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Research Article

A New Biparametric Family of Two-Point Optimal Fourth-Order Multiple-Root Finders

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We construct a biparametric family of fourth-order iterative methods to compute multiple roots of nonlinear equations. This method is verified to be optimally convergent. Various nonlinear equations confirm our proposed method with order of convergence of four and show that the computed asymptotic error constant agrees with the theoretical one.

1. Introduction

It is not surprising that modified Newton's method [1] in the simple form

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)} \tag{1}$$

is most widely used to find the approximate multiple root α of known multiplicity m for a given nonlinear equation $f(x) = 0$. Recall that numerical scheme (1) is a one-point optimal method with quadratic convergence. In order to find numerical solution for multiple roots of nonlinear equations more accurately, many researchers have made enormous efforts in developing higher-order methods with improved convergence.

In this paper, we extend modified Newton's method and propose two-point optimal fourth-order multiple-root finders by evaluating two derivatives and one function per iteration. The optimality will be pursued based on Kung-Traub's conjecture [2] in which the convergence order of any multipoint method [3] without memory can reach at most 2^{k-1} for k evaluations of functions or derivatives.

The contents of this paper consist of what follows. Described in Section 2 are previous studies on multiple-root finders. Section 3 proposes a new biparametric family of two-point optimal fourth-order multiple-root finders. It fully treats method development and convergence analysis. Derivation of the error equations for the proposed schemes is an important task for ensuring convergence behavior. In Section 4, a variety of numerical examples are presented for a wide selection of test functions. It is important to compare the convergence behavior of the proposed schemes with that of existing methods. We confirm that the proposed methods well show the convergence behavior predicted by the developed theory.

2. Preliminary Review of Previous Studies

A number of interesting fourth-order multiple-root finders can be found in papers [4–16]. Among these, we especially introduce five studies as follows. Shengguo et al. [14] introduced the following fourth-order method which needs evaluations of one function and two derivatives per iteration for x_0 chosen in a neighborhood of the sought zero α of $f(x)$ with known multiplicity $m \ge 1$:

$$x_{n+1} = x_n - \frac{\beta f'(x_n) + \phi f'(y_n)}{f'(x_n) + \delta f'(y_n)}\frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots, \tag{2}$$

where $y_n = x_n - (2m/(m+2))(f(x_n)/f'(x_n))$, $\beta = -m^2/2$, $\phi = (1/2)(m(m-2)/(m/(m+2))^m)$, and $\delta = -(m/(m+2))^{-m}$.

J. R. Sharma and R. Sharma [12] constructed the following fourth-order scheme with $A = (1/8)m(m^3 - 4m + 8)$, $B = -(1/4)m(m-1)(m+2)^2(m/(m+2))^2$, $C = (1/8)m(m+2)^3(m/(m+2))^{2m}$, and $y_n = x_n - (2m/(m+2))(f(x_n)/f'(x_n))$:

$$x_{n+1} = x_n - A\frac{f(x_n)}{f'(x_n)} - B\frac{f(x_n)}{f'(y_n)} - C\left(\frac{f(x_n)}{f'(y_n)}\right)^2\left(\frac{f(x_n)}{f'(x_n)}\right)^{-1}, \quad n = 0, 1, 2, \dots. \tag{3}$$

Li et al. [15] presented the fourth-order method with $y_n = x_n - (2m/(m+2))(f(x_n)/f'(x_n))$:

$$x_{n+1} = x_n - \left(m - \frac{m^2}{2}\right)\frac{f(x_n)}{f'(x_n)} - \frac{mf(x_n)}{-f'(x_n) + (m/(m+2))^{-m}f'(y_n)}. \tag{4}$$

Zhou et al. [16] proposed the following fourth-order iterative scheme with $y_n = x_n - (2m/(m+2))(f(x_n)/f'(x_n))$:

$$x_{n+1} = x_n - \frac{m}{8}\left[m^3\left(\frac{m+2}{m}\right)^{2m}\left(\frac{f'(y_n)}{f'(x_n)}\right)^2 - 2m^2(m+3)\left(\frac{2+m}{m}\right)^m\frac{f'(y_n)}{f'(x_n)} + \left(m^3 + 6m^2 + 8m + 8\right)\right]\frac{f(x_n)}{f'(x_n)}. \tag{5}$$

Kanwar et al. [8] developed the fourth-order optimal multipoint iterative method for multiple zeros:

$$\begin{aligned} x_{n+1} &= x_n - \Big(mf(x_n)\Big(m^2(3m-2)\{f'(y_n)\}^2 - C_1f'(x_n)f'(y_n) + C_2\{f'(x_n)\}^2\Big)\Big) \\ &\times \Big(2f'(x_n)\Big((m-1)\left(\frac{m}{m+2}\right)^m f'(x_n) - mf'(y_n)\Big) \\ &\times \Big(mf'(y_n) - \left(\frac{m}{m+2}\right)^m (m+8)f'(x_n)\Big)\Big)^{-1}, \end{aligned} \tag{6}$$

where $y_n = x_n - (2m/(m+2))(f(x_n)/f'(x_n))$, $C_1 = m(m/(m+2))^m(6m^2 + 17m - 14)$, and $C_2 = (m/(m+2))^{2m}(3m^3 + 19m^2 + 16m + 16)$.

3. Method Development and Convergence Analysis

We first suppose that a function $f : \mathbb{C} \to \mathbb{C}$ has a multiple root α with integer multiplicity $m \geq 1$ and is analytic in a small neighborhood of α. Then a new iteration method free of second derivatives is proposed below to find an approximate root α of multiplicity m, given an initial guess x_0 sufficiently close to α:

$$x_{n+1} = y_n - a\frac{f(x_n)}{f'(x_n)} - c\frac{f(x_n)}{f'(y_n)} - d\frac{F(y_n)}{f'(y_n)}, \quad n = 0, 1, 2, \dots, \tag{7}$$

where

$$\begin{aligned} y_n &= x_n - \gamma\frac{f(x_n)}{f'(x_n)}, \\ F(y_n) &= f(x_n) - (y_n - x_n)\frac{\lambda f'(x_n)f'(y_n)}{f'(x_n) + \rho f'(y_n)}, \end{aligned} \tag{8}$$

with a, c, d, γ, λ, and ρ are parameters to be chosen for maximal order of convergence [17, 18]. We establish a main theorem describing the convergence analysis regarding proposed scheme (7) and find out how to select parameters a, c, and d for optimal fourth-order convergence.

Definition 1 (error equation, asymptotic error constant, and order of convergence). Let $x_0, x_1, x_2, \dots, x_n, \dots$ be a sequence converging to α and let $e_n = x_n - \alpha$ be the nth iterate error. If there exist real numbers $p \in \mathbb{R}$ and $b \in \mathbb{R} - \{0\}$ such that the following error equation holds

$$e_{n+1} = be_n^p + O\left(e_n^{p+1}\right), \tag{9}$$

then b or $|b|$ is called the asymptotic error constant and p is called the order of convergence [17, 18].

Theorem 2. *Let $f : \mathbb{C} \to \mathbb{C}$ have a zero α with integer multiplicity $m \geq 1$ and be analytic in a small neighborhood of α. Let $\kappa = (m/(m+2))^m$ and $\theta_j = f^{(m+j)}(\alpha)/f^{(m)}(\alpha)$ for $j \in \mathbb{N}$. Let x_0 be an initial guess chosen in a sufficiently small neighborhood of α. Let λ, $\rho \in \mathbb{R}$ be two free constant parameters. Let $a = m^3(-4\kappa\rho + m(m+2)(1+\kappa\rho))/8(m+2)\kappa\rho$, $c = (m+2)\{-m^3 + m(m+2)\kappa(-3m + 2(m+2)\kappa\lambda)\rho + m(m+2)^2\kappa^2(-3+2\kappa\rho)\rho^2 - (m+2)^3\kappa^3\rho^3\}/16\kappa\lambda\rho$, $d = (2+m)(m+(2+m)\kappa\rho))^3/16\kappa\lambda\rho$, and $\gamma = 2m/(2+m)$. Then iterative methods (7) are optimal and of order four and possess the following error equation:*

$$e_{n+1} = \psi_4 e_n^4 + O\left(e_n^5\right), \tag{10}$$

where $\psi_4 = ((m^4 + 4m^3 + 6m^2 + 2m + 8 - (24\kappa\rho/(m + (m+2)\kappa\rho)))/3m^2(m+1)^3(m+2))\theta_1^3 - (1/m(m+1)^2(m+2))\theta_1\theta_2 + (m/(m+2)^3(m+1)(m+3))\theta_3$.

Proof. The optimality on convergence order of proposed scheme (7) is clear in the sense of Kung-Traub due to three functional evaluations. Hence, it suffices to determine the constant parameters for fourth-order convergence.

Applying the Taylor's series expansion about α, we get the following relations:

$$f(x_n) = \frac{f^{(m)}(\alpha)}{m!} e_n^m \left[1 + A_1 e_n + A_2 e_n^2 + A_3 e_n^3 + A_4 e_n^4 + O\left(e_n^5\right)\right], \quad (11)$$

$$f'(x_n) = \frac{f^{(m)}(\alpha)}{(m-1)!} e_n^{m-1} \left[1 + B_1 e_n + B_2 e_n^2 + B_3 e_n^3 + B_4 e_n^4 + O\left(e_n^5\right)\right], \quad (12)$$

where $A_k = (m!/(m+k)!)\theta_k$, $B_k = ((m-1)!/(m+k-1)!)\theta_k$, and $\theta_k = f^{(m+k)}(\alpha)/f^{(m)}(\alpha)$ for $k \in \mathbb{N}$.

Dividing (11) by (12), we have

$$\frac{f(x_n)}{f'(x_n)} = \frac{1}{m}\left[e_n - K_1 e_n^2 - K_2 e_n^3 - K_3 e_n^4 + O\left(e_n^5\right)\right], \quad (13)$$

where $K_1 = -A_1 + B_1$, $K_2 = -A_2 + A_1 B_1 - B_1^2 + B_2$, $K_3 = -A_3 + A_2 B_1 - A_1 B_1^2 + B_1^3 + A_1 B_2 - 2B_1 B_2 + B_3$.

For algebraic convenience, we introduce a parameter t defined by $\gamma = m(1-t)$; that is, $t = 1 - \gamma/m$ to obtain

$$\begin{aligned} y_n &= x_n - \gamma \frac{f(x_n)}{f'(x_n)} = e_n + \alpha - \gamma \frac{f(x_n)}{f'(x_n)} \\ &= \alpha + te + K_1(1-t)e_n^2 + K_2(1-t)e_n^3 \\ &\quad + K_3(1-t)e_n^4 + O\left(e_n^5\right). \end{aligned} \quad (14)$$

Evaluating $f(y_n)$ from (11) with e_n being replaced by $y_n - \alpha$ in (14), we find

$$\begin{aligned} f(y_n) = \frac{f^{(m)}(\alpha) e^m}{m!} &\times \Big\{ t^m + t^{m-1}\left(A_1 t^2 + K_1 m(1-t)\right) e_n \\ &+ \frac{1}{2}\Big(t^{m-2}\Big(K_1^2 (m-1) m (t-1)^2 \\ &\quad - 2A_1 K_1 (m+1)(t-1) t^2 \\ &\quad + 2t\left(A_2 t^3 + K_2 m(1-t)\right)\Big)\Big) e_n^2 \\ &+ \frac{1}{6} t^{m-3}\Big(-K_1^3 m\left(2 - 3m + m^2\right)(t-1)^3 \\ &\quad + 3A_1 K_1^2 m(m+1)(t-1)^2 t^2 \\ &\quad - 6K_1 (t-1) t \\ &\quad \times \Big(A_2 (m+2) t^3 \\ &\quad\quad + K_2 m(m-1)(1-t) + 6t^2 \\ &\quad\quad \times \Big(K_3 m(1-t) \\ &\quad\quad\quad + t\left(A_1 K_2 (m+1)(t-1) \right. \\ &\quad\quad\quad\quad \left. + A_3 t^3\right)\Big)\Big)\Big) e_n^3\Big) + O\left(e_n^4\right)\Big\}. \end{aligned} \quad (15)$$

Substituting (11)–(15) into (7), we obtain the error equation:

$$e_{n+1} = y_n - \alpha - K_f = \psi_1 e_n + \psi_2 e_n^2 + \psi_3 e_n^3 + \psi_4 e_n^4 + O\left(e_n^5\right), \quad (16)$$

where $K_f = a(f(x_n)/f'(x_n)) + c(f(x_n)/f'(y_n)) + d(F(y_n)/f'(y_n))$, $\psi_1 = -(a/m) + t(1 - ((c+d)t^{-m}/m) - (d(t-1)\lambda/(t + t^m \rho)))$, and coefficients ψ_i, $(i = 2, 3, 4)$ depend on the parameters t, a, c, d, λ, and ρ and the function $f(x)$.

Solving $\psi_1 = 0$, $\psi_2 = 0$ for a and c, respectively, we get

$$\begin{aligned} a &= mt\left(1 - \frac{(c+d)t^{-m}}{m} - \frac{d(t-1)\lambda}{t + t^m \rho}\right), \\ c &= -\Big(mt^m (t + t^m \rho)^2 + d(t-1)(1 + m(t-1) + t) \\ &\quad \times \left(t^2 + 2t^{m+1}\rho + t^{2m}\rho\left(m(t-1)\lambda + \rho\right)\right)\Big) \\ &\quad \times \left((t-1)(1 + m(t-1) + t)(t + t^m \rho)^2\right)^{-1}. \end{aligned} \quad (17)$$

We substitute a, c into ψ_3 and put $\psi_3 = \psi_{31}\theta_1^2 + \psi_{32}\theta_2$. Solving $\psi_3 = 0$ independently of θ_1 and θ_2, that is, solving $\psi_{31} = \psi_{32} = 0$ for d and t, we obtain

$$\begin{aligned} d &= -\Big(t^{-1-m}\Big(2t^2(2+t) + m^2(t-1)(1 + 2(t-1)t) \\ &\quad + m\left(1 - 3t + 4t^3\right)\Big)(t + t^m \rho)^3\Big) \\ &\quad \times \left(2(t-1)^3 (1 + m(t-1) + t)^3 \lambda \rho\right)^{-1}, \\ t &= \frac{m}{m+2}. \end{aligned} \quad (18)$$

Table 1: Typical methods with interesting parameters (λ, ρ) and constants (a, c, d).

Method	(λ, ρ)	(a, c, d)
Y1	$(-10, 10)$	$a = \frac{m^3\{-40\kappa + m(m+2)(10\kappa+1)\}}{80(m+2)\kappa}$, $d = -\frac{(m+2)(m+10(m+2)\kappa)^3}{1600\kappa}$, $c = \frac{(m+2)\{m^3 + 30m^2(m+2)\kappa + 500m(m+2)^2\kappa^2 + 1000(m+2)^2(3m+2)\kappa^3\}}{1600\kappa}$
Y2	$\left(-1, -\frac{m(m+2)}{(-4+2m+m^2)\kappa}\right)$	$\left(0, -\frac{m\{m(m+4)^3 - (m+2)^3(m^2+2m-4)\kappa\}}{2(m^2+2m-4)^2}, -\frac{m^2(m+4)^3}{2(m^2+2m-4)^2}\right)$
Y3	$\left(-\frac{m(m+4)^3}{(m+2)^3(m^2+2m-4)\kappa}, -\frac{m(m+2)}{(m^2+2m-4)\kappa}\right)$	$\left(0, 0, -\frac{m(m+2)^3\kappa}{2(m^2+2m-4)}\right)$
Y4	$\left(\frac{(m+(m+2)\kappa\rho)^3}{2m(m+2)^2\kappa^2\rho(1+\kappa\rho)}, -\frac{1}{2}\right)$	$\left(\frac{m^3(m(m+2)(\kappa-2)-4\kappa)}{8(m+2)\kappa}, 0, -\frac{1}{16}m(m+2)^3\kappa(\kappa+2)\right)$

Substituting $t = m/(m+2)$ into (17) and (18) with $\kappa = (m/(2+m))^m$, we get the following relations:

$$a = \frac{m^3\left(-4\kappa\rho + m(m+2)(1+\kappa\rho)\right)}{8(m+2)\kappa\rho},$$
$$c = \Big((m+2)\Big\{-m^3 + m(m+2)\kappa(-3m + 2(m+2)\kappa\lambda)\rho + m(m+2)^2\kappa^2(-3+2\kappa\rho)\rho^2 - (m+2)^3\kappa^3\rho^3\Big\}\Big) \times (16\kappa\lambda\rho)^{-1},$$
$$d = \frac{(2+m)(m+(2+m)\kappa\rho)^3}{16\kappa\lambda\rho}. \tag{19}$$

By the aid of symbolic computation of *Mathematica* [19], we arrive at the relation below:

$$e_{n+1} = \psi_4 e_n^4 + O\left(e_n^5\right), \tag{20}$$

where $\psi_4 = ((m^4 + 4m^3 + 6m^2 + 2m + 8 - (24\kappa\rho/(m + (m+2)\kappa\rho)))/3m^2(m+1)^3(m+2))\theta_1^3 - (1/m(m+1)^2(m+2))\theta_1\theta_2 + (m/(m+2)^3(m+1)(m+3))\theta_3$. As a result, the proof is completed. □

Remark 3. We observe that error equation (20) contains only one free parameter ρ, being independent of λ. Table 1 shows typically chosen parameters λ and ρ and defines various methods Y_k, $(k = 1, 2, 3, 4)$.

4. Numerical Examples and Conclusion

We have performed a variety of numerical experiments with Mathematica Version 5 [19] to confirm the theory developed in Section 3. In these experiments, we assign 300, via Mathematica command *$MinPrecision* = 300, as the minimum number of precision digits to achieve the specified sufficient accuracy. It is crucial to compute $e_n = x_n - \alpha$ with high accuracy for desired numerical results. When zero α is not exactly known, it is replaced by a highly accurate value $\tilde{\alpha}$ which has larger number of significant digits than the assigned minimum number of precision digits. To deal with numerical results more effectively, we first define

$$\overline{\alpha} = \begin{cases} \alpha, & \text{if } \alpha \text{ is exactly known,} \\ \tilde{\alpha}, & \text{if } \alpha \text{ is not exactly known.} \end{cases} \tag{21}$$

To properly display numerical results, we need to define the nth computational error $\overline{e}_n = x_n - \overline{\alpha}$ for $n = 0, 1, 2\ldots$. We need further terminologies as defined below.

Definition 4 (computational asymptotic error constant and computational convergence order). Assume that theoretical asymptotic error constant $\eta = \lim_{n\to\infty}|e_n|/|e_{n-1}|^p$ and convergence order $p \geq 1$ are known (usually via main theorem). Define $q_n = |\overline{e}_n|/|\overline{e}_{n-1}|^p$ as the computational asymptotic error constant and $p_n = \log|\overline{e}_n/\eta|/\log|\overline{e}_{n-1}|$ as the computational convergence order. Then we find that $\lim_{n\to\infty}q_n$ is equal or close to η, while $\lim_{n\to\infty}p_n$ is equal or close to p.

If $\tilde{\alpha}$ has the same accuracy of *$MinPrecision* as that of x_n, then $\overline{e}_n = x_n - \tilde{\alpha}$ would be nearly zero and hence computing $|\overline{e}_{n+1}/\overline{e}_n{}^4|$ would unfavorably break down. Computed values of x_n are accurate up to 300 significant digits. For current experiments $\tilde{\alpha}$ is found to be accurate enough about up to 400 significant digits. To supply such $\tilde{\alpha}$, a set of following Mathematica commands are used:

$$\begin{aligned} sol = FindRoot\left[f[x], \{x, x_0\}, PrecisionGoal \longrightarrow 100 + \$MinPrecision,\right. \\ WorkingPrecision \longrightarrow 2 * \$MinPrecision\left.\right]; \\ \tilde{\alpha} = sol[[1, 2]]. \end{aligned} \tag{22}$$

Although the number of significant digits of x_n and $\tilde{\alpha}$ is 300 and 400, respectively, the limited paper space allows us to list both of them only up to 15 significant digits. We set the error bound ε to 10^{-250} for $|x_n - \overline{\alpha}| < \varepsilon$.

As a first example, we select a function $f(x) = (x^2 + 7)^2 \csc x \log(x^2 + 8)$ having a multiple zero $\alpha = -\sqrt{7}i$ with

TABLE 2: Convergence behavior with $f(x)=(x^2+7)^2\csc x\log(x^2+8)$. $(m,\lambda,\rho)=(3,1/2,-1)$, $\alpha=-\sqrt{7}i=-2.64i$.

n	x_n	$\lvert f(x_n)\rvert$	$\lvert x_n-\bar{\alpha}\rvert$	$\lvert \bar{e}_{n+1}/\bar{e}_n{}^4\rvert$	η	$\log(\bar{e}_n/\eta)/\log(\bar{e}_{n-1})$
0	$-2.71i$	0.00667483	0.0642487		0.2482221894	
1	$-2.64575363508446i$	2.65244×10^{-16}	2.32402×10^{-6}	0.1363900716		4.21814
2	$-2.64575131106459i$	8.02272×10^{-69}	7.24101×10^{-24}	0.2482218636		4.00000
3	$-2.64575131106459i$	6.71488×10^{-279}	6.82399×10^{-94}	0.2482221894		4.00000
4	$-2.64575131106459i$	$0.\times10^{899}$	$0.\times10^{-299}$			

TABLE 3: Convergence behavior with $f(x)=(x^2+4x-1)^4e^{-x}$. $(m,\lambda,\rho)=(4,-10,10)$, $\alpha=-2-\sqrt{5}=-4.23607$.

n	x_n	$\lvert f(x_n)\rvert$	$\lvert x_n-\bar{\alpha}\rvert$	$\lvert \bar{e}_{n+1}/\bar{e}_n{}^4\rvert$	η	$\log(\bar{e}_n/\eta)/\log(\bar{e}_{n-1})$
0	-4.35000000000000	5.77466	0.113932		0.2202805193	
1	-4.23609825091956	2.32291×10^{-14}	0.0000302734	0.1796711117		4.09332
2	-4.23606797749979	3.24005×10^{-71}	1.85011×10^{-19}	0.2202681897		4.00001
3	-4.23606797749979	1.22696×10^{-298}	2.58087×10^{-76}	0.2202805193		4.00000
4	-4.23606797749979	2.52311×10^{-1208}	$0.\times10^{-299}$			

TABLE 4: Convergence behavior with $f(x)=(e^{x^2+3x+7}-1)^5(x^2+3x+7)^3$. $(m,\lambda,\rho)=(8,1/7,-3)$, $\alpha=(-3-\sqrt{19}i)/2$.

n	x_n	$\lvert f(x_n)\rvert$	$\lvert x_n-\bar{\alpha}\rvert$	$\lvert \bar{e}_{n+1}/\bar{e}_n{}^4\rvert$	η	$\log(\bar{e}_n/\eta)/\log(\bar{e}_{n-1})$
0	$-1.46000000000000-2.09000000000000i$	0.00250952	0.0979858		2.458883957	
1	$-1.49981340826556-2.17945296114250i$	1.91755×10^{-25}	0.000186624	2.024494357		4.08382
2	$-1.50000000000000-2.17944947177033i$	8.16451×10^{-112}	2.98273×10^{-15}	2.458901785		4.00000
3	$-1.50000000000000-2.17944947177034i$	2.68276×10^{-457}	1.94624×10^{-58}	2.458883957		4.00000
4	$-1.50000000000000-2.17944947177034i$	3.12745×10^{-1839}	3.52795×10^{-231}	2.458883957		4.00000
5	$-1.50000000000000-2.17944947177034i$	$0.\times10^{-3888}$	$0.\times10^{-299}$			

$i=\sqrt{-1}$. We choose $x_0=-2.71i$ as an initial guess. We take another function $f(x)=(x^2+4x-1)^4e^{-x}$ with a root $\alpha=-2-\sqrt{5}$. We select $x_0=-4.35$ as an initial value. The order of convergence and the asymptotic error constant are clearly shown in Tables 2 and 3 revealing a good agreement with the theory in Section 3. Taking another function $f(x)=(e^{x^2+3x+7}-1)^5(x^2+3x+7)^3$ with a root $\alpha=(-3-\sqrt{19}i)/2$ with multiplicity $m=8$, we select $x_0=-1.46-2.09i$ as an initial value. In this example, we also find that the order of convergence is four and the computational asymptotic error constant $\lvert\bar{e}_{n+1}/\bar{e}_n^4\rvert$ well approaches the theoretical value η. The computational convergence order and the computational asymptotic error constant are certainly shown in Tables 2–4 reaching a good agreement with the theory. It is certain that these methods need one evaluation of the function f and two evaluations of the first derivative f'.

Additional test functions below are used to display the convergence behavior of proposed scheme (7):

$$
\begin{aligned}
f_1(x)&=\left(-1+3x+x^2\right)\log\left(3+x^2\right),\\
&\quad\tilde{\alpha}=0.302775637731995,\quad m=1,\quad x_0=0.29,\\
f_2(x)&=\left(x^2-5\right)\log\left(x^2-4\right),\\
&\quad\alpha=-\sqrt{5},\quad m=2,\quad x_0=-2.19,\\
f_3(x)&=\left(x^2-25\right)^2e^{x^2+1}\csc x\log\left(x^2+3x-9\right),\\
&\quad\alpha=-5,\quad m=3,\quad x_0=-5,\\
f_4(x)&=\left(x-\frac{7}{x}\right)^4,\\
&\quad\alpha=\sqrt{7},\quad m=4,\quad x_0=2.77,\\
f_5(x)&=\log^3\left(x^2-1\right)\left(x^2-2\right)^2,\\
&\quad\alpha=-\sqrt{2},\quad m=5,\quad x_0=-1.38,\\
f_6(x)&=\left(x^2+3\right)^4\left(\log\left(x^4+x^2-5\right)\right)^2,\\
&\quad\alpha=\sqrt{3}i=1.73i,\quad m=6,\quad x_0=1.68i,\\
f_7(x)&=\log\left(x^2-8\right)\left(e^{x^2+7x-30}-1\right)^3(x-3)^3,\\
&\quad\alpha=3,\quad m=7,\quad x_0=2.88.
\end{aligned}
\tag{23}
$$

In Table 5, we compare numerical errors $\lvert x_n-\alpha\rvert$ of proposed methods Y1–Y4 with those of existing optimal fourth-order multiple-root finders. Abbreviations S, J, L, Z, and K denote optimal fourth-order multiple-root finders obtained by Shengguo et al., Sharma et al., Li et al., Zhou et al., and Kanwar et al., respectively.

The least errors within the prescribed error bound are highlighted in boldface. Method Y2 shows best convergence

Table 5: Comparison of $|x_n - \overline{\alpha}|$ for various multiple-root finders.

$f(x)$	x_0	$\lvert x_n - \alpha \rvert$	**S**	**J**	**L**	**Z**	**K**	**Y1**	**Y2**	**Y3**	**Y4**
f_1	0.29	$\lvert x_1 - \overline{\alpha} \rvert$	1.35e − 9	2.90e − 10	2.90e − 10	1.85e − 9	2.93e − 9	9.70e − 11	7.76e − 11	7.76e − 11	4.02e − 9
		$\lvert x_2 - \overline{\alpha} \rvert$	1.63e − 37	6.07e − 41	6.07e − 41	8.60e − 37	8.46e − 36	9.92e − 44	1.38e − 44	1.38e − 44	4.07e − 35
		$\lvert x_3 - \overline{\alpha} \rvert$	3.53e − 149	1.16e − 163	1.16e − 163	3.99e − 146	5.83e − 142	1.09e − 175	**1.39e − 179**	1.39e − 179	4.24e − 139
		$\lvert x_4 - \overline{\alpha} \rvert$	0.e − 300	0.e − 300	0.e − 300	0.e − 300	0.e − 300	0.e − 300	0.e − 300	0.e − 300	0.e − 300
f_2	−2.19	$\lvert x_1 - \overline{\alpha} \rvert$	1.83e − 6	1.67e − 6	3.03e − 6	3.09e − 7	7.02e − 7	1.45e − 6	1.45 − 6	2.12 − 6	2.46e − 8
		$\lvert x_2 - \overline{\alpha} \rvert$	2.60e − 23	1.97e − 24	3.06e − 23	1.62e − 27	2.66e − 26	9.92e − 25	9.92e − 24	6.24e − 24	3.80e − 33
		$\lvert x_3 - \overline{\alpha} \rvert$	1.05e − 91	3.76e − 96	7.24e − 91	4.92e − 109	5.51e − 104	2.12e − 97	2.12e − 94	4.69e − 94	**2.15e − 132**
		$\lvert x_4 - \overline{\alpha} \rvert$	0.e − 300	0.e − 299	0.e − 300	0.e − 300	0.e − 300	0.e − 300	0.e − 300	0.e − 300	0.e − 300
f_3	−5	$\lvert x_1 - \overline{\alpha} \rvert$	0.011	0.013	0.011	0.013	0.013	0.012	0.011	0.012	0.014
		$\lvert x_2 - \overline{\alpha} \rvert$	2.73e − 8	1.26e − 7	2.73e − 8	2.26e − 7	1.50e − 7	6.32e − 8	2.22e − 8	6.67e − 8	2.51e − 7
		$\lvert x_3 - \overline{\alpha} \rvert$	1.05e − 30	6.94e − 29	1.05e − 30	4.47e − 27	4.01e − 28	9.72e − 30	4.61e − 31	1.70e − 29	8.06e − 27
		$\lvert x_4 - \overline{\alpha} \rvert$	2.35e − 120	6.28e − 114	2.35e − 120	6.83e − 106	2.02e − 110	5.42e − 117	**8.51e − 122**	7.20e − 116	8.55e − 105
		$\lvert x_5 - \overline{\alpha} \rvert$	0.e − 299	0.e − 299	0.e − 299	0.e − 299	0.e − 299	0.e − 299	0.e − 299	0.e − 299	0.e − 299
f_4	2.77	$\lvert x_1 - \overline{\alpha} \rvert$	9.18e − 7	7.67e − 7	9.18e − 7	6.90e − 7	7.67e − 7	7.48e − 7	8.62e − 7	8.62e − 7	6.78e − 7
		$\lvert x_2 - \overline{\alpha} \rvert$	2.93e − 27	1.17e − 27	2.93e − 27	6.86e − 28	1.16e − 27	1.03e − 27	2.12e − 27	2.12e − 27	6.21e − 28
		$\lvert x_3 - \overline{\alpha} \rvert$	3.07e − 109	6.39e − 111	3.07e − 109	6.65e − 112	6.32e − 111	3.70e − 111	7.85e − 110	7.85e − 110	**4.36e − 112**
		$\lvert x_4 - \overline{\alpha} \rvert$	0.e − 299	0.e − 299	0.e − 299	0.e − 299	0.e − 299	0.e − 299	0.e − 299	0.e − 299	0.e − 299
f_5	−1.38	$\lvert x_1 - \overline{\alpha} \rvert$	1.04e − 7	8.69e − 8	1.04e − 7	8.00e − 8	8.91e − 8	8.49e − 8	9.88e − 8	9.88e − 8	7.89e − 8
		$\lvert x_2 - \overline{\alpha} \rvert$	6.35e − 30	2.50e − 30	6.35e − 30	1.63e − 30	2.85e − 30	2.23e − 30	4.83e − 30	4.83e − 30	1.53e − 30
		$\lvert x_3 - \overline{\alpha} \rvert$	8.75e − 119	1.72e − 120	8.75e − 119	2.86e − 121	2.99e − 120	1.06e − 120	2.77e − 119	2.77e − 119	**2.16e − 121**
		$\lvert x_4 - \overline{\alpha} \rvert$	0.e − 299	0.e − 299	0.e − 299	0.e − 299	0.e − 299	0.e − 299	0.e − 299	0.e − 299	0.e − 299
f_6	1.68i	$\lvert x_1 - \overline{\alpha} \rvert$	1.79e − 3	1.32e − 3	1.79e − 3	1.20e − 3	1.43e − 3	1.29e − 3	1.16e − 3	1.65e − 3	1.18e − 3
		$\lvert x_2 - \overline{\alpha} \rvert$	1.36e − 10	4.38e − 11	1.36e − 10	2.99e − 11	5.88e − 11	3.91e − 11	2.66e − 11	1.008e − 10	2.84e − 11
		$\lvert x_3 - \overline{\alpha} \rvert$	4.76e − 39	5.29e − 41	4.76e − 39	1.18e − 41	1.69e − 40	3.39e − 41	7.38e − 42	1.42e − 39	9.63e − 42
		$\lvert x_4 - \overline{\alpha} \rvert$	6.94e − 153	1.13e − 160	6.94e − 153	2.84e − 163	1.18e − 158	1.92e − 161	**4.37e − 164**	5.67e − 155	1.26e − 163
		$\lvert x_5 - \overline{\alpha} \rvert$	0.e − 299	0.e − 299	0.e − 299	0.e − 299		0.e − 299	0.e − 299	0.e − 299	0.e − 299
f_7	2.88	$\lvert x_1 - \overline{\alpha} \rvert$	1.380e − 2	1.372e − 2	1.380e − 2	1.370e − 2	1.374e − 2	1.372e − 2	1.378e − 2	1.378e − 2	1.370e − 2
		$\lvert x_2 - \overline{\alpha} \rvert$	2.21e − 7	1.92e − 7	2.21e − 7	1.84e − 7	2.00e − 7	1.91e − 7	2.14e − 7	2.14e − 7	1.83e − 7
		$\lvert x_3 - \overline{\alpha} \rvert$	4.23e − 27	1.48e − 27	4.23e − 27	1.02e − 27	2.02e − 27	1.33e − 27	3.40e − 27	3.40e − 27	9.73e − 28
		$\lvert x_4 - \overline{\alpha} \rvert$	5.69e − 106	5.27e − 108	5.69e − 106	9.61e − 109	2.11e − 107	3.18e − 108	2.16e − 106	2.16e − 106	**7.64e − 109**
		$\lvert x_5 - \overline{\alpha} \rvert$	0.e − 299	0.e − 299	0.e − 299	0.e − 299	0.e − 299	0.e − 299	0.e − 299	0.e − 299	0.e − 299

Here, 1.35e − 9 denotes 1.35×10^{-9}.

for f_1, f_3, f_6, while method Y4 does show for f_2, f_4, f_5, f_7. It must be kept in mind that such favorable performance is shown only in the current numerical experiments with the particular choice of test functions. For a different choice of test functions and initial values, it is hardly expected that each of listed methods would always give rise to better convergence behavior. One should be aware that no numerical method exhibits better performance for all the test functions as compared with other numerical methods. With the same order of convergence, one should notice that the speed of local convergence is dependent on the function $f(x)$, an initial value x_0, and a multiple zero α itself.

For efficiency check of multipoint iterative methods, we need to calculate the efficiency index [17] defined by $\mathrm{EI} = p^{1/d}$, where p is the order of convergence and d is the number of distinct functional or derivative evaluations per iteration. The proposed iterative methods (7) have $\mathrm{EI} = 4^{1/3} \approx 1.587$ describing the optimality, which is the same as that of any other listed method. This paper confirms optimal fourth-order convergence and derives the correct error equation for proposed iterative methods, using the weighted harmonic mean of two derivatives to find approximate multiple zeros of nonlinear equations. Proposed optimal fourth-order schemes efficiently solve given problems without any difficulty, with a wide selection of free parameters λ and ρ. In future work, we will pursue higher-order optimal methods by extending the methods developed here.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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