \geq \epsilon$ for $n \geq n_1 + [\rho h] + h$. Therefore, for $n \geq n_1 + [\rho h] + h$,

$$\Delta H > (\mu_2 + P + \xi_1) \left\{ \frac{A\beta_1(0) S^\Delta}{\mu_2 + P + \xi_1} - 1 \right\} \epsilon > 0, \tag{56}$$

which implies that $H_n \rightarrow +\infty$ as $n \rightarrow +\infty$. But, from Lemma 2 and (46), there exists a sufficiently large integer $n'_1 > 0$ such that, for $n > n'_1$,

$$\begin{aligned} H_n &\leq \frac{\lambda}{\mu_1} + \frac{\mu_2 + P + \xi_1}{A} \sum_{k=0}^h \left(\sum_{l=n-k}^n \frac{\lambda}{\mu_1} \right) \eta_k \\ &\leq \frac{\lambda}{\mu_1} \{1 + (\mu_2 + P + \xi_1)(h+1)\}, \end{aligned} \tag{57}$$

which is a contradiction. Hence, the claim is proved.

In the rest, we only need to consider the following two cases:

- (i) $U_n \geq \alpha$ for all large n .
- (ii) U_n oscillates about α for all large n .

We show that $U_n \geq m_u$ for all large n , where $0 < m_u \leq \alpha$, is a constant which will be given later. Clearly, we only need to consider case (ii). Let positive integers n_1 and n_2 be sufficiently large that $U_{n_1} \geq \alpha$, $U_{n_2} \geq \alpha$, and $U_n < \alpha$, for $n_1 < n < n_2$.

If $n_2 - n_1 < h + [\rho h]$, since

$$\begin{aligned} U_n &= \frac{\beta_1(U_{n-1}) S_n \sum_{k=0}^h U_{n-k-1} \eta_k + U_{n-1}}{1 + \mu_2 + P + \xi_1 - \beta_3 V_n} \\ &\geq \frac{U_{n-1}}{1 + \mu_2 + P + \xi_1}, \end{aligned} \tag{58}$$

we have

$$\begin{aligned} U_n &\geq \left(\frac{1}{1 + \mu_2 + P + \xi_1} \right)^{n-n_1} U_{n_1} \\ &> \left(\frac{1}{1 + \mu_2 + P + \xi_1} \right)^{n_2-n_1} U_{n_1} \\ &> m_u = \left(\frac{1}{1 + \mu_2 + P + \xi_1} \right)^{h+[\rho h]} \alpha. \end{aligned} \tag{59}$$

Hence, $U_n > m_u$ for $n \in [n_1, n_2]$.

If $n_2 - n_1 > h + [\rho h]$, we can easily obtain that $U_n > m_u$ for $n \in [n_1, n_1 + h + [\rho h]]$. Assume that there exists an integer $\hat{n} > 0$ such that

$$\begin{aligned} U_n &\geq m_u, & n_1 + h + [\rho h] \leq n \leq n_1 + h + [\rho h] + \hat{n}, \\ U_{n+1} &< m_u, & n = n_1 + h + [\rho h] + \hat{n}. \end{aligned} \tag{60}$$

However, for $n = n_1 + h + [\rho h] + \hat{n}$,

$$U_{n+1} - m_u \geq \frac{\mu_2 + P + \xi_1}{1 + \mu_2 + P + \xi_1} \left\{ \frac{\beta_1(0) S^\Delta A}{\mu_2 + P + \xi_1} - 1 \right\} m_u > 0. \tag{61}$$

This is a contradiction to the proposition that $U_{n+1} < m_u$. Therefore, $U_n \geq m_u$ for $n \in [n_1, n_2]$. Since these positive integers n_1 and n_2 are chosen in an arbitrary way, we conclude that $U_n \geq m_u$ for all large n in case (ii). Hence, $\liminf_{n \rightarrow \infty} U_n \geq m_u$.

Note that, from that third equation of system (2), we have

$$\liminf_{n \rightarrow \infty} V_n \geq m_v = \frac{P}{\mu_3 + \xi_2 + \lambda\beta_3/\mu_1} m_u = \frac{P}{\mu_3 + \xi_2 + \beta_3 S^0} m_u. \tag{62}$$

From Lemma 2 and the discussion above, we have

$$\begin{aligned} m_s &\leq \liminf_{n \rightarrow \infty} S_n \leq \limsup_{n \rightarrow \infty} S_n \leq S^0, \\ m_u &\leq \liminf_{n \rightarrow \infty} U_n \leq \limsup_{n \rightarrow \infty} U_n \leq S^0, \\ m_v &\leq \liminf_{n \rightarrow \infty} V_n \leq \limsup_{n \rightarrow \infty} V_n \leq S^0. \end{aligned} \tag{63}$$

The proof is completed. □

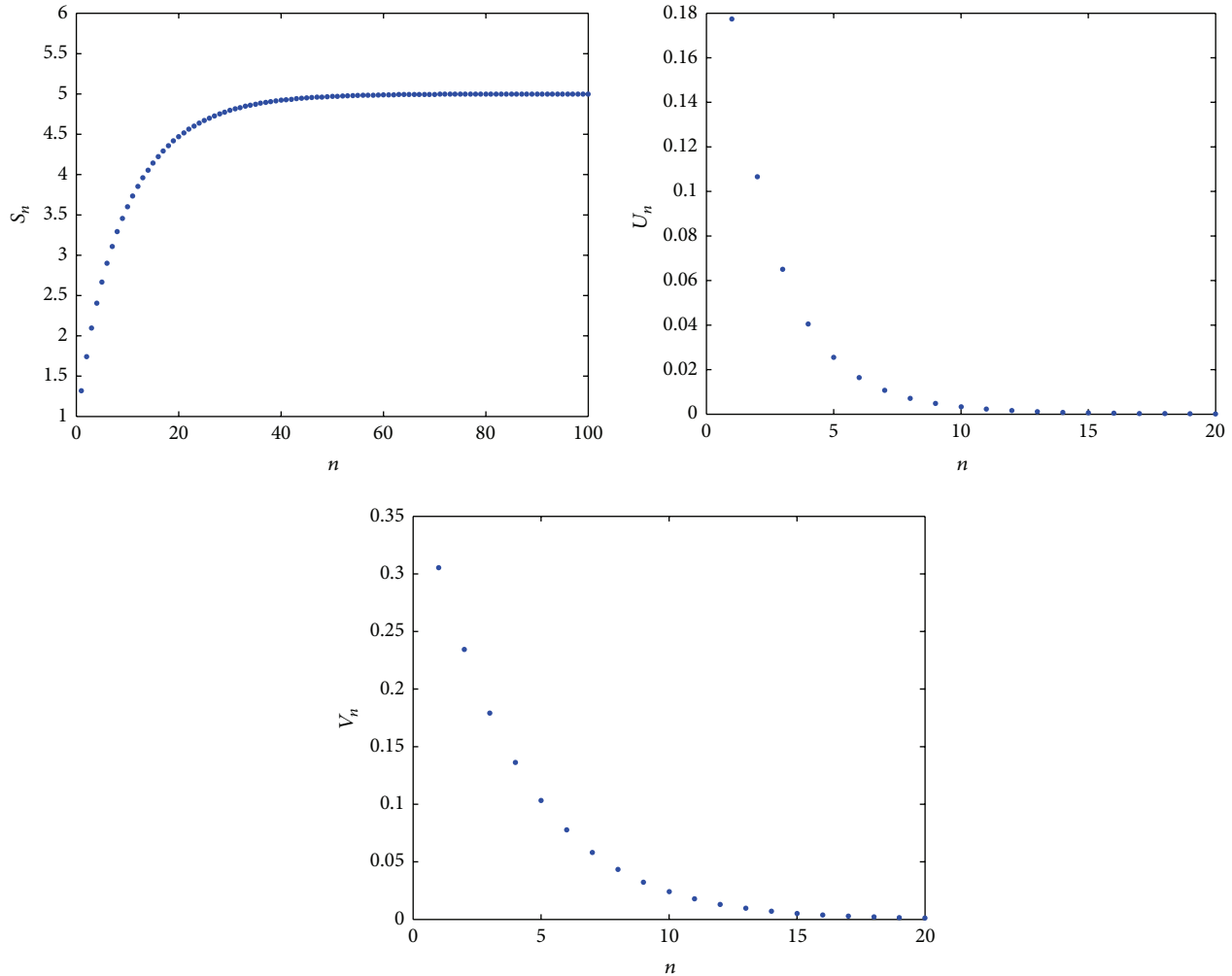


FIGURE 1

5. Numerical Example

In order to confirm the validity of our results, we consider the following heroin epidemic model with a discrete time delay:

$$\begin{aligned}
 S_{n+1} - S_n &= \lambda - \beta_1 S_{n+1} U_{n-h} - \mu_1 S_{n+1} + \xi_1 U_{n+1} + \xi_2 V_{n+1}, \\
 U_{n+1} - U_n &= \beta_1 S_{n+1} U_{n-h} + \beta_3 U_{n+1} V_{n+1} \\
 &\quad - (\mu_2 + P + \xi_1) U_{n+1}, \\
 V_{n+1} - V_n &= P U_{n+1} - \beta_3 U_{n+1} V_{n+1} - (\mu_3 + \xi_2) V_{n+1}.
 \end{aligned}
 \tag{64}$$

Now, we present a numerical example. For the sake of simplicity, we choose the parameters as $\beta_1 = 0.9$, $\beta_3 = 0.8$, $\lambda = 2$, $\mu_1 = 0.1$, $\mu_2 = 0.2$, $\mu_3 = 0.1$, $P = 0.4$, $\xi_1 = 0.1$, and $\xi_2 = 0.2$; we get $R_0 = 25.7143 < 1$. Figure 1 shows that the disease free equilibrium E^0 of the system (64) is globally asymptotically stable when $R_0 < 1$. Figure 2 shows that the system (64) is permanent when $R_0 > 1$.

6. Conclusions

In this paper, we have modified the Samanta heroin epidemic model into an autonomous heroin epidemic model with distributed time delay. Further, we established a discretized heroin epidemic model with time delay, sufficient conditions have been obtained to ensure the global asymptotic stability of heroin-using free equilibrium when $R_0 \leq 1$ and $\beta_1(U)$ is replaced by a positive constant. We also carried out some discussion about the heroin-using equilibrium, but our results are only restricted to the existence of this equilibrium for $\beta_1(U) = \beta_1 > 0$, a special case of system (2). The stability of heroin-using equilibrium is yet to be studied. As a main result of this paper, we obtained the permanence of the system (2). From the expression of $R_0 = \beta_1(0)\lambda A / \mu_1(\mu_2 + P + \xi_1)$, we see that a decrease in β_1 (transmission coefficient from susceptible population) will cause a decrease of the same proportion in R_0 . If the rate of migration or recruitment is restricted into susceptible community, the spread of the disease can also be kept under control by reducing λ and thereby decreasing R_0 . The spread of the heroin users can also be controlled by educators, epidemiologists, and treatment

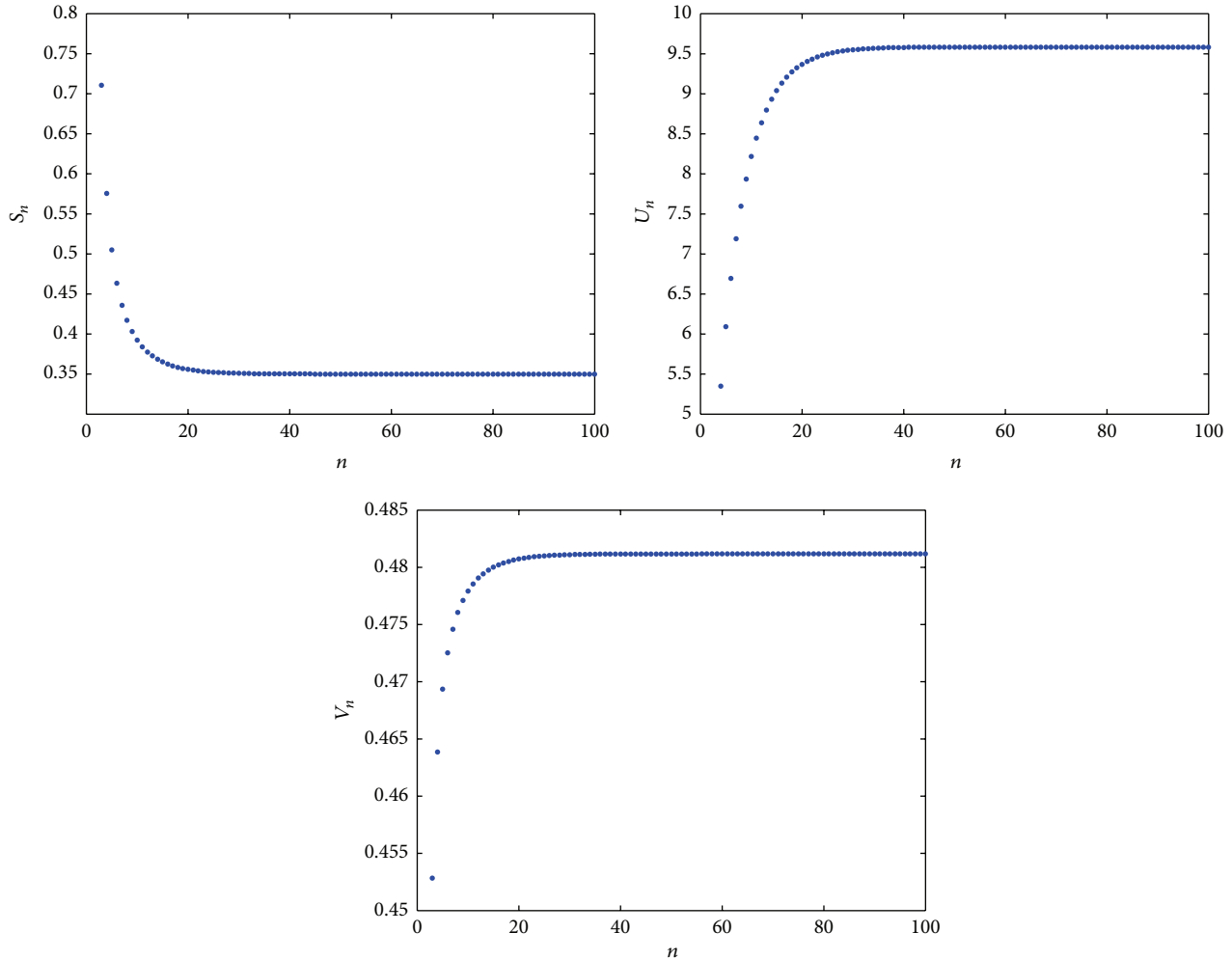


FIGURE 2

providers to increase the values of ξ (removal rate of heroin users not in treatment who stop using heroin but are susceptible) and P (proportion of heroin users who enter treatment) and thereby to decrease R_0 . This analysis tells us that prevention is better than cure; efforts to increase prevention are more effective in controlling the spread of habitual drug use than efforts to increase the numbers of individuals accessing treatment.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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