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*Research Article*

# Demiclosedness Principle for Total Asymptotically Nonexpansive Mappings in $CAT(0)$ Spaces

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We prove the demiclosedness principle for a class of mappings which is a generalization of all the forms of nonexpansive, asymptotically nonexpansive, and nearly asymptotically nonexpansive mappings. Moreover, we establish the existence theorem and convergence theorems for modified Ishikawa iterative process in the framework of $CAT(0)$ spaces. Our results generalize, extend, and unify the corresponding results on the topic in the literature.

## 1. Introduction

A self-mapping $T$, on a metric space $(X, d)$, is called nonexpansive, if

$$d(Tx, Ty) \le d(x, y), \quad x, y \in X, \tag{1}$$

and asymptotically nonexpansive, introduced by Goebel and Kirk [1], if there exists a nonnegative sequence $\{k_n\}_{n\ge1}$ with $\lim_{n\to\infty} k_n = 1$ such that

$$d(T^n x, T^n y) \le k_n d(x, y), \quad n \ge 1, \ x, y \in X. \tag{2}$$

A self-mapping $T$, on a metric space $(X, d)$, is called asymptotic point-wise nonexpansive, introduced by Hussain and Khamsi [2], if there exists a sequence of mappings $\alpha_n : K \to [0, \infty)$ with $\limsup_{n\to\infty} \alpha_n(x) \le 1$ such that

$$d(T^n x, T^n y) \le \alpha_n(x) d(x, y), \quad n \ge 1, \ x, y \in X. \tag{3}$$

It is quite natural to extend (2) and (3) in the following way: a self-mapping $T$, on a metric space $(X, d)$, is called asymptotically $\psi$-nonexpansive, if there exists a nonnegative sequence $\{k_n\}_{n\ge1}$ with $\lim_{n\to\infty} k_n = 1$ such that

$$d(T^n x, T^n y) \le k_n \psi(d(x, y)), \quad n \ge 1, \ x, y \in X, \tag{4}$$

where $\psi : R^+ \to R^+$ is a strictly increasing and continuous mapping with $\psi(0) = 0$. Notice that an asymptotically $\psi$-nonexpansive is a generalization of an asymptotically nonexpansive. Indeed, if we take $\psi(\lambda) = \lambda$, we get inequality (2). Analogously, we consider the extension of (3) as follows. A self-mapping $T$, on a metric space $(X, d)$, is called asymptotic point-wise $\psi$-nonexpansive if there exists a sequence of mappings $\alpha_n : K \to [0, \infty)$ with $\limsup_{n\to\infty} \alpha_n(x) \le 1$ such that

$$d(T^n x, T^n y) \le \alpha_n(x) \psi(d(x, y)), \quad n \ge 1, \ x, y \in X. \tag{5}$$

It is evident that if we replace $\psi(\lambda) = \lambda$ in (5), then we derive (3).

A self-mapping $T$, on a metric space $(X, d)$, is called nearly Lipschitzian with respect to a fix sequence $\{a_n\}$, introduced by Sahu [3], if, for each $n \in N$, there exists a constant $k_n \ge 0$ such that

$$d(T^n x, T^n y) \le k_n (d(x, y) + a_n), \tag{6}$$

for all $x, y \in X$, where $a_n \in [0, 1)$ for each $n$ and $a_n \to 0$. The infimum of constants $k_n$ satisfying (6) is called the nearly Lipschitz constant of $T^n$ and is denoted by $\eta(T^n)$. Furthermore, $T$ is called nearly nonexpansive if $\eta(T^n) = 1$ for all $n \in N$ and nearly asymptotically nonexpansive if $\eta(T^n) \ge 1$ for all $n \in N$ and $\lim_{n\to\infty} \eta(T^n) = 1$.

A self-mapping $T$, on a metric space $(X, d)$, is said to be asymptotically nonexpansive in the intermediate sense, introduced by Bruck et al. [4], if it is continuous and the following inequality holds:

$$\limsup_{n\to\infty} \sup_{x,y\in C} \left(d\left(T^n x, T^n y\right) - d\left(x, y\right)\right) \le 0. \quad (7)$$

We note that if we set

$$\xi_n = \max\left\{0, \sup_{x,y\in C} \left(d\left(T^n x, T^n y\right) - d\left(x, y\right)\right)\right\}, \quad (8)$$

then $\xi_n \to 0$ as $n \to \infty$. It follows that (7) is reduced to

$$d\left(T^n x, T^n y\right) \le d\left(x, y\right) + \xi_n, \quad (9)$$

for $n \ge 1$ and $x, y \in K$.

For the examples which are asymptotically nonexpansive in the intermediate sense but not asymptotically nonexpansive, see, for example, [5]. In fact, the class of nearly asymptotically nonexpansive mappings is intermediate classes between the class of asymptotically nonexpansive mappings and that of asymptotically nonexpansive in the intermediate sense mappings.

In 2006, Alber et al. [6] introduced the notion of total asymptotically nonexpansive mappings. The class of such mappings includes the asymptotically nonexpansive mappings; for more details, see, for example, [7]. This new notion unifies various definitions mentioned above.

On the context of uniformly convex Banach spaces, several papers appeared on the topic of the approximation of fixed points of mappings in the classes of nonexpansive and asymptotically nonexpansive mappings. Motivated by these results, we investigate the existence of fixed points of total asymptotically nonexpansive mappings in the context of $CAT(0)$ spaces that attracted attention of several authors; see, for example, [8–18].

More precisely, we prove the convergence of modified Ishikawa iterative process, introduced by Schu [19],

$$\begin{aligned} x_{n+1} &= \left(1 - \alpha_n\right) x_n \oplus \alpha_n T^n y_n, \\ y_n &= \left(1 - \beta_n\right) x_n \oplus \beta_n T^n x_n, \end{aligned} \quad (10)$$

where $x_1$ lies in a nonempty closed convex subset $K$ of a $CAT(0)$ space $X$, $\{\alpha_n\}, \{\beta_n\}$ are real sequences in $(0, 1)$ for each $n \ge 1$, and $T : K \to K$ is a total asymptotically nonexpansive mapping. The notation "$\oplus$" is introduced in the next section.

Notice that it is not possible to get the following modified Mann iterative process:

$$x_{n+1} = \left(1 - \alpha_n\right) x_n \oplus \alpha_n T^n x_n, \quad (11)$$

from the modified Ishikawa iterative process, since we can take $\beta_n = 0, (n \ge 1)$, in (10). Also, as a special case the results remain true for modified Mann iteration. Our results generalize, extend, and unify the corresponding results of [13, 20–22] and the references contained therein.

## 2. Preliminary Remarks

Throughout the paper, the set of real numbers will be denoted by $\mathbb{R}$. Suppose that $(X, d)$ is a metric space, $x, y \in X$, and $[0, l] \subset \mathbb{R}$. A map $c : [0, l] \to X$ is said to be a geodesic path joining the point $x$ to $y$ if $c(0) = x$ and $c(l) = y$, with $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In short, we use a geodesic from $x$ to $y$ instead of a geodesic path joining $x$ to $y$. Notice that if $c$ is an isometry, then $d(x, y) = l$. The image of $c$ is called a geodesic segment (or metric segment) joining $x$ and $y$. If it is unique, this geodesic is denoted by $[x, y]$. A metric space $(X, d)$ is called a geodesic space if every two points of $X$ are joined by a geodesic. Furthermore, $X$ is called uniquely geodesic if there is exactly one geodesic joining $x$ to $y$ for each $x, y \in X$. A subset $Y \subseteq X$ is called convex if $Y$ includes every geodesic segment joining any two of its points.

In a geodesic metric space $(X, d)$, geodesic triangle $\triangle(x_1, x_2, x_3)$ consists of three points in $X$ and a geodesic segment between each pair of vertices. Here, the points are also called vertices of $\triangle$ and a geodesic segment is said to be the edge of $\triangle$. A triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\overline{x}_1, \overline{x}_2, \overline{x}_3)$, in the Euclidean plane $E^2$, is called a comparison triangle for geodesic triangle $\triangle(x_1, x_2, x_3)$ in $(X, d)$ is where $d_{E^2}(\overline{x}_i, \overline{x}_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$.

*Comparison Axiom.* Let $(X, d)$ be a geodesic metric space $(X, d)$ and let $\overline{\triangle}$ be a comparison triangle for a geodesic triangle $\triangle$ in $X$. We say that $\triangle$ satisfies the $CAT(0)$ inequality if

$$d\left(x, y\right) \le d_{E^2}\left(\overline{x}, \overline{y}\right), \quad (12)$$

for all $x, y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \overline{\triangle}$.

A geodesic metric space is called a $CAT(0)$ space [23] if all geodesic triangles of appropriate size satisfy the comparison axiom. A complete $CAT(0)$ space $(X, d)$ is called "Hadamard space."

**Lemma 1** (see [20]). *Let $(X, d)$ be a $CAT(0)$ space. Then,*

(1) *$(X, d)$ is uniquely geodesic;*

(2) *let $x, y \in X$ with $x \ne y$. If $z, w \in [x, y]$ such that $d(x, z) = d(x, w)$, then $z = w$;*

(3) *let $x, y \in X$. For each $t \in [0, 1]$, there exists a unique point $z \in X$ such that*

$$d\left(x, z\right) = td\left(x, y\right); \qquad d\left(y, z\right) = \left(1 - t\right) d\left(x, y\right). \quad (13)$$

In the sequel, we use the notation $(1 - t)x \oplus ty$ for the unique point $z \in X$ satisfying (13).

Assume that $(X, d)$ is a Hadamard space. Suppose that $(x_n)$ is a bounded sequence in $X$. Define

$$r\left(x, \left(x_n\right)\right) = \limsup_{n\to\infty} d\left(x, x_n\right), \quad (14)$$

for $x \in X$. The asymptotic radius $r((x_n))$ of $(x_n)$ is given by

$$r\left(\left(x_n\right)\right) = \inf\left\{r\left(x, \left(x_n\right)\right) : x \in X\right\}. \quad (15)$$

The asymptotic center $A((x_n))$ of $(x_n)$ is defined as follows:

$$A((x_n)) = \{x \in X : r(x, x_n) = r((x_n))\}. \quad (16)$$

We denote by $\mathscr{K}$ the collection of all bounded closed convex subsets of a Hadamard space $(X, d)$.

Asymptotic center is exactly one point in a $CAT(0)$ space (see, e.g., [24]). Furthermore, the distance function is convex in complete $CAT(0)$ spaces (see, e.g., [23]).

Notice that if $x, y_1, y_2$ are points of a $CAT(0)$ space and if $y_0$ is the midpoint of the segment $[y_1, y_2]$, which we will denote by $(y_1 \oplus y_2)/2$, then the $CAT(0)$ inequality implies that

$$d\left(x, \frac{y_1 \oplus y_2}{2}\right)^2 \le \frac{1}{2}d(x, y_1)^2 + \frac{1}{2}d(x, y_2)^2 - \frac{1}{4}d(y_1, y_2)^2, \quad (17)$$

because equality holds in the Euclidean metric. Here, (17) is known as the $CN$ inequality; see Bruhat and Tits [25].

Finally, we note that a geodesic metric space is a $CAT(0)$ space if and only if it satisfies inequality (17) (see, e.g., [23]).

**Lemma 2** (see [8]). *Let $(X, d)$ be a $CAT(0)$ space. Then, the following inequality,*

$$d^2((1-t)x \oplus ty, z) \le (1-t)d^2(x, z) + td^2(y, z) - t(1-t)d^2(x, y), \quad (18)$$

*is satisfied for all $x, y, z \in X$ and $t \in [0, 1]$.*

*Definition 3* (see [14]). A sequence $(x_n)$ in $X$ is said to $\Delta$-converge to $x \in X$ if $x$ is the unique asymptotic center of $(u_n)$ for every subsequence $(u_n)$ of $(x_n)$. In this case, one writes $\Delta - \lim_n x_n = x$ and call $x$ the $\Delta -$ Limit of $(x_n)$.

**Lemma 4** (see [8]). *Assume that $(X, d)$ is a Hadamard space and $K \in \mathscr{K}$. If $\{x_n\}$ is a bounded sequence in $K$, then the asymptotic center of $\{x_n\}$ is in $K$.*

**Lemma 5** (see [8]). *Assume that $(X, d)$ is a Hadamard space. Each bounded sequence $\{x_n\}$ in $X$ has a $\Delta$-convergent subsequence.*

$CAT(0)$ space has the *Opial* property; that is, for a given $(x_n) \subset X$, we have

$$\limsup_n d(x_n, x) < \limsup_n d(x_n, y), \quad (19)$$

where $(x_n)$ $\Delta$-converges to $x$ and given $y \in X$ with $y \ne x$. Moreover, these metric spaces offer a nice example of uniformly convex metric spaces.

## 3. Convergent Theorems

In this section, we first recollect some elementary definitions and basic results on the topic in the framework of $CAT(0)$ spaces.

*Definition 6* (see [6]). Let $X$ be a $CAT(0)$ space and $K$ be a subset of $X$. A self-mapping $T$ on a subset $K$ is called total asymptotically nonexpansive if there are nonnegative real sequences $\{k_n^{(1)}\}$ and $\{k_n^{(2)}\}$, $n \ge 1$, with $k_n^{(1)}$, $k_n^{(2)} \to 0$ as $n \to \infty$, and strictly increasing and continuous function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\psi(0) = 0$ such that

$$d(T^n x, T^n y) \le d(x, y) + k_n^{(1)}\psi(d(x, y)) + k_n^{(2)}. \quad (20)$$

*Remark 7.* If $\psi(\lambda) = \lambda$, then inequality (20) turns into

$$d(T^n x, T^n y) \le \left(1 + k_n^{(1)}\right)d(x, y) + k_n^{(2)}, \quad (21)$$

which is nearly asymptotically nonexpansive (6).

In addition, if $k_n^{(2)} = 0$ for all $n \ge 1$, then total asymptotically nonexpansive mappings coincide with asymptotically nonexpansive mappings. In the case $k_n^{(2)} = 0$, a self-mapping $T$ is uniformly continuous. Notice that a self-mapping $T$ can be uniformly continuous even if $k_n^{(2)} \ne 0$. If $k_n^{(1)} = 0$ and $k_n^{(2)} = 0$ for all $n \ge 1$, then we obtain the class of nonexpansive mappings from (20).

*Definition 8* (see [18, 26]). Assume that $(X, d)$ is a Hadamard space and $K \in \mathscr{K}$. A self-mapping $I - T$ on $K$ is called demiclosed at zero, if, for each sequence $(x_n) \subset K$ that $\Delta$-converges to a point $x_0$ in $K$, $I - T(x_n) \to 0$, implies that $I - T(x_0) \to 0$ or let one formally say that $I - T$ is demiclosed at zero if the conditions $\{x_n\}$, $\Delta$-converges to $x$ and $d(x_n, Tx_n) \to 0$, imply $x \in F(T)$.

**Lemma 9.** *Assume that $(X, d)$ is a Hadamard space and $K \in \mathscr{K}$. For a bounded sequence $(x_n)$ in $K$, one sets $\phi(u) = \limsup_{n\to\infty} d(x_n, u)$. Then there is a point $x_0 \in K$ such that*

$$\phi(x_0) = \inf\{\phi(x) : x \in K\}. \quad (22)$$

*Proof.* It is immediate consequence of existence of the asymptotic center and Lemma 4. □

**Lemma 10.** *Assume that $(X, d)$ is a Hadamard space and $K \in \mathscr{K}$. Suppose that a self-mapping $T$ on $K$ is a total asymptotically nonexpansive mapping. For a point $x$ in $K$, let $x_n = T^n(x)$. Then $\lim_{m\to\infty} \phi(T^m(w)) = \phi(w)$, where $w \in K$ is such that $\phi(w) = \inf\{\phi(u) : u \in K\}$ for the same $\phi$ in Lemma 9.*

*Proof.* Since $T$ is total asymptotically nonexpansive, we have

$$d(T^{n+m}(x), T^m(w)) \le d(T^n(x), w) + k_m^{(1)}\psi(d(T^n(x), w)) + k_m^{(2)}, \quad (23)$$

for any $n, m \ge 1$. If we let $n$ go to infinity, we get

$$\phi(T^m(w)) \le \phi(w) + k_m^{(1)}\psi(\phi(w)) + k_m^{(2)}. \quad (24)$$

Let $m$ go to infinity, which implies that $\lim_{m\to\infty} \phi(T^m(w)) = \phi(w)$. □

**Theorem 11.** *Assume that $(X, d)$ is a Hadamard space and $K \in \mathscr{K}$. If $T : K \to K$ is a uniformly continuous total asymptotically nonexpansive mapping, then $T$ has a fixed point. Moreover, the fixed point set $F(T)$ is closed and convex.*

*Proof.* Define $\phi(u) = \limsup_{n\to\infty} d(T^n(x), u)$, for each $u \in K$. Let $w \in K$ such that $\phi(w) = \inf\{\phi(u) : u \in K\}$. We have seen that $\phi(T^m(w)) = \phi(w)$ as $m \to \infty$. The $CN$ inequality implies the following:

$$\begin{aligned} d&\left(T^n(x), \frac{T^m(w) \oplus T^h(w)}{2}\right)^2 \\ &\le \frac{1}{2}d(T^n(x), T^m(w))^2 + \frac{1}{2}d\left(T^n(x), T^h(w)\right)^2 \\ &\quad - \frac{1}{4}d\left(T^m(w), T^h(w)\right)^2. \end{aligned} \tag{25}$$

If we let $n$ go to infinity, we get

$$\begin{aligned} \phi(w)^2 &\le \phi\left(\frac{T^m(w) \oplus T^h(w)}{2}\right)^2 \\ &\le \frac{1}{2}\phi(T^m(w))^2 + \frac{1}{2}\phi\left(T^h(w)\right)^2 \\ &\quad - \frac{1}{4}d\left(T^m(w), T^h(w)\right)^2, \end{aligned} \tag{26}$$

which implies that

$$\limsup_{m,h\to\infty} d\left(T^m w, T^h w\right)^2 \le 0. \tag{27}$$

Therefore, $(T^n(w))$ is a Cauchy sequence in $K$ and hence converges to some $v \in K$; that is, $v = \lim_{n\to\infty} T^n(w)$. Since $T$ is continuous, then

$$Tv = T\left(\lim_{n\to\infty} T^n v\right) = \lim_{n\to\infty} T^{n+1} v = v, \tag{28}$$

and this proves that $F(T) \neq \emptyset$. Again, since $T$ is continuous, $F(T)$ is closed. In order to prove that $F(T)$ is convex, it is enough to prove that $(x \oplus y)/2 \in F(T)$, whenever $x, y \in F(T)$. Indeed, set $w = (x \oplus y)/2$. The $CN$ inequality implies that

$$d(T^n w, w)^2 \le \frac{1}{2}d(x, T^n w)^2 + \frac{1}{2}d(y, T^n w)^2 - \frac{1}{4}d(x, y)^2, \tag{29}$$

for any $n \ge 1$. Since

$$\begin{aligned} d(x, T^n w)^2 &= d(T^n x, T^n w)^2 \\ &\le \left(d(x, w) + k_n^{(1)}\psi(d(x, w)) + k_n^{(2)}\right)^2 \\ &= \left(\frac{1}{2}d(x, y) + k_n^{(1)}\psi\left(\frac{1}{2}d(x, y)\right) + k_n^{(2)}\right)^2, \\ d(y, T^n w)^2 &= d(T^n y, T^n w)^2 \\ &\le \left(d(y, w) + k_n^{(1)}\psi(d(y, w)) + k_n^{(2)}\right)^2 \\ &= \left(\frac{1}{2}d(x, y) + k_n^{(1)}\psi\left(\frac{1}{2}d(x, y)\right) + k_n^{(2)}\right)^2, \end{aligned} \tag{30}$$

when $n$ go to infinity, $\lim_{n\to\infty} T^n(w) = w$, which implies $Tw = w$. □

It is known that the demiclosed principle plays an important role in studying the asymptotic behavior for nonexpansive mappings (see [12, 27–30]). In [29], Xu proved the demiclosed principle for asymptotically nonexpansive mappings in the setting of a uniformly convex Banach space. Nanjaras and Panyanak [12] extended Xu's result to $CAT(0)$ spaces. A demiclosed principle for asymptotically nonexpansive mappings in the intermediate sense on a real uniformly convex Banach space was proved by Yanga et al. [30]. Motivated by them we will establish demiclosed principle and existence theorem for total asymptotically nonexpansive mappings in the context of $CAT(0)$ spaces. Also the next theorem shows that the result of Theorem 11 holds without the boundness condition imposed on $K$, provided that there exists a bounded approximate fixed point sequence $\{x_n\}$; that is, $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

**Theorem 12.** *Assume that $(X, d)$ is a Hadamard space and $K \in \mathscr{K}$. Suppose that $T : K \to K$ is a uniformly continuous total asymptotically nonexpansive mapping. Let $(x_n) \in K$ be a bounded approximate fixed point sequence. If $\Delta - \lim_n x_n = w$, then we have $Tw = w$.*

*Proof.* Since $(x_n)$ is an approximate fixed point sequence, then we have

$$\phi(x) = \limsup_{n\to\infty} d(T^m x_n, x) = \limsup_{n\to\infty} d(x_n, x), \tag{31}$$

for any $m \ge 1$. Hence, $\phi(T^m x) \le \phi(x) + k_m^{(1)}\psi(\phi(x)) + k_m^{(2)}$, for each $x \in K$. In particular, we have $\lim_{m\to\infty} \phi(T^m(w)) = \phi(w)$. The $CN$ inequality implies that

$$\begin{aligned} d&\left(x_n, \frac{w \oplus T^m(w)}{2}\right)^2 \\ &\le \frac{1}{2}d(x_n, w)^2 + \frac{1}{2}d(x_n, T^m(w))^2 - \frac{1}{4}d(w, T^m(w))^2, \end{aligned} \tag{32}$$

for any $n, m \ge 1$. If we let $n \to \infty$, we will get

$$\begin{aligned} \phi&\left(\frac{w \oplus T^m(w)}{2}\right)^2 \\ &\le \frac{1}{2}\phi(w)^2 + \frac{1}{2}\phi(T^m(w))^2 - \frac{1}{4}d(w, T^m(w))^2, \end{aligned} \tag{33}$$

for any $m \ge 1$. The definition of $w$ implies that

$$\phi(w)^2 \le \frac{1}{2}\phi(w)^2 + \frac{1}{2}\phi(T^m(w))^2 - \frac{1}{4}d(w, T^m(w))^2, \tag{34}$$

for any $m \ge 1$, or

$$d(w, T^m(w))^2 \le 2\phi(T^m w)^2 - 2\phi(w)^2. \tag{35}$$

Letting $m \to \infty$, we will get $\lim_{m\to\infty} d(w, T^m w) = 0$. By the continuity of $T$,

$$Tw = T\left(\lim_{m\to\infty} T^m w\right) = \lim_{m\to\infty} T^{m+1} w = w. \tag{36}$$

□

Consequently, we derive the following corollaries which can be found in [20].

**Corollary 13.** *Assume that $(X, d)$ is a Hadamard space and $K \in \mathscr{K}$. Suppose that $T : K \to K$ is a uniformly continuous nearly asymptotically nonexpansive mapping. If $\{y_n\}$ is a bounded sequence in $K$ such that $\lim_n d(y_n, Ty_n) = 0$, then $T$ has a fixed point.*

*Proof.* Every bounded sequence $\{y_n\}$ in $K$ has a $\Delta$-convergent subsequence, by Lemma 5, which can be showed again by $\{y_n\}$. Now apply Theorem 12. □

**Corollary 14.** *Assume that $(X, d)$ is a Hadamard space and $K \in \mathscr{K}$. Suppose that $T : K \to K$ is a uniformly continuous nearly asymptotically nonexpansive mapping. If $\{x_n\}$ is a bounded sequence in $K$ which $\Delta$-converges to $x$ and $\lim_n d(x_n, Tx_n) = 0$, then $x \in K$ and $x = Tx$.*

## 4. Approximation

Assume that $(X, d)$ is a Hadamard space and $K \in \mathscr{K}$. Suppose that a self-mapping $T : K \to K$ is total asymptotically nonexpansive. Consider the following iteration process, namely, modified Ishikawa iteration scheme:

$$x_{n+1} = (1 - \alpha_n) x_n \oplus \alpha_n T^n y_n, \qquad y_n = (1 - \beta_n) x_n \oplus \beta_n T^n x_n, \tag{37}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0, 1)$ for each $n \geq 1$.

Note that the modified Ishikawa iterative process coincides with the following modified Mann iterative process if $\beta_n = 0$ for each $n \geq 1$ then

$$x_{n+1} = (1 - \alpha_n) x_n \oplus \alpha_n T^n x_n. \tag{38}$$

In this section we want to show that $\{x_n\}$ is an approximate fixed point sequence. Due to this, we use the following lemma which can be found in [31].

**Lemma 15.** *Let $\{\lambda_n\}_{n\geq1}$, $\{\kappa_n\}_{n\geq1}$, and $\{\gamma_n\}_{n\geq1}$ be sequences of nonnegative real numbers such that, for all $n \geq 1$,*

$$\lambda_{n+1} \leq (1 + \kappa_n) \lambda_n + \gamma_n. \tag{39}$$

*Let $\sum_1^\infty \kappa_n < \infty$ and $\sum_1^\infty \gamma_n < \infty$. Then $\lim_n \lambda_n$ exists.*

**Lemma 16.** *Assume that $(X, d)$ is a Hadamard space and $K \in \mathscr{K}$. Suppose that a self-mapping $T : K \to K$ is a uniformly continuous total asymptotically nonexpansive with $F(T) \neq \emptyset$. Suppose also that there exist constants $M_0$, $M \geq 0$ such that $\psi(\lambda) \leq M_0 \lambda$ for all $\lambda \geq M$. Let $x^* \in \mathrm{Fix}(T)$. Starting from arbitrary $x_1 \in K$ define the sequence $\{x_n\}$ by (37). Suppose that $\sum_1^\infty k_n^{(1)} < \infty$ and $\sum_1^\infty k_n^{(2)} < \infty$. Then $\lim_n d(x_n, x^*)$ exists and $\sum_1^\infty \alpha_n \beta_n (1 - \beta_n) d^2(x_n, T^n x_n) < \infty$.*

*Proof.* Let $x^* \in \mathrm{Fix}(T)$; then

$$\begin{aligned} d(y_n, x^*) &= d((1 - \beta_n) x_n \oplus \beta_n T^n x_n, x^*) \\ &\leq (1 - \beta_n) d(x_n, x^*) + \beta_n d(T^n x_n, T^n x^*) \\ &\leq d(x_n, x^*) + \beta_n k_n^{(1)} \psi(d(x_n, x^*)) + \beta_n k_n^{(2)}. \end{aligned} \tag{40}$$

Since $\psi$ is increasing function, it results that $\psi(\lambda) \leq \psi(M)$ if $\lambda \leq M$ and $\psi(\lambda) \leq M_0 \lambda$ if $\lambda \geq M$. In either case we obtain

$$\psi(d(x_n, x^*)) \leq \psi(M) + M_0 d(x_n, x^*) \tag{41}$$

for each $n \geq 1$. Therefore,

$$\begin{aligned} d(y_n, x^*) \leq{}& d(x_n, x^*) + \beta_n k_n^{(1)} [\psi(M) + M_0 d(x_n, x^*)] \\ &+ \beta_n k_n^{(2)}. \end{aligned} \tag{42}$$

Thus,

$$\begin{aligned} &\psi(d(y_n, x^*)) \\ &\quad\leq \psi(M) + \left(M_0 + \beta_n k_n^{(1)} (M_0)^2\right) d(x_n, x^*) \\ &\qquad + M_0 \beta_n k_n^{(1)} \psi(M) + M_0 \beta_n k_n^{(2)}, \end{aligned} \tag{43}$$

so one can write

$$\begin{aligned} &d(x_{n+1}, x^*) \\ &\quad= d((1 - \alpha_n) x_n \oplus \alpha_n T^n y_n, x^*) \\ &\quad\leq (1 - \alpha_n) d(x_n, x^*) + \alpha_n d(T^n y_n, T^n x^*) \\ &\quad\leq (1 - \alpha_n) d(x_n, x^*) + \alpha_n d(y_n, x^*) \\ &\qquad + \alpha_n k_n^{(1)} \psi(d(y_n, x^*)) + \alpha_n k_n^{(2)} \\ &\quad\leq \left[1 + \alpha_n \beta_n k_n^{(1)} M_0 + \alpha_n k_n^{(1)} M_0 + \alpha_n \beta_n \left(k_n^{(1)}\right)^2 (M_0)^2\right] \\ &\qquad\times d(x_n, x^*) + \alpha_n \beta_n k_n^{(1)} \psi(M) \\ &\qquad + \alpha_n \beta_n k_n^{(2)} + \alpha_n k_n^{(2)} + \alpha_n k_n^{(1)} \psi(M) \\ &\qquad + \alpha_n \beta_n \left(k_n^{(1)}\right)^2 M_0 \psi(M) + \alpha_n \beta_n k_n^{(1)} k_n^{(2)} M_0. \end{aligned} \tag{44}$$

Thus, we get the following inequality:

$$d(x_{n+1}, x^*) \leq \left(1 + A k_n^{(1)}\right) d(x_n, x^*) + B k_n^{(1)} + C k_n^{(2)}. \tag{45}$$

For some $A, B, C \geq 0$, since $\sum_1^\infty k_n^{(1)} < \infty$ and $\sum_1^\infty k_n^{(2)} < \infty$, $k_n^{(1)}, k_n^{(2)}$ are bounded, due to Lemma 15 the sequence

$d(x_n, x^*)$ has a limit and so it is bounded. By Lemma 2, we have

$$
\begin{aligned}
&d^2(x_{n+1}, x^*) \\
&\quad = d^2((1-\alpha_n)x_n \oplus \alpha_n T^n y_n, x^*) \\
&\quad \le (1-\alpha_n) d^2(x_n, x^*) + \alpha_n d^2(T^n y_n, x^*) \\
&\qquad - \alpha_n(1-\alpha_n) d^2(x_n, T^n y_n) \\
&\quad \le (1-\alpha_n) d^2(x_n, x^*) + \alpha_n d^2(T^n y_n, x^*) \\
&\quad \le (1-\alpha_n) d^2(x_n, x^*) \\
&\qquad + \alpha_n \left[ d(y_n, x^*) + k_n^{(1)} \psi(d(y_n, x^*)) + k_n^{(2)} \right]^2 \\
&\quad \le (1-\alpha_n) d^2(x_n, x^*) \\
&\qquad + \alpha_n \Big[ d^2(y_n, x^*) \\
&\qquad\qquad + k_n^{(1)2} \{\psi(M) + M_0 d(y_n, x^*)\}^2 \\
&\qquad\qquad + k_n^{(2)2} + 2k_n^{(1)} d(y_n, x^*) \\
&\qquad\qquad \times \{\psi(M) + M_0 d(y_n, x^*)\} + 2k_n^{(2)} d(y_n, x^*) \\
&\qquad\qquad + 2k_n^{(2)} \{\psi(M) + M_0 d(y_n, x^*)\} \Big] \\
&\quad = (1-\alpha_n) d^2(x_n, x^*) + \alpha_n \left(1 + M_0 k_n^{(1)}\right)^2 d^2(y_n, x^*) \\
&\qquad + \alpha_n \Big( 2k_n^{(1)2} M_0 \psi(M) + 2k_n^{(1)} \psi(M) \\
&\qquad\qquad + 2k_n^{(2)} + 2k_n^{(1)} k_n^{(2)} M_0 \Big) d(y_n, x^*) \\
&\qquad + \alpha_n \left( k_n^{(1)} \psi(M) + k_n^{(2)} \right)^2.
\end{aligned} \tag{46}
$$

Since $\lim_n d(x_n, x^*)$ exists, $\{x_n\}$ is bounded and it follows from (42) that $\{y_n\}$ is also bounded. Then, there exist constants $A, B \ge 0$ such that

$$
\begin{aligned}
&d^2(x_{n+1}, x^*) \\
&\quad \le (1-\alpha_n) d^2(x_n, x^*) \\
&\qquad + \alpha_n \left(1 + M_0 k_n^{(1)}\right)^2 d^2(y_n, x^*) + A k_n^{(1)} + B k_n^{(2)} \\
&\quad \le (1-\alpha_n) d^2(x_n, x^*) \\
&\qquad + \alpha_n \left(1 + M_0 k_n^{(1)}\right)^2 \\
&\qquad \times \Big[ (1-\beta_n) d^2(x_n, x^*) \\
&\qquad\qquad + \beta_n d^2(T^n x_n, x^*) - \beta_n(1-\beta_n) d^2(x_n, T^n x_n) \Big] \\
&\qquad + A k_n^{(1)} + B k_n^{(2)} \\
&\quad \le (1-\alpha_n) d^2(x_n, x^*) \\
&\qquad + \alpha_n \left(1 + M_0 k_n^{(1)}\right)^2 \\
&\qquad \times \Big[ (1-\beta_n) d^2(x_n, x^*) \\
&\qquad\qquad + \beta_n \left[ d(x_n, x^*) + k_n^{(1)} \psi(d(x_n, x^*)) + k_n^{(2)} \right]^2 \\
&\qquad\qquad - \beta_n(1-\beta_n) d^2(x_n, T^n x_n) \Big] \\
&\qquad + A k_n^{(1)} + B k_n^{(2)} \\
&\quad \le (1-\alpha_n) d^2(x_n, x^*) + \alpha_n \left(1 + M_0 k_n^{(1)}\right)^2 \\
&\qquad \times \Big[ (1-\beta_n) d^2(x_n, x^*) \\
&\qquad\qquad + \beta_n \Big\{ d^2(x_n, x^*) \\
&\qquad\qquad\qquad + \left(k_n^{(1)}\right)^2 [\psi(M) + M_0 d(x_n, x^*)]^2 + \left(k_n^{(2)}\right)^2 \\
&\qquad\qquad\qquad + 2k_n^{(1)} k_n^{(2)} [\psi(M) + M_0 d(x_n, x^*)] \\
&\qquad\qquad\qquad + 2k_n^{(1)} d(x_n, x^*) [\psi(M) + M_0 d(x_n, x^*)] \\
&\qquad\qquad\qquad + 2k_n^{(2)} d(x_n, x^*) \Big\} \\
&\qquad\qquad - \beta_n(1-\beta_n) d^2(x_n, T^n x_n) \Big] + A k_n^{(1)} + B k_n^{(2)} \\
&\quad \le (1-\alpha_n) d^2(x_n, x^*) + \alpha_n \left(1 + M_0 k_n^{(1)}\right)^2 \\
&\qquad \times \Big[ (1-\beta_n) d^2(x_n, x^*) \\
&\qquad\qquad + \beta_n \left\{ \left(1 + M_0 k_n^{(1)}\right)^2 d^2(x_n, x^*) + C k_n^{(1)} + D k_n^{(2)} \right\} \\
&\qquad\qquad - \beta_n(1-\beta_n) d^2(x_n, T^n x_n) \Big] + A k_n^{(1)} + B k_n^{(2)} \\
&\quad \le (1-\alpha_n) d^2(x_n, x^*) + \alpha_n \left(1 + M_0 k_n^{(1)}\right)^2 \\
&\qquad \times \Big[ (1-\beta_n) \left(1 + M_0 k_n^{(1)}\right)^2 d^2(x_n, x^*) \\
&\qquad\qquad + \beta_n \left(1 + M_0 k_n^{(1)}\right)^2 d^2(x_n, x^*) \\
&\qquad\qquad + C k_n^{(1)} + D k_n^{(2)} - \beta_n(1-\beta_n) d^2(x_n, T^n x_n) \Big] \\
&\qquad + A k_n^{(1)} + B k_n^{(2)} \\
&\quad \le (1-\alpha_n) d^2(x_n, x^*) \\
&\qquad + \alpha_n \left(1 + M_0 k_n^{(1)}\right)^4 d^2(x_n, x^*) + E k_n^{(1)} + F k_n^{(2)} \\
&\qquad - \alpha_n \beta_n (1-\beta_n) \left(1 + M_0 k_n^{(1)}\right)^2 d^2(x_n, T^n x_n) \\
&\quad \le \left(1 + M_0 k_n^{(1)}\right)^4 d^2(x_n, x^*) + E k_n^{(1)} \\
&\qquad + F k_n^{(2)} - \alpha_n \beta_n (1-\beta_n) d^2(x_n, T^n x_n) \\
&\quad \le \left(1 + K M_0 k_n^{(1)}\right) d^2(x_n, x^*) + E k_n^{(1)} + F k_n^{(2)}
\end{aligned}
$$

$$- \alpha_n \beta_n (1-\beta_n) d^2 (x_n, T^n x_n)$$
$$\leq d^2 (x_n, x^*) + G k_n^{(1)} + F k_n^{(2)}$$
$$- \alpha_n \beta_n (1-\beta_n) d^2 (x_n, T^n x_n), \tag{47}$$

for some $C, D, E, F, G \geq 0$. Thus,

$$\alpha_n \beta_n (1-\beta_n) d^2 (x_n, T^n x_n) \leq d^2 (x_n, x^*) - d^2 (x_{n+1}, x^*) + G k_n^{(1)} + F k_n^{(2)}, \tag{48}$$

which implies that

$$\sum_{n=1}^{m} \alpha_n \beta_n (1-\beta_n) d^2 (x_n, T^n x_n) \leq \sum_{n=1}^{m} \left[ d^2 (x_n, x^*) - d^2 (x_{n+1}, x^*) \right] + \sum_{n=1}^{m} \left( E k_n^{(1)} + F k_n^{(2)} \right). \tag{49}$$

Since $\sum_{n=1}^{\infty} (E k_n^{(1)} + F k_n^{(2)}) < \infty$ and $\lim_n d(x_n, x^*)$ exists, therefore $\sum_{n=1}^{\infty} \alpha_n \beta_n (1-\beta_n) d^2 (x_n, T^n x_n) < \infty$. □

Note that if the domain of $T$ is bounded, we can omit the conditions of existence of constants $M_0, M \geq 0$ such that $\psi(\lambda) \leq M_0 \lambda$ for all $\lambda \geq M$ and $F(T) \neq \emptyset$.

**Theorem 17.** *Assume that $(X, d)$ is a Hadamard space and $K \in \mathcal{K}$. Assume that $T : K \to K$ is a uniformly continuous total asymptotically nonexpansive mapping with $F(T) \neq \emptyset$ and there exist constants $M_0, M \geq 0$ such that $\psi(\lambda) \leq M_0 \lambda$ for all $\lambda \geq M$. Let $x^* \in \mathrm{Fix}(T)$. Starting from arbitrary $x_1 \in K$ define the sequence $\{x_n\}$ by (37), where $\{\alpha_n\}, \{\beta_n\}$ are sequences in $(0,1)$ such that $\lim_{n\to\infty} \alpha_n \beta_n (1-\beta_n) \neq 0$. Suppose that $\sum_1^{\infty} k_n^{(1)} < \infty$ and $\sum_1^{\infty} k_n^{(2)} < \infty$. Then $\{x_n\}$ is $\Delta$-convergent to a fixed point of $T$.*

*Proof.* Since $\lim_{n\to\infty} \alpha_n \beta_n (1-\beta_n) \neq 0$, by Lemma 16, $\lim_{n\to\infty} d(x_n, T^n x_n) = 0$, so by uniform continuity of $T$, $\lim_{n\to\infty} d(T x_n, T^{n+1} x_n) = 0$. Therefore, one can write

$$d(x_{n+1}, x_n)$$
$$= d((1-\alpha_n) x_n \oplus \alpha_n T^n y_n, x_n)$$
$$\leq d(x_n, T^n y_n)$$
$$\leq d(x_n, T^n x_n) + d(T^n x_n, T^n y_n)$$
$$\leq d(x_n, T^n x_n) + d(x_n, y_n)$$
$$+ k_n^{(1)} \psi(d(x_n, y_n)) + k_n^{(2)}$$
$$\leq d(x_n, T^n x_n) + d(x_n, (1-\beta_n) x_n \oplus \beta_n T^n x_n)$$
$$+ k_n^{(1)} \psi(d(x_n, y_n)) + k_n^{(2)}$$
$$\leq (1+\beta_n) d(x_n, T^n x_n) + k_n^{(1)} \psi(d(x_n, y_n))$$
$$+ k_n^{(2)} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{50}$$

Also

$$d(x_n, T x_n)$$
$$\leq d(x_n, x_{n+1}) + d\left(x_{n+1}, T^{n+1} x_{n+1}\right)$$
$$+ d\left(T^{n+1} x_{n+1}, T^{n+1} x_n\right) + d\left(T^{n+1} x_n, T x_n\right) \tag{51}$$
$$\leq 2d(x_n, x_{n+1}) + d\left(x_{n+1}, T^{n+1} x_{n+1}\right)$$
$$+ k_{n+1}^{(1)} \psi(d(x_n, x_{n+1})) + k_n^{(2)} + d\left(T^{n+1} x_n, T x_n\right),$$

and hence

$$\lim_{n\to\infty} d(x_n, T x_n) = 0. \tag{52}$$

Set $w_w(x_n) := \bigcup A(\{u_n\})$, where the union is taken over by all subsequences $\{u_n\}$ of $\{x_n\}$. We assert that $w_w(x_n) \subset F(T)$. Let $u \in w_w(x_n)$; then there is a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemmas 4 and 5 there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta - \lim_n v_n = v \in K$. We have seen that $\lim_n d(T v_n, v_n) = 0$, so $v \in F(T)$ by Theorem 12 and $\lim_n d(x_n, v)$ exists by Lemma 16. We will show that $u = v$. Suppose, on the contrary, that $u \neq v$. By the uniqueness of asymptotic centers,

$$\limsup_n d(v_n, v) < \limsup_n d(v_n, u)$$
$$\leq \limsup_n d(u_n, u)$$
$$< \limsup_n d(u_n, v) \tag{53}$$
$$= \limsup_n d(x_n, v)$$
$$= \limsup_n d(v_n, v),$$

which is a contradiction. Hence, we get that $u = v \in F(T)$. To show that $\{x_n\}$ $\Delta$-converges to a fixed point of $T$, it suffices to show that $w_w(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$. By Lemmas 4 and 5 there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta - \lim_n v_n = v \in K$. Let $A(\{u_n\}) = \{u\}$ and $A(\{x_n\}) = \{x\}$. We have seen that $u = v$ and $v \in F(T)$. It is sufficient to show that $x = v$ to finalize the proof. Suppose, on the contrary, $x$ is not equal to $y$. Since

$\{d(x_n, v)\}$ is convergent, then by the uniqueness of asymptotic centers,

$$\begin{aligned}\limsup_n d(v_n, v) &< \limsup_n d(v_n, x)\\ &\leq \limsup_n d(x_n, x)\\ &< \limsup_n d(x_n, v)\\ &= \limsup_n d(v_n, v),\end{aligned} \tag{54}$$

which is a contradiction, and hence the conclusion follows. □

**Corollary 18.** *Assume that $(X, d)$ is a Hadamard space and $K \in \mathcal{K}$. Suppose that $T : K \to K$ is a uniformly continuous total asymptotically nonexpansive mapping with $F(T) \neq \emptyset$ and there exist constants $M_0, M \geq 0$ such that $\psi(\lambda) \leq M_0\lambda$ for all $\lambda \geq M$. Let $x^* \in \mathrm{Fix}(T)$. Starting from arbitrary $x_1 \in K$ define the sequence $\{x_n\}$ by (37), where $\{\alpha_n\}, \{\beta_n\}$ are sequences in $(0, 1)$. Suppose that $\sum_1^\infty k_n^{(1)} < \infty$ and $\sum_1^\infty k_n^{(2)} < \infty$. Then the condition $d(T^n x_n, x_n) \to 0$ as $n \to \infty$ implies that*

$$\lim_n d(x_n, x_{n+1}) = 0, \qquad \lim_n d(x_n, Tx_n) = 0. \tag{55}$$

Note that, in the case $\beta_n = 0$, we can state Theorem 17 in the following manner.

**Lemma 19.** *Assume that $(X, d)$ is a Hadamard space and $K \in \mathcal{K}$. Assume that $T : K \to K$ is a uniformly continuous total asymptotically nonexpansive mapping with $F(T) \neq \emptyset$. Suppose also that there exist constants $M_0, M \geq 0$ such that $\psi(\lambda) \leq M_0\lambda$ for all $\lambda \geq M$. Let $x^* \in \mathrm{Fix}(T)$. Starting from arbitrary $x_1 \in K$ define the sequence $\{x_n\}$ by (38), where $\{\alpha_n\}$ is a sequence in $(0, 1)$. Suppose that $\sum_1^\infty k_n^{(1)} < \infty$ and $\sum_1^\infty k_n^{(2)} < \infty$. Then the condition $d(T^n x_n, x_n) \to 0$ as $n \to \infty$ implies that*

$$\lim_n d(x_n, x_{n+1}) = 0, \qquad \lim_n d(x_n, Tx_n) = 0. \tag{56}$$

*Proof.* We have from (38)

$$\begin{aligned}d(x_{n+1}, x_n) &= d((1-\alpha_n) x_n \oplus \alpha_n T^n x_n, x_n)\\ &\leq \alpha_n d(T^n x_n, x_n).\end{aligned} \tag{57}$$

Therefore, $\lim_n d(x_{n+1}, x_n) = 0$, and also

$$\begin{aligned}&d(x_n, Tx_n)\\ &\quad\leq d(x_n, x_{n+1}) + d\left(x_{n+1}, T^{n+1} x_{n+1}\right)\\ &\qquad + d\left(T^{n+1} x_{n+1}, T^{n+1} x_n\right) + d\left(T^{n+1} x_n, Tx_n\right)\\ &\quad\leq 2d(x_n, x_{n+1}) + k_{n+1}^{(1)} \psi(d(x_n, x_{n+1})) + k_{n+1}^{(2)}\\ &\qquad + d\left(x_{n+1}, T^{n+1} x_{n+1}\right) + d\left(T^{n+1} x_n, Tx_n\right).\end{aligned} \tag{58}$$

Since $T$ is uniformly continuous, the hypotheses $d(T^n x_n, x_n) \to 0$ as $n \to \infty$ implies that

$$d\left(T^{n+1} x_n, Tx_n\right) \longrightarrow 0, \qquad d\left(x_{n+1}, T^{n+1} x_{n+1}\right) \longrightarrow 0. \tag{59}$$

□

**Proposition 20** (see [32, Lemma 2.9]). *Let $(X, d)$ be a complete CAT(0) space and let $x \in X$. Suppose that $\{t_n\}$ is a sequence in $[a, b]$ for some $a, b \in (0, 1)$ and $\{x_n\}, \{y_n\}$ are sequences in $X$ such that $\limsup_{n\to\infty} d(x_n, x) \leq r$, $\limsup_{n\to\infty} d(y_n, x) \leq r$, and $\lim_{n\to\infty} d((1 - t_n)x_n \oplus t_n y_n, x) = r$ for some $r \geq 0$. Then*

$$\lim_{n\to\infty} d(x_n, y_n) = 0. \tag{60}$$

**Theorem 21.** *Assume that $(X, d)$ is a Hadamard space and $K \in \mathcal{K}$. Suppose that a self-mapping $T : K \to K$ is uniformly continuous total asymptotically nonexpansive mapping with $F(T) \neq \emptyset$ and suppose that there exist constants $M_0, M \geq 0$ such that $\psi(\lambda) \leq M_0\lambda$ for all $\lambda \geq M$. Let $x^* \in \mathrm{Fix}(T)$, and $\{\alpha_n\}_{n\geq1}$ is a sequence in $(0, 1)$ for all $n \geq 1$. Starting from arbitrary $x_1 \in K$ define the sequence $\{x_n\}$ by (38). Suppose that $\sum_1^\infty k_n^{(1)} < \infty$ and $\sum_1^\infty k_n^{(2)} < \infty$. Then $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.*

*Proof.* First we show that $d(x_n, x^*)$ for $x^* \in F(T)$ is bounded and it has a limit

$$\begin{aligned}&d(x_{n+1}, x^*)\\ &\quad\leq d((1-\alpha_n) x_n \oplus T^n x_n, x^*)\\ &\quad\leq (1-\alpha_n) d(x_n, x^*) + \alpha_n d(T^n x_n, T^n x^*)\\ &\quad\leq d(x_n, x^*) + \alpha_n k_n^{(1)} \psi(d(x_n, x^*)) + \alpha_n k_n^{(2)}.\end{aligned} \tag{61}$$

Since $\psi$ is increasing function, it results that $\psi(\lambda) \leq \psi(M)$ if $\lambda \leq M$ and $\psi(\lambda) \leq M_0\lambda$ if $\lambda \geq M$. In either case we obtain

$$\psi(d(x_n, x^*)) \leq \psi(M) + M_0 d(x_n, x^*) \tag{62}$$

for each $n \geq 1$. Thus, we get the following inequality:

$$\begin{aligned}d(x_{n+1}, x^*) \leq{}& \left(1 + M_0\alpha_n k_n^{(1)}\right) d(x_n, x^*)\\ &+ \alpha_n k_n^{(1)} \psi(M) + \alpha_n k_n^{(2)}.\end{aligned} \tag{63}$$

However, $\sum_{n=1}^\infty k_n^{(1)} < \infty$ and $\sum_{n=1}^\infty k_n^{(2)} < \infty$; therefore, due to Lemma 15 the sequence $d(x_n, x^*)$ has a limit and it is bounded. Assume that $\lim_{n\to\infty} d(x_n, x^*) = c$. Since

$$\begin{aligned}d(T^n x_n, x^*) &= d(T^n x_n, T^n x^*)\\ &\leq d(x_n, x^*) + k_n^{(1)} \psi(d(x_n, x^*)) + k_n^{(2)}\end{aligned} \tag{64}$$

for all $n \in \mathbb{N}$, then

$$\limsup_{n\to\infty} d(T^n x_n, x^*) \leq c. \tag{65}$$

Additionally, since

$$\begin{aligned}&d(x_{n+1}, x^*)\\ &\quad= d(\alpha_n T^n x_n \oplus (1-\alpha_n) x_n, x^*)\\ &\quad\leq \alpha_n d(T^n x_n, x^*) + (1-\alpha_n) d(x_n, x^*)\\ &\quad\leq \alpha_n d(T^n x_n, T^n x^*) + (1-\alpha_n) d(x_n, x^*)\\ &\quad\leq \alpha_n \left[d(x_n, x^*) + k_n^{(1)} \psi(d(x_n, x^*)) + k_n^{(2)}\right]\\ &\qquad + (1-\alpha_n) d(x_n, x^*)\\ &\quad\leq d(x_n, x^*) + \alpha_n \left[k_n^{(1)} \psi(d(x_n, x^*)) + k_n^{(2)}\right],\end{aligned} \tag{66}$$

then

$$d(x_{n+1}, x^*) = d(\alpha_n T^n x_n \oplus (1-\alpha_n) x_n, x^*) \leq d(x_n, x^*) + \alpha_n \left[k_n^{(1)} \psi(d(x_n, x^*)) + k_n^{(2)}\right]. \quad (67)$$

Hence,

$$\limsup_{n\to\infty} (d(\alpha_n T^n x_n \oplus (1-\alpha_n) x_n, x^*)) = c. \quad (68)$$

By Proposition 20, we have $\lim_{n\to\infty} d(T^n x_n, x_n) = 0$. By Lemma 19, $\lim_{n\to\infty} d(Tx_n, x_n) = 0$. This completes the proof. □

Recall that a mapping $T : C \to C$ is said to be semicompact if $C$ is closed and for any bounded sequence $\{x_n\} \subset C$ with $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, there exist $z \in C$ and $\{x_{n_j}\} \subset \{x_n\}$ satisfying $\lim_{j\to\infty} x_{n_j} = z$.

The next theorem extends corresponding results of Beg [33], Chang [34], and Osilike and Aniagbosor [22] for a more general class of non-Lipschitzian mappings in the framework of $CAT(0)$ spaces. It also extends corresponding results of Dhompongsa and Panyanak [8] from the class of nonexpansive mappings to a more general class of non-Lipschitzian mappings in the same space setting. Moreover, it extends corresponding results of Abbas et al. [20].

**Theorem 22.** *Assume that $(X, d)$ is a Hadamard space and $K \in \mathscr{K}$. Suppose that a self-mapping $T : K \to K$ is uniformly continuous total asymptotically nonexpansive mapping with $F(T) \neq \emptyset$; suppose that there exist constants $M_0, M \geq 0$ such that $\psi(\lambda) \leq M_0\lambda$ for all $\lambda \geq M$. Let $x^* \in \mathrm{Fix}(T)$. Starting from arbitrary $x_1 \in K$ define the sequence $\{x_n\}$ by (37), where $\{\alpha_n\}, \{\beta_n\}$ are sequences in $(0, 1)$ for all $n \geq 1$, such that $\lim_{n\to\infty} \alpha_n\beta_n(1-\beta_n) \neq 0$. Suppose that $\sum_1^\infty k_n^{(1)} < \infty$ and $\sum_1^\infty k_n^{(2)} < \infty$, and also suppose that $T^m$ is semicompact for some $m \in \mathbb{N}$. Then the sequence $\{x_n\}$ converges strongly to some fixed point of $T$.*

*Proof.* By Theorem 17, we have $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Since $T$ is uniform continuous, it follows the estimation

$$d(x_n, T^m x_n) \leq d(x_n, Tx_n) + d(Tx_n, T^2 x_n) + \cdots + d(T^{m-1} x_n, T^m x_n) \quad (69)$$

that $\lim_{n\to\infty} d(x_n, T^m x_n) = 0$. Since $T^m$ is semicompact, there exist a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and $x \in K$ with $\lim_{j\to\infty} x_{n_j} = \lim_{j\to\infty} T^m x_{n_j} = x$. Again since $T$ is uniformly continuous $\lim_{n\to\infty} d(Tx, Tx_{n_j}) = 0$ and it follows from the estimation,

$$d(Tx, x) \leq d(Tx, Tx_{n_j}) + d(Tx_{n_j}, x_{n_j}) + d(x_{n_j}, x) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \quad (70)$$

that $d(x, Tx) \to 0$; that is, $x \in F(T)$. By Lemma 16, the limit of $d(x_n, x) = c$ exists as $n \to \infty$. Since $\lim_{n\to\infty} x_{n_j} = x$, therefore $c = 0$. This accomplishes the proof. □

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

## Authors' Contribution

All authors contributed equally and significantly in writing this paper. All authors read and approved the final paper.

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