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Research Article

Strong Convergence for Hybrid Implicit *S*-Iteration Scheme of Nonexpansive and Strongly Pseudocontractive Mappings

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Let K be a nonempty closed convex subset of a real Banach space E, let $S : K \to K$ be nonexpansive, and let $T : K \to K$ be Lipschitz strongly pseudocontractive mappings such that $p \in F(S) \cap F(T) = \{x \in K : Sx = Tx = x\}$ and $\|x - Sy\| \le \|Sx - Sy\|$ and $\|x - Ty\| \le \|Tx - Ty\|$ for all $x, y \in K$. Let $\{\beta_n\}$ be a sequence in $[0, 1]$ satisfying (i) $\sum_{n=1}^{\infty} \beta_n = \infty$; (ii) $\lim_{n\to\infty} \beta_n = 0$. For arbitrary $x_0 \in K$, let $\{x_n\}$ be a sequence iteratively defined by $x_n = Sy_n$, $y_n = (1 - \beta_n) x_{n-1} + \beta_n T x_n$, $n \ge 1$. Then the sequence $\{x_n\}$ converges strongly to a common fixed point p of S and T.

1. Introduction and Preliminaries

Let E be a real Banach space and let K be a nonempty convex subset of E. Let J denote the normalized duality mapping from E to 2^{E^*} defined by

$$J(x) = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2, \ \|f^*\| = \|x\|\}, \quad x \in E, \tag{1}$$

where E^* denotes the dual space of E and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. We will denote the single-valued duality map by j.

Let $T : K \to K$ be a mapping.

Definition 1. The mapping T is said to be *Lipschitzian* if there exists $L > 1$ such that

$$\|Tx - Ty\| \le L \|x - y\| \tag{2}$$

for all $x, y \in K$.

Definition 2. The mapping T is said to be *nonexpansive* if

$$\|Tx - Ty\| \le \|x - y\| \tag{3}$$

for all $x, y \in K$.

Definition 3. The mapping T is said to be *pseudocontractive* if

$$\|x - y\| \le \|x - y + t((I - T)x - (I - T)y)\| \tag{4}$$

for all $x, y \in K$ and $t > 0$.

Remark 4. As a consequence of a result of Kato [1], it follows from the inequality that T is pseudocontractive if and only if there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y) \rangle \le \|x - y\|^2 \tag{5}$$

for all $x, y \in K$.

Definition 5. The mapping T is said to be *strongly pseudocontractive* if there exists a constant $t > 1$ such that

$$\|x - y\| \le \|(1 + r)(x - y) - rt(Tx - Ty)\| \tag{6}$$

for all $x, y \in K$ and $r > 0$. Or equivalently (see [2]) one has for $0 < k < 1$

$$\langle Tx - Ty, j(x - y) \rangle \le k\|x - y\|^2 \tag{7}$$

for all $x, y \in K$.

For a nonempty convex subset K of a normed space E, $T : K \to K$ is a mapping.

(I) The sequence $\{x_n\}$, defined by, for arbitrary $x_1 \in K$,

$$x_{n+1} = (1 - a_n) x_n + a_n T y_n, \\ y_n = (1 - b_n) x_n + b_n T x_n, \quad n \geq 1, \tag{8}$$

where $\{a_n\}$ and $\{b_n\}$ are sequences in $[0, 1]$, is known as the Ishikawa iteration process [3].

If $b_n = 0$ for $n \geq 1$, then the Ishikawa iteration scheme becomes the Mann iteration process [4].

(S) The sequence $\{x_n\}$, defined by, for arbitrary $x_1 \in K$,

$$x_{n+1} = T y_n, \\ y_n = (1 - b_n) x_n + b_n T x_n, \quad n \geq 1, \tag{9}$$

where $\{b_n\}$ is a sequence in $[0, 1]$, is known as the S-iteration process [5, 6].

In the last few years or so, numerous papers have been published on the iterative approximation of fixed points of Lipschitz strongly pseudocontractive mappings using the Ishikawa iteration scheme (see, e.g., [3]). Results which had been known only in Hilbert spaces and only for Lipschitz mappings have been extended to more general Banach spaces (see, e.g., [7–13] and the references cited therein).

In 1974, Ishikawa [3] proved the following result.

Theorem 6. *Let K be a compact convex subset of a Hilbert space H and let $T : K \to K$ be a Lipschitzian pseudocontractive mapping. For arbitrary $x_1 \in K$, let $\{x_n\}$ be a sequence defined iteratively by*

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T y_n, \\ y_n = (1 - \beta_n) x_n + \beta_n T x_n, \quad n \geq 1, \tag{10}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences satisfying

(i) $0 \leq \alpha_n \leq \beta_n < 1$;

(ii) $\lim_{n \to \infty} \beta_n = 0$;

(iii) $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$.

Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

In [7], Chidume extended the results of Schu [12] from Hilbert spaces to the much more general class of real Banach spaces and approximate the fixed points of pseudocontractive mappings. Also, in [14], he investigated the approximation of the fixed points of strongly pseudocontractive mappings.

In [15], Zhou and Jia gave the answer of the question raised by Chidume [14] and proved the following.

If X is a real Banach space with a uniformly convex dual X^*, K is a nonempty bounded closed convex subset of X, and $T : K \to K$ is a continuous strongly pseudocontractive mapping, then the Ishikawa iteration scheme converges strongly to the unique fixed point of T.

In [16], Liu et al. introduced the following condition.

Remark 7. Let $S, T : K \to K$ be two mappings. The mappings S and T are said to satisfy *condition* ($C1$) if

$$\|x - Ty\| \leq \|Sx - Ty\| \tag{C1}$$

for all $x, y \in K$.

In 2012, Kang et al. [17] established the strong convergence for the implicit S-iterative process associated with Lipschitzian hemicontractive mappings in Hilbert spaces.

Theorem 8. *Let K be a compact convex subset of a real Hilbert space H and let $T : K \to K$ be a Lipschitzian hemicontractive mapping satisfying*

$$\|x - Ty\| \leq \|Tx - Ty\| \tag{C2}$$

for all $x, y \in K$. Let $\{\beta_n\}$ be a sequence in $[0, 1]$ satisfying

(i) $\sum_{n=1}^{\infty} \beta_n = \infty$;

(ii) $\sum_{n=1}^{\infty} \beta_n^2 < \infty$.

For arbitrary $x_0 \in K$, let $\{x_n\}$ be a sequence iteratively defined by

$$x_n = T y_n, \\ y_n = (1 - \beta_n) x_{n-1} + \beta_n T x_n, \quad n \geq 1. \tag{11}$$

Then the sequence $\{x_n\}$ converges strongly to the fixed point x^* of T.

In 2013, Kang et al. [18] proved the following result.

Theorem 9. *Let K be a nonempty closed convex subset of a real Banach space E, let $S : K \to K$ be a nonexpansive mapping, and let $T : K \to K$ be a Lipschitz strongly pseudocontractive mapping such that $p \in F(S) \cap F(T) = \{x \in K : Sx = Tx = x\}$ and*

$$\|x - Sy\| \leq \|Sx - Sy\|, \qquad \|x - Ty\| \leq \|Tx - Ty\| \tag{C3}$$

for all $x, y \in K$. Let $\{\beta_n\}$ be a sequence in $[0, 1]$ satisfying

(i) $\sum_{n=1}^{\infty} \beta_n = \infty$;

(ii) $\lim_{n \to \infty} \beta_n = 0$.

For arbitrary $x_1 \in K$, let $\{x_n\}$ be a sequence iteratively defined by

$$x_{n+1} = S y_n, \\ y_n = (1 - \beta_n) x_n + \beta_n T x_n, \quad n \geq 1. \tag{12}$$

Then the sequence $\{x_n\}$ converges strongly to a common fixed point p of S and T.

Keeping in view the importance of the implicit iteration schemes (see [17]) in this paper we establish the strong convergence theorem for the hybrid implicit S-iterative scheme associated with nonexpansive and Lipschitz strongly pseudocontractive mappings in real Banach spaces.

2. Main Results

We will need the following results.

Lemma 10 (see [19, 20]). *Let $J : E \to 2^{E^*}$ be the normalized duality mapping. Then for any $x, y \in E$, one has*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle, \quad \forall j(x + y) \in J(x + y). \tag{13}$$

Lemma 11 (see [13]). *Let $\{\rho_n\}$ and $\{\theta_n\}$ be nonnegative sequences satisfying*

$$\rho_{n+1} \leq (1 - \theta_n)\rho_n + b_n, \tag{14}$$

where $\theta_n \in [0, 1)$, $\sum_{n=1}^{\infty}\theta_n = \infty$, and $b_n = o(\theta_n)$. Then $\lim_{n\to\infty}\rho_n = 0$.

The following is our main result.

Theorem 12. *Let K be a nonempty closed convex subset of a real Banach space E, let $S : K \to K$ be a nonexpansive mapping, and let $T : K \to K$ be a Lipschitz strongly pseudocontractive mapping such that $p \in F(S) \cap F(T) = \{x \in K : Sx = Tx = x\}$ and condition* (C3).

Let $\{\beta_n\}$ be a sequence in $[0, 1]$ satisfying

(i) $\sum_{n=1}^{\infty}\beta_n = \infty$;

(ii) $\lim_{n\to\infty}\beta_n = 0$.

For arbitrary $x_0 \in K$, let $\{x_n\}$ be a sequence iteratively defined by

$$x_n = Sy_n, \quad y_n = (1 - \beta_n)x_{n-1} + \beta_n Tx_n, \quad n \geq 1. \tag{15}$$

Then the sequence $\{x_n\}$ converges strongly to a common fixed point p of S and T.

Proof. For strongly pseudocontractive mappings, the existence of a fixed point follows from Deimling [21]. It is shown in [15] that the set of fixed points for strongly pseudocontractions is a singleton.

By (ii), since $\lim_{n\to\infty}\beta_n = 0$, there exists $n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$,

$$\beta_n \leq \min\left\{\frac{1}{4k}, \frac{1-k}{2(1+L)(1+2L)}\right\}, \tag{16}$$

where $k < 1/2$ and L is a Lipschitz constant of T. Consider

$$\begin{aligned}\|x_n - p\|^2 &= \langle x_n - p, j(x_n - p)\rangle \\ &= \langle Sy_n - p, j(x_n - p)\rangle \\ &= \langle Tx_n - p, j(x_n - p)\rangle + \langle Sy_n - Tx_n, j(x_n - p)\rangle \\ &\leq k\|x_n - p\|^2 + \|Sy_n - Tx_n\|\|x_n - p\|,\end{aligned} \tag{17}$$

where

$$\begin{aligned}\|Sy_n - Tx_n\| &\leq \|Sy_n - Ty_n\| + \|Ty_n - Tx_n\| \\ &\leq \|x_n - Sy_n\| + \|x_n - Ty_n\| + \|Ty_n - Tx_n\| \\ &\leq \|Sx_n - Sy_n\| + \|Tx_n - Ty_n\| + \|Ty_n - Tx_n\| \\ &= \|Sx_n - Sy_n\| + 2\|Tx_n - Ty_n\| \\ &\leq (1 + 2L)\|x_n - y_n\|,\end{aligned} \tag{18}$$

$$\begin{aligned}\|x_n - y_n\| &\leq \|x_n - x_{n-1}\| + \|x_{n-1} - y_n\| \\ &= \|Sy_n - x_{n-1}\| + \|x_{n-1} - y_n\| \\ &\leq \|Sx_{n-1} - Sy_n\| + \|x_{n-1} - y_n\| \\ &\leq 2\|x_{n-1} - y_n\| \\ &= 2\beta_n\|x_{n-1} - Tx_n\| \\ &\leq 2\beta_n(\|x_{n-1} - p\| + \|p - Tx_n\|) \\ &\leq 2\beta_n(\|x_{n-1} - p\| + L\|x_n - p\|),\end{aligned} \tag{19}$$

and consequently from (18) and (19), we obtain

$$\|Sy_n - Tx_n\| \leq 2(1 + 2L)\beta_n\|x_{n-1} - p\| + 2L(1 + 2L)\beta_n\|x_n - p\|. \tag{20}$$

Substituting (20) in (17) and using (16), we get

$$\begin{aligned}\|x_n - p\| &\leq \frac{2(1 + 2L)\beta_n}{1 - k - 2L(1 + 2L)\beta_n}\|x_{n-1} - p\| \\ &\leq \|x_{n-1} - p\| \quad \text{for } n \geq n_0.\end{aligned} \tag{21}$$

So, from the above discussion, we can conclude that the sequence $\{x_n - p\}$ is bounded. Since T is Lipschitzian, so $\{Tx_n - p\}$ is also bounded. Let $M_1 = \sup_{n\geq1}\|x_n - p\| + \sup_{n\geq1}\|Tx_n - p\|$. Also by (ii), we have

$$\begin{aligned}\|x_{n-1} - y_n\| &= \beta_n\|x_{n-1} - Tx_n\| \\ &\leq M_1\beta_n \\ &\longrightarrow 0\end{aligned} \tag{22}$$

as $n \to \infty$, which implies that $\{x_{n-1} - y_n\}$ is bounded, so let $M_2 = \sup_{n\geq1}\|x_{n-1} - y_n\| + M_1$. Further

$$\begin{aligned}\|y_n - p\| &\leq \|y_n - x_{n-1}\| + \|x_{n-1} - p\| \\ &\leq M_2,\end{aligned} \tag{23}$$

which implies that $\{y_n - p\}$ is bounded. Therefore $\{Ty_n - p\}$ is also bounded.

Set

$$M_3 = \sup_{n\ge1} \|y_n - p\| + \sup_{n\ge1} \|Ty_n - p\|. \quad (24)$$

Denote $M = M_1 + M_2 + M_3$. Obviously $M < \infty$.

Now, from (15), for all $n \ge 1$, we obtain

$$\|x_n - p\|^2 = \|Sy_n - p\|^2 \le \|y_n - p\|^2, \quad (25)$$

and by Lemma 10,

$$\begin{aligned}
&\|y_n - p\|^2 \\
&\quad = \|(1-\beta_n)x_{n-1} + \beta_n Tx_n - p\|^2 \\
&\quad = \|(1-\beta_n)(x_{n-1} - p) + \beta_n (Tx_n - p)\|^2 \\
&\quad \le (1-\beta_n)^2 \|x_{n-1} - p\|^2 + 2\beta_n \langle Tx_n - p, j(y_n - p)\rangle \\
&\quad = (1-\beta_n)^2 \|x_{n-1} - p\|^2 + 2\beta_n \langle Ty_n - p, j(y_n - p)\rangle \\
&\qquad + 2\beta_n \langle Tx_n - Ty_n, j(y_n - p)\rangle \\
&\quad \le (1-\beta_n)^2 \|x_{n-1} - p\|^2 + 2k\beta_n \|y_n - p\|^2 \\
&\qquad + 2\beta_n \|Tx_n - Ty_n\| \|y_n - p\| \\
&\quad \le (1-\beta_n)^2 \|x_{n-1} - p\|^2 + 2k\beta_n \|y_n - p\|^2 \\
&\qquad + 2ML\beta_n \|x_n - y_n\|, \quad \forall j(y_n - p) \in J(y_n - p),
\end{aligned} \quad (26)$$

which implies that

$$\begin{aligned}
&\|y_n - p\|^2 \\
&\le \frac{(1-\beta_n)^2}{1-2k\beta_n} \|x_{n-1} - p\|^2 + \frac{2ML\beta_n}{1-2k\beta_n} \|x_n - y_n\| \\
&\le (1-\beta_n) \|x_{n-1} - p\|^2 + 4ML\beta_n \|x_n - y_n\| \quad \text{for } n \ge n_0.
\end{aligned} \quad (27)$$

Because of (16), we have $(1-\beta_n)/(1-2k\beta_n) \le 1$ and $1/(1-2k\beta_n) \le 2$. Also, by (ii) and (19), $\|x_n - y_n\| \le 2M(1+L)\beta_n \to 0$ as $n \to \infty$.

Hence (25) and (27) give

$$\|x_n - p\|^2 \le (1-\beta_n) \|x_{n-1} - p\|^2 + 4ML\beta_n \|x_n - y_n\|. \quad (28)$$

For all $n \ge 1$, put

$$\begin{aligned}
\rho_n &= \|x_{n-1} - p\|, \\
\theta_n &= \beta_n, \\
b_n &= 4ML\beta_n \|x_n - y_n\|;
\end{aligned} \quad (29)$$

then according to Lemma 11, we obtain from (28) that

$$\lim_{n\to\infty} \|x_n - p\| = 0. \quad (30)$$

This completes the proof. □

Corollary 13. *Let K be a nonempty closed convex subset of a real Hilbert space H, let $S : K \to K$ be a nonexpansive mapping, and let $T : K \to K$ be a Lipschitz strongly pseudocontractive mapping such that $p \in F(S) \cap F(T) = \{x \in K : Sx = Tx = x\}$ and condition ($C3$). Let $\{\beta_n\}$ be a sequence in $[0,1]$ satisfying conditions (i) and (ii) in Theorem 12.*

For arbitrary $x_0 \in K$, let $\{x_n\}$ be a sequence iteratively defined by (15). Then the sequence $\{x_n\}$ converges strongly to a common fixed point p of S and T.

Example 14. As a particular case, we may choose, for instance, $\beta_n = 1/n$.

Remark 15. (1) Condition ($C2$) is due to Kang et al. [17] and condition ($C1$) with $S = T$ becomes condition ($C2$).

(2) Condition ($C3$) is due to Kang et al. [18] and condition ($C3$) with $S = T$ becomes condition ($C2$).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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