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Research Article

Hybrid Iterations for the Fixed Point Problem and Variational Inequalities

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A hybrid iterative algorithm with Meir-Keeler contraction is presented for solving the fixed point problem of the pseudocontractive mappings and the variational inequalities. Strong convergence analysis is given as $\lim_{n\to\infty} d(STx_n, TSx_n)$.

1. Introduction

Throughout, we assume that $\mathbb{H}$ is a real Hilbert space with the inner $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$ and $\mathbb{C}\subset\mathbb{H}$ is a nonempty closed convex set.

Definition 1. A mapping $\mathbb{T}:\mathbb{C}\to\mathbb{C}$ is said to be *pseudocontractive* if

$$\left\langle \mathbb{T}u-\mathbb{T}u^{\dagger}, u-u^{\dagger}\right\rangle \le \left\|u-u^{\dagger}\right\|^2 \tag{1}$$

for all $u,u^{\dagger}\in\mathbb{C}$.

We use $\mathrm{Fix}(\mathbb{T})$ to denote the set of fixed points of $\mathbb{T}$.

Remark 2. It is easily seen that (1) is equivalent to the following:

$$\left\|\mathbb{T}u-\mathbb{T}u^{\dagger}\right\|^2 \le \left\|u-u^{\dagger}\right\|^2 + \left\|(\mathbb{I}-\mathbb{T})u-(\mathbb{I}-\mathbb{T})u^{\dagger}\right\|^2 \tag{2}$$

for all $u,u^{\dagger}\in\mathbb{C}$.

Definition 3. A mapping $\mathbb{T}:\mathbb{C}\to\mathbb{C}$ is said to be *L-Lipschitzian* if

$$\left\|\mathbb{T}u-\mathbb{T}u^{\dagger}\right\| \le L\left\|u-u^{\dagger}\right\| \tag{3}$$

for all $u,u^{\dagger}\in\mathbb{C}$, where $L>0$ is a constant.

If $L=1$, $\mathbb{T}$ is said to be *nonexpansive*.

One of our purposes of this paper is to find the fixed points of the pseudocontractive mappings in Hilbert spaces. In the literature, there are a large number references associated with the fixed point algorithms for the pseudocontractive mappings. See, for instance, [1–9]. The first interesting algorithm for finding the fixed points of the Lipschitz pseudocontractive mappings in Hilbert spaces was presented by Ishikawa [4] in 1974.

Ishikawa's Algorithm. For any $x_0\in\mathbb{C}$, define the sequence $\{x_n\}$ iteratively by

$$\begin{aligned} y_n &= \left(1-\varpi_n\right)x_n+\varpi_n\mathbb{T}x_n,\\ x_{n+1} &= \left(1-\varrho_n\right)x_n+\varrho_n\mathbb{T}y_n \end{aligned} \tag{4}$$

for all $n \in \mathbb{N}$, where $\{\varpi_n\} \subset [0,1]$ and $\{\varrho_n\} \subset [0,1]$ satisfy the following conditions:

(a) $\lim_{n\to\infty}\varpi_n = 0$;

(b) $\sum_{n=1}^{\infty} \varpi_n \varrho_n = \infty$.

Ishikawa proved that the sequence $\{x_n\}$ generated by (4) converges strongly to a fixed point of $\mathbb{T}$ provided $\mathbb{C}$ is a compact set.

Recently, Zhou [9] suggested the following algorithm.

Zhou's Algorithm. For any $x_0 \in \mathbb{C}$, define the sequence $\{x_n\}$ iteratively by

$$\begin{aligned} y_n &= (1-\varrho_n) x_n + \varrho_n \mathbb{T} x_n, \\ z_n &= (1-\varpi_n) x_n + \varpi_n \mathbb{T} y_n, \\ \mathbb{C}_n &= \Big\{ z \in \mathbb{C} : \|z_n - z\|^2 \le \|x_n - z\|^2 \\ &\qquad - \varpi_n \varrho_n \left(1 - 2\varrho_n - \varrho_n^2 L^2\right) \|x_n - \mathbb{T} x_n\|^2 \Big\}, \\ \mathbb{Q}_n &= \{ z \in \mathbb{C} : \langle x_n - z, x_0 - x_n \rangle \ge 0 \}, \\ x_{n+1} &= \mathrm{proj}_{\mathbb{C}_n \cap \mathbb{Q}_n}(x_0), \quad n \in \mathbb{N}, \end{aligned} \tag{5}$$

where $\{\varpi_n\}$ and $\{\varrho_n\}$ are two real sequences in $(0,1)$ satisfying the following conditions:

(a) $\varpi_n \le \varrho_n$ for all $n \in \mathbb{N}$;

(b) $0 < \liminf_{n\to\infty}\varrho_n \le \limsup_{n\to\infty}\varrho_n \le \varrho < 1/(\sqrt{1+L^2}+1)$.

Zhou proved that the sequence $\{x_n\}$ generated by (5) converges strongly to $\mathrm{proj}_{\mathrm{Fix}(\mathbb{T})}(x_0)$ without the compactness assumption.

Definition 4. A mapping $\mathbb{A} : \mathbb{C} \to \mathbb{H}$ is said to be *inverse strongly monotone* if there exists $\zeta > 0$ such that

$$\langle u - v, \mathbb{A}u - \mathbb{A}v \rangle \ge \zeta \|\mathbb{A}u - \mathbb{A}v\|^2 \tag{6}$$

for all $u, v \in \mathbb{C}$.

The variational inequality problem is to find $u \in \mathbb{C}$ such that

$$\langle \mathbb{A}u, v - u \rangle \ge 0, \quad \forall v \in \mathbb{C}. \tag{7}$$

The set of solutions of the variational inequality problem is denoted by $\mathrm{VI}(\mathbb{C}, \mathbb{A})$. It is well known that variational inequality theory has emerged as an important tool in studying a wide class of obstacles, unilateral and equilibrium problems, which arise in several branches of pure and applied sciences in a unified and general framework. For related work, please refer to [10–18] and the references therein.

Motivated and inspired by the related work on the fixed point problem and the variational inequality problem in the literature, the purpose of this paper is continuous to study algorithmic approach to the fixed point problem of the pseudocontractive mappings and the variational inequality problem in Hilbert spaces. We suggest a hybrid algorithm with Meir-Keeler contraction and consequently we prove the strong convergence of the presented algorithm.

2. Preliminaries

Recall that the metric projection $\mathrm{proj}_{\mathbb{C}} : \mathbb{H} \to \mathbb{C}$ satisfies

$$\|u - \mathrm{proj}_{\mathbb{C}}(u)\| = \inf\left\{ \left\|u - u^\dagger\right\| : u^\dagger \in \mathbb{C} \right\}. \tag{8}$$

The metric projection $\mathrm{proj}_{\mathbb{C}}$ is a typical firmly nonexpansive mapping, that is,

$$\begin{aligned} &\left\|\mathrm{proj}_{\mathbb{C}}(u) - \mathrm{proj}_{\mathbb{C}}\left(u^\dagger\right)\right\|^2 \\ &\quad \le \left\langle \mathrm{proj}_{\mathbb{C}}(u) - \mathrm{proj}_{\mathbb{C}}\left(u^\dagger\right), u - u^\dagger \right\rangle \end{aligned} \tag{9}$$

for all $u, u^\dagger \in \mathbb{H}$.

It is well known that, in a real Hilbert space $\mathbb{H}$, the following equality holds:

$$\begin{aligned} &\left\|\xi u + (1-\xi) u^\dagger\right\|^2 \\ &\quad = \xi\|u\|^2 + (1-\xi)\left\|u^\dagger\right\|^2 - \xi(1-\xi)\left\|u - u^\dagger\right\|^2 \end{aligned} \tag{10}$$

for all $u, u^\dagger \in \mathbb{H}$ and $\xi \in [0,1]$.

Lemma 5 (see [9]). *Let $\mathbb{H}$ be a real Hilbert space and let $\mathbb{C}$ be a closed convex subset of $\mathbb{H}$. Let $\mathbb{T} : \mathbb{C} \to \mathbb{C}$ be a continuous pseudocontractive mapping. Then,*

(i) $\mathrm{Fix}(\mathbb{T}) \subset \mathbb{C}$ *is a closed convex set;*

(ii) $(\mathbb{I} - \mathbb{T})$ *is demiclosed at zero.*

Let $\{\mathbb{C}_n\} \subset \mathbb{H}$ be a sequence of nonempty closed convex sets. We define the symbols $s\text{-}Li_n\mathbb{C}_n$ and $w\text{-}Ls_n\mathbb{C}_n$ as follows.

(1) $x^* \in s\text{-}Li_n\mathbb{C}_n \Leftrightarrow$ there exists $\{x_n\} \subset \mathbb{C}_n$ such that $x_n \to x^*$ strongly.

(2) $x^\dagger \in w\text{-}Ls_n\mathbb{C}_n \Leftrightarrow$ there exist a subsequence $\{\mathbb{C}_{n_i}\}$ of $\{\mathbb{C}_n\}$ and a sequence $\{x_n\}$ in $\mathbb{C}_{n_i}$ such that $x_n \rightharpoonup x^\dagger$ weakly.

If $\mathbb{C}_0$ satisfies the following:

$$\mathbb{C}_0 = s\text{-}Li_n\mathbb{C}_n = w\text{-}Ls_n\mathbb{C}_n, \tag{11}$$

then we say that $\{\mathbb{C}_n\}$ converges to $\mathbb{C}_0$ in the sense of Mosco [19] and we write $\mathbb{C}_0 = M\text{-}\lim_{n\to\infty}\mathbb{C}_n$. It is easy to show that if $\{\mathbb{C}_n\}$ is nonincreasing with respect to inclusion, then $\{\mathbb{C}_n\}$ converges to $\bigcap_{n=1}^{\infty}\mathbb{C}_n$ in the sense of Mosco.

Tsukada [20] proved the following theorem for the metric projection.

Lemma 6 (see [20]). *Let $\mathbb{H}$ be a Hilbert space. Let $\{\mathbb{C}_n\}$ be a sequence of nonempty closed convex subsets of $\mathbb{H}$. If $\mathbb{C}_0 = M\text{-}\lim_{n\to\infty}\mathbb{C}_n$ exists and is nonempty, then, for each $x \in \mathbb{H}$, $\{proj_{\mathbb{C}_n}(x)\}$ converges strongly to $proj_{\mathbb{C}_0}(x)$, where $proj_{\mathbb{C}_n}$ and $proj_{\mathbb{C}_0}$ are the metric projections of $\mathbb{H}$ onto $\mathbb{C}_n$ and $\mathbb{C}_0$, respectively.*

Let $(\mathbb{E}, d)$ be a complete metric space. A mapping $\psi : \mathbb{E} \to \mathbb{E}$ is called a *Meir-Keeler contraction* [21] if, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$d\left(u, u^\dagger\right) < \epsilon + \delta \Longrightarrow d\left(\psi(u), \psi\left(u^\dagger\right)\right) < \epsilon \tag{12}$$

for all $u, u^\dagger \in \mathbb{E}$. It is well known that the Meir-Keeler contraction is a generalization of the contraction.

Lemma 7 (see [21]). *A Meir-Keeler contraction defined on a complete metric space has a unique fixed point.*

Lemma 8 (see [22]). *Let ψ be a Meir-Keeler contraction on a convex subset $\mathbb{C}$ of a Banach space $\mathbb{E}$. Then, for any $\epsilon > 0$, there exists $\sigma \in (0,1)$ such that*

$$\left\|u - u^\dagger\right\| \geq \epsilon \Longrightarrow \left\|\psi(u) - \psi\left(u^\dagger\right)\right\| \leq \sigma \left\|u - u^\dagger\right\| \quad (13)$$

for all $u, u^\dagger \in \mathbb{C}$.

Lemma 9 (see [22]). *Let $\mathbb{C}$ be a convex subset of a Banach space $\mathbb{E}$. Let $\mathbb{T}$ be a nonexpansive mapping on $\mathbb{C}$ and let ψ be a Meir-Keeler contraction on $\mathbb{C}$. Then the following holds.*

(i) *$\mathbb{T}\psi$ is a Meir-Keeler contraction on $\mathbb{C}$.*

(ii) *For each $\zeta \in (0,1)$, $(1-\zeta)\mathbb{T} + \alpha\psi$ is a Meir-Keeler contraction on $\mathbb{C}$.*

3. Main Results

In this section, we firstly introduce a hybrid iterative algorithm for finding the common element of the fixed point problem and the variational inequality problem.

Algorithm 10. Let $\mathbb{H}$ be a real Hilbert space and $\mathbb{C} \subset \mathbb{H}$ a nonempty closed convex set. Let $\psi : \mathbb{C} \to \mathbb{C}$ be a Meir-Keeler contractive mapping. Let $\mathbb{A} : \mathbb{C} \to \mathbb{H}$ be a inverse strongly monotone mapping. Let $\mathbb{T} : \mathbb{C} \to \mathbb{C}$ be a κ-Lipschitz pseudocontractive mapping with $\kappa > 1$. For $x_0 \in \mathbb{C}_0 = \mathbb{C}$ arbitrarily, define a sequence $\{x_n\}$ iteratively by

$$\begin{aligned} u_n &= \operatorname{proj}_{\mathbb{C}}\left[x_n - \alpha\mathbb{A}x_n\right], \\ v_n &= \left(1-\varrho_n\right)u_n + \varrho_n\mathbb{T}u_n, \\ w_n &= \left(1-\varpi_n\right)u_n + \varpi_n\mathbb{T}v_n, \\ \mathbb{C}_{n+1} &= \left\{\mu \in \mathbb{C}_n : \left\|w_n - \mu\right\| \leq \left\|x_n - \mu\right\|\right\}, \\ x_{n+1} &= \operatorname{proj}_{\mathbb{C}_{n+1}}\psi\left(x_n\right), \quad n \in \mathbb{N}, \end{aligned} \quad (14)$$

where $\alpha \in (0, 2\lambda)$ is a constant and $\{\varpi_n\}$ and $\{\varrho_n\}$ are two real number sequences in $(0,1)$ satisfying $0 < c_1 < \varpi_n \leq \varrho_n < c_2 < 1/(\sqrt{1+\kappa^2}+1)$.

Next, we show the strong convergence of (14).

Theorem 11. *Suppose that $\Lambda = \mathrm{VI}(\mathbb{C}, \mathbb{A}) \cap \mathrm{Fix}(\mathbb{T}) \neq \emptyset$. Then the sequence $\{x_n\}$ defined by (14) converges strongly to $x^\dagger = proj_\Lambda\psi(x^\dagger)$.*

Remark 12. Note that Λ is a closed convex subset of $\mathbb{C}$. Thus $\operatorname{proj}_\Lambda$ is well defined. Since ψ is a Meir-Keeler contraction of $\mathbb{C}$, it follows that $\operatorname{proj}_\Lambda\psi$ is a Meir-Keeler contraction of $\mathbb{C}$ by Lemma 9. According to Lemma 7, there exists a unique fixed point $x^\dagger \in \mathbb{C}$ such that $x^\dagger = \operatorname{proj}_\Lambda\psi(x^\dagger)$.

Proof. The outline of our proof is as follows.

Step 1. $\Lambda \subset \mathbb{C}_n$ for all $n \in \mathbb{N}$;

Step 2. $\mathbb{C}_n$ is closed and convex for all $n \in \mathbb{N}$;

Step 3. $\lim_{n\to\infty}\|x_n - \nu\| = 0$ where $\nu = \operatorname{proj}_{\bigcap_{n=1}^{\infty}\mathbb{C}_n}\psi(\nu)$;

Step 4. $\nu \in \mathrm{Fix}(\mathbb{T})$;

Step 5. $\nu \in \mathrm{VI}(\mathbb{C}, \mathbb{A})$;

Step 6. $\nu = \operatorname{proj}_\Lambda\psi(\nu) = x^\dagger$.

Proof of Step 1. We prove this step by induction. (i) $\Lambda \subset \mathbb{C}_0$ is obvious. (ii) Suppose that $\Lambda \subset \mathbb{C}_k$ for some $k \in \mathbb{N}$. Pick up $x^* \in \Lambda \subset \mathbb{C}_k$. Then, we have

$$\begin{aligned} \left\|u_n - x^*\right\| &= \left\|\operatorname{proj}_{\mathbb{C}}\left[x_n - \alpha\mathbb{A}x_n\right] - \operatorname{proj}_{\mathbb{C}}\left[x^* - \alpha\mathbb{A}x^*\right]\right\| \\ &\leq \left\|\left(x_n - \alpha\mathbb{A}x_n\right) - \left(x^* - \alpha\mathbb{A}x^*\right)\right\| \\ &\leq \left\|x_n - x^*\right\|. \end{aligned} \quad (15)$$

By (2), we have

$$\left\|\mathbb{T}u_n - x^*\right\|^2 \leq \left\|u_n - x^*\right\|^2 + \left\|\mathbb{T}u_n - u_n\right\|^2, \quad (16)$$

$$\begin{aligned} \left\|\mathbb{T}v_n - x^*\right\|^2 &= \left\|\mathbb{T}\left(\left(1-\varrho_n\right)u_n + \varrho_n\mathbb{T}u_n\right) - x^*\right\|^2 \\ &\leq \left\|\left(1-\varrho_n\right)\left(u_n - x^*\right) + \varrho_n\left(\mathbb{T}u_n - x^*\right)\right\|^2 \\ &\quad + \left\|\left(1-\varrho_n\right)u_n + \varrho_n\mathbb{T}u_n - \mathbb{T}v_n\right\|^2. \end{aligned} \quad (17)$$

From (10), we obtain

$$\begin{aligned} &\left\|\left(1-\varrho_n\right)u_n + \varrho_n\mathbb{T}u_n - \mathbb{T}v_n\right\|^2 \\ &\quad = \left\|\left(1-\varrho_n\right)\left(u_n - \mathbb{T}v_n\right) + \varrho_n\left(\mathbb{T}u_n - \mathbb{T}v_n\right)\right\|^2 \\ &\quad = \left(1-\varrho_n\right)\left\|u_n - \mathbb{T}v_n\right\|^2 + \varrho_n\left\|\mathbb{T}u_n - \mathbb{T}v_n\right\|^2 \\ &\qquad - \varrho_n\left(1-\varrho_n\right)\left\|u_n - \mathbb{T}u_n\right\|^2. \end{aligned} \quad (18)$$

Since T is κ-Lipschitzian and $u_n - v_n = \varrho_n(u_n - \mathbb{T}u_n)$, by (18), we get

$$\begin{aligned} &\left\|\left(1-\varrho_n\right)u_n + \varrho_n\mathbb{T}u_n - \mathbb{T}v_n\right\|^2 \\ &\quad \leq \left(1-\varrho_n\right)\left\|u_n - \mathbb{T}v_n\right\|^2 + \varrho_n^3\kappa^2\left\|u_n - \mathbb{T}u_n\right\|^2 \\ &\qquad - \varrho_n\left(1-\varrho_n\right)\left\|u_n - \mathbb{T}u_n\right\|^2 \\ &\quad = \left(1-\varrho_n\right)\left\|u_n - \mathbb{T}v_n\right\|^2 \\ &\qquad + \left(\varrho_n^3\kappa^2 + \varrho_n^2 - \varrho_n\right)\left\|u_n - \mathbb{T}u_n\right\|^2. \end{aligned} \quad (19)$$

By (10) and (16), we have

$$\begin{aligned}\|(1-\varrho_n)(u_n-x^*)+\varrho_n(\mathbb{T}u_n-x^*)\|^2 &= \|(1-\varrho_n)(u_n-x^*)+\varrho_n(\mathbb{T}u_n-x^*)\|^2 \\ &= (1-\varrho_n)\|u_n-x^*\|^2+\varrho_n\|\mathbb{T}u_n-x^*\|^2 \\ &\quad -\varrho_n(1-\varrho_n)\|u_n-\mathbb{T}u_n\|^2 \\ &\le (1-\varrho_n)\|u_n-x^*\|^2+\varrho_n\left(\|u_n-x^*\|^2+\|u_n-\mathbb{T}u_n\|^2\right) \\ &\quad -\varrho_n(1-\varrho_n)\|u_n-\mathbb{T}u_n\|^2 \\ &= \|u_n-x^*\|^2+\varrho_n^2\|u_n-\mathbb{T}u_n\|^2.\end{aligned} \tag{20}$$

From (17), (19), and (20), we deduce

$$\begin{aligned}\|\mathbb{T}v_n-x^*\|^2 \le{}& \|u_n-x^*\|^2+(1-\varrho_n)\|u_n-\mathbb{T}v_n\|^2 \\ &-\varrho_n\left(1-2\varrho_n-\varrho_n^2\kappa^2\right)\|u_n-\mathbb{T}u_n\|^2.\end{aligned} \tag{21}$$

Since $\varrho_n < c_2 < 1/(\sqrt{1+\kappa^2}+1)$, we have

$$1-2\varrho_n-\varrho_n^2\kappa^2 > 0 \tag{22}$$

for all $n \in \mathbb{N}$. This together with (21) implies that

$$\|\mathbb{T}v_n-x^*\|^2 \le \|u_n-x^*\|^2+(1-\varrho_n)\|u_n-\mathbb{T}v_n\|^2. \tag{23}$$

By (10), (15), and (23) and noting that $\varpi_n \le \varrho_n$, we have

$$\begin{aligned}\|w_n-x^*\|^2 &= \|(1-\varpi_n)u_n+\varpi_n\mathbb{T}v_n-x^*\|^2 \\ &= (1-\varpi_n)\|u_n-x^*\|^2+\varpi_n\|\mathbb{T}v_n-x^*\|^2 \\ &\quad -\varpi_n(1-\varpi_n)\|u_n-\mathbb{T}v_n\|^2 \\ &\le \|u_n-x^*\|^2-\varpi_n(\varrho_n-\varpi_n)\|\mathbb{T}v_n-x^*\|^2 \\ &\le \|u_n-x^*\|^2 \\ &\le \|x_n-x^*\|^2\end{aligned} \tag{24}$$

and hence $x^* \in \mathbb{C}_{k+1}$. This indicates that $\Lambda \subset \mathbb{C}_n$ for all $n \in \mathbb{N}$.

Proof of Step 2. In fact, it is obvious from the assumption that $\mathbb{C}_0 = \mathbb{C}$ is closed convex. Suppose that $\mathbb{C}_k$ is closed and convex for some $k \in \mathbb{N}$. For any $\mu \in \mathbb{C}_k$, we know that $\|y_k-\mu\| \le \|x_k-\mu\|$ is equivalent to

$$\|y_k-x_k\|^2+2\langle y_k-x_k, x_k-\mu\rangle \le 0. \tag{25}$$

So $\mathbb{C}_{k+1}$ is closed and convex. By induction, we deduce that $\mathbb{C}_n$ is closed and convex for all $n \in \mathbb{N}$.

Proof of Step 3. Firstly, from Step 2, we note that $\{x_n\}$ is well defined. Since $\bigcap_{n=1}^{\infty}\mathbb{C}_n$ is closed convex, we also have that $\mathrm{proj}_{\bigcap_{n=1}^{\infty}\mathbb{C}_n}$ is well defined and so $\mathrm{proj}_{\bigcap_{n=1}^{\infty}\mathbb{C}_n}\psi$ is a Meir-Keeler contraction on $\mathbb{C}$. By Lemma 7, there exists a unique fixed point $\nu \in \bigcap_{n=1}^{\infty}\mathbb{C}_n$ of $\mathrm{proj}_{\bigcap_{n=1}^{\infty}\mathbb{C}_n}\psi$. Since $\mathbb{C}_n$ is a nonincreasing sequence of nonempty closed convex subsets of $\mathbb{H}$ with respect to inclusion, it follows that

$$\emptyset \ne \Lambda \subset \bigcap_{n=1}^{\infty}\mathbb{C}_n = M\text{-}\lim_{n\to\infty}\mathbb{C}_n. \tag{26}$$

Setting $s_n := \mathrm{proj}_{\mathbb{C}_n}\psi(\nu)$ and applying Lemma 6, we can conclude that

$$\lim_{n\to\infty}s_n = \mathrm{proj}_{\bigcap_{n=1}^{\infty}\mathbb{C}_n}\psi(\nu) = \nu. \tag{27}$$

Now, we show that $\lim_{n\to\infty}\|x_n-\nu\| = 0$. Assume that $M = \overline{\lim}_{n\to\infty}\|x_n-\nu\| > 0$. Then, for any ϵ with $0 < \epsilon < M$, we can choose $\delta_1 > 0$ such that

$$\overline{\lim_{n\to\infty}}\|x_n-\nu\| > \epsilon+\delta_1. \tag{28}$$

Since ψ is a Meir-Keeler contraction, for the positive ϵ, there exists another $\delta_2 > 0$ such that

$$\|x-y\| < \epsilon+\delta_2 \Longrightarrow \|\psi(x)-\psi(y)\| < \epsilon \tag{29}$$

for all $x, y \in \mathbb{C}$.

In fact, we can choose a common $\delta > 0$ such that (28) and (29) hold. If $\delta_1 > \delta_2$, then

$$\overline{\lim_{n\to\infty}}\|x_n-\nu\| > \epsilon+\delta_1 > \epsilon+\delta_2. \tag{30}$$

If $\delta_1 \le \delta_2$, then, from (29), it follows that

$$\|x-y\| < \epsilon+\delta_1 \Longrightarrow \|\psi(x)-\psi(y)\| < \epsilon \tag{31}$$

for all $x, y \in \mathbb{C}$. Thus, we have

$$\overline{\lim_{n\to\infty}}\|x_n-\nu\| > \epsilon+\delta, \tag{32}$$

$$\|x-y\| < \epsilon+\delta \Longrightarrow \|\psi(x)-\psi(y)\| < \epsilon \tag{33}$$

for all $x, y \in \mathbb{C}$. Since $s_n \to \nu$, there exists $n_0 \in \mathbb{N}$ such that

$$\|s_n-\nu\| < \delta \tag{34}$$

for all $n \ge n_0$.

Now, we consider two possible cases.

Case 1. There exists $n_1 \ge n_0$ such that

$$\|x_{n_1}-\nu\| \le \epsilon+\delta. \tag{35}$$

By (33) and (34), we get

$$\begin{aligned}\|x_{n_1+1}-\nu\| &\le \|x_{n_1+1}-s_{n_1+1}\|+\|s_{n_1+1}-\nu\| \\ &= \left\|\mathrm{proj}_{\mathbb{C}_{n_1+1}}\psi(x_{n_1})-\mathrm{proj}_{\mathbb{C}_{n_1+1}}\psi(\nu)\right\| \\ &\quad +\|s_{n_1+1}-\nu\| \\ &\le \|\psi(x_{n_1})-\psi(\nu)\|+\|s_{n_1+1}-\nu\| \\ &\le \epsilon+\delta.\end{aligned} \tag{36}$$

By induction, we can obtain that

$$\left\|x_{n_1+m} - \nu\right\| \leq \epsilon + \delta \tag{37}$$

for all $m \geq 1$, which implies that

$$\overline{\lim_{n\to\infty}} \left\|x_n - \nu\right\| \leq \epsilon + \delta, \tag{38}$$

which contradicts (32). Therefore, we conclude that $\|x_n - \nu\| \to 0$ as $n \to \infty$.

Case 2 ($\|x_n - \nu\| > \epsilon + \delta$ for all $n \geq n_0$). Now, we prove that Case 2 is impossible. Suppose that Case 2 is true. By Lemma 8, there exists $\sigma \in (0, 1)$ such that

$$\left\|\psi(x_n) - \psi(\nu)\right\| \leq \sigma \left\|x_n - \nu\right\| \tag{39}$$

for all $n \geq n_0$. Thus we have

$$\begin{aligned} \left\|x_{n+1} - s_{n+1}\right\| &= \left\|\text{proj}_{\mathbb{C}_{n+1}} \psi(x_n) - \text{proj}_{\mathbb{C}_{n+1}} \psi(\nu)\right\| \\ &\leq \left\|\psi(x_n) - \psi(\nu)\right\| \\ &\leq \sigma \left\|x_n - \nu\right\| \end{aligned} \tag{40}$$

for all $n \geq n_0$. It follows that

$$\begin{aligned} \overline{\lim_{n\to\infty}} \left\|x_{n+1} - \nu\right\| &= \overline{\lim_{n\to\infty}} \left\|x_{n+1} - s_{n+1}\right\| \\ &\leq \sigma \overline{\lim_{n\to\infty}} \left\|x_n - \nu\right\| \\ &< \overline{\lim_{n\to\infty}} \left\|x_n - \nu\right\|, \end{aligned} \tag{41}$$

which gives a contradiction. Hence we obtain

$$\lim_{n\to\infty} \left\|x_n - \nu\right\| = 0. \tag{42}$$

Proof of Step 4. By Step 3, we deduce immediately that $\{x_n\}$ is bounded. Observe that

$$\begin{aligned} \left\|x_{n+1} - x_n\right\| &\leq \left\|x_n - \nu\right\| + \left\|\nu - s_{n+1}\right\| + \left\|s_{n+1} - x_{n+1}\right\| \\ &= \left\|x_n - \nu\right\| + \left\|\nu - s_{n+1}\right\| \\ &\quad + \left\|\text{proj}_{\mathbb{C}_{n+1}} \psi(x_n) - \text{proj}_{\mathbb{C}_{n+1}} \psi(\nu)\right\| \\ &\leq \left\|x_n - \nu\right\| + \left\|\nu - s_{n+1}\right\| + \left\|\psi(x_n) - \psi(\nu)\right\|. \end{aligned} \tag{43}$$

Therefore, we have

$$\lim_{n\to\infty} \left\|x_{n+1} - x_n\right\| = 0. \tag{44}$$

Since $x_{n+1} \in \mathbb{C}_{n+1}$, we have

$$\left\|w_n - x_{n+1}\right\| \leq \left\|x_n - x_{n+1}\right\|. \tag{45}$$

This together with (44) implies that

$$\lim_{n\to\infty} \left\|w_n - x_{n+1}\right\| = \lim_{n\to\infty} \left\|w_n - x_n\right\| = 0. \tag{46}$$

From (15) and (24), we have

$$\begin{aligned} \left\|w_n - x^*\right\|^2 &\leq \left\|u_n - x^*\right\|^2 \\ &\leq \left\|(x_n - \alpha\mathbb{A}x_n) - (x^* - \alpha\mathbb{A}x^*)\right\|^2 \\ &= \left\|x_n - x^*\right\|^2 + \alpha^2\left\|\mathbb{A}x_n - \mathbb{A}x^*\right\|^2 \\ &\quad - 2\left\langle \mathbb{A}x_n - \mathbb{A}x^*, x_n - x^*\right\rangle \\ &\leq \left\|x_n - x^*\right\|^2 + \alpha(\alpha - 2\lambda)\left\|\mathbb{A}x_n - \mathbb{A}x^*\right\|^2. \end{aligned} \tag{47}$$

Then we have

$$\begin{aligned} &(2\lambda - \alpha)\alpha\left\|\mathbb{A}x_n - \mathbb{A}x^*\right\|^2 \\ &\quad \leq \left\|x_n - x^*\right\|^2 - \left\|w_n - x^*\right\|^2 \\ &\quad \leq \left\|x_n - w_n\right\|\left(\left\|x_n - x^*\right\| + \left\|w_n - x^*\right\|\right). \end{aligned} \tag{48}$$

By (46) and (48), we obtain

$$\lim_{n\to\infty} \left\|\mathbb{A}x_n - \mathbb{A}x^*\right\| = 0. \tag{49}$$

Since $\text{proj}_{\mathbb{C}}$ is firmly nonexpansive, we have

$$\begin{aligned} &\left\|u_n - x^*\right\|^2 \\ &\quad = \left\|\text{proj}_{\mathbb{C}}\left[x_n - \alpha\mathbb{A}x_n\right] - \text{proj}_{\mathbb{C}}\left[x^* - \alpha\mathbb{A}x^*\right]\right\|^2 \\ &\quad \leq \left\langle (x_n - \alpha\mathbb{A}x_n) - (x^* - \alpha\mathbb{A}x^*), u_n - x^*\right\rangle \\ &\quad = \frac{1}{2}\left(\left\|(x_n - \alpha\mathbb{A}x_n) - (x^* - \alpha\mathbb{A}x^*)\right\|^2 + \left\|u_n - x^*\right\|^2\right. \\ &\qquad \left. - \left\|(x_n - \alpha\mathbb{A}x_n) - (x^* - \alpha\mathbb{A}x^*) + x^* - u_n\right\|^2\right) \\ &\quad \leq \frac{1}{2}\left(\left\|x_n - x^*\right\|^2 + \left\|u_n - x^*\right\|^2\right. \\ &\qquad \left. - \left\|(x_n - u_n) - \alpha(\mathbb{A}x_n - \mathbb{A}x^*)\right\|^2\right) \\ &\quad = \frac{1}{2}\left(\left\|x_n - x^*\right\|^2 + \left\|u_n - x^*\right\|^2 - \left\|x_n - u_n\right\|^2\right. \\ &\qquad \left. + 2\alpha\left\langle x_n - u_n, \mathbb{A}x_n - \mathbb{A}x^*\right\rangle - \alpha^2\left\|\mathbb{A}x_n - \mathbb{A}x^*\right\|^2\right). \end{aligned} \tag{50}$$

It follows that

$$\begin{aligned} \left\|u_n - x^*\right\|^2 &\leq \left\|x_n - x^*\right\|^2 - \left\|x_n - u_n\right\|^2 \\ &\quad + 2\alpha\left\langle x_n - u_n, \mathbb{A}x_n - \mathbb{A}x^*\right\rangle \\ &\quad - \alpha^2\left\|\mathbb{A}x_n - \mathbb{A}x^*\right\|^2. \end{aligned} \tag{51}$$

From (24) and (51), we get

$$\begin{aligned}\|w_n - x^*\|^2 &\le \|u_n - x^*\|^2 \\ &\le \|x_n - x^*\|^2 - \|x_n - u_n\|^2 \\ &\quad + 2\alpha \langle x_n - u_n, \mathbb{A}x_n - \mathbb{A}x^* \rangle - \alpha^2 \|\mathbb{A}x_n - \mathbb{A}x^*\|^2 \\ &\le \|x_n - x^*\|^2 - \|x_n - u_n\|^2 \\ &\quad + 2\alpha \|x_n - u_n\| \|\mathbb{A}x_n - \mathbb{A}x^*\|\end{aligned} \tag{52}$$

and so

$$\begin{aligned}\|x_n - u_n\|^2 &\le \|x_n - x^*\|^2 - \|w_n - x^*\|^2 \\ &\quad + 2\alpha \|x_n - u_n\| \|\mathbb{A}x_n - \mathbb{A}x^*\| \\ &\le \|x_n - w_n\| (\|x_n - x^*\| + \|w_n - x^*\|) \\ &\quad + 2\alpha \|x_n - u_n\| \|\mathbb{A}x_n - \mathbb{A}x^*\|.\end{aligned} \tag{53}$$

This together with (46) and (49) implies that

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{54}$$

Note that

$$\begin{aligned}\|u_n - \mathbb{T}u_n\| &\le \|u_n - w_n\| + \|w_n - \mathbb{T}u_n\| \\ &\le \|u_n - w_n\| + (1 - \varpi_n) \|u_n - \mathbb{T}u_n\| \\ &\quad + \varpi_n \|\mathbb{T}v_n - \mathbb{T}u_n\| \\ &\le \|u_n - w_n\| + (1 - \varpi_n) \|u_n - \mathbb{T}u_n\| \\ &\quad + \varpi_n \kappa \|v_n - u_n\| \\ &\le \|u_n - w_n\| + (1 - \varpi_n) \|u_n - \mathbb{T}u_n\| \\ &\quad + \varpi_n \varrho_n \kappa \|u_n - \mathbb{T}u_n\|.\end{aligned} \tag{55}$$

It follows that

$$\begin{aligned}\|u_n - \mathbb{T}u_n\| &\le \frac{1}{\varpi_n (1 - \varrho_n \kappa)} \|u_n - w_n\| \\ &\le \frac{1}{c_1 (1 - c_2 \kappa)} \|u_n - w_n\| \longrightarrow 0.\end{aligned} \tag{56}$$

Since $x_n \to \nu$, we have $u_n \to \nu$ by (54). So, from (56) and Lemma 5, we deduce that $\nu \in \mathrm{Fix}(\mathbb{T})$.

Proof of Step 5. Define a mapping ϕ by

$$\phi(\nu) = \begin{cases} \mathbb{A}\nu + N_{\mathbb{C}}\nu, & \nu \in \mathbb{C}, \\ \emptyset, & \nu \notin \mathbb{C}. \end{cases} \tag{57}$$

Then ϕ is maximal monotone (see [15]). Let $(\nu, w) \in G(\phi)$. Since $w - \mathbb{A}\nu \in N_{\mathbb{C}}\nu$ and $u_n \in \mathbb{C}$, we have $\langle \nu - u_n, w - \mathbb{A}\nu \rangle \ge 0$. On the other hand, from $u_n = \mathrm{proj}_{\mathbb{C}}[x_n - \alpha \mathbb{A}x_n]$, we have

$$\langle \nu - u_n, u_n - (x_n - \alpha \mathbb{A}x_n) \rangle \ge 0, \tag{58}$$

that is,

$$\left\langle \nu - u_n, \frac{u_n - x_n}{\alpha} + \mathbb{A}x_n \right\rangle \ge 0. \tag{59}$$

Therefore, we have

$$\begin{aligned}\langle \nu - u_n, w \rangle &\ge \langle \nu - u_n, \mathbb{A}\nu \rangle \\ &\ge \langle \nu - u_n, \mathbb{A}\nu \rangle - \left\langle \nu - u_n, \frac{u_n - x_n}{\alpha} + \mathbb{A}x_n \right\rangle \\ &= \left\langle \nu - u_n, \mathbb{A}\nu - \mathbb{A}x_n - \frac{u_n - x_n}{\alpha} \right\rangle \\ &= \langle \nu - u_n, \mathbb{A}\nu - \mathbb{A}u_n \rangle + \langle \nu - u_n, \mathbb{A}u_n - \mathbb{A}x_n \rangle \\ &\quad - \left\langle \nu - u_n, \frac{u_n - x_n}{\alpha} \right\rangle \\ &\ge \langle \nu - u_n, \mathbb{A}u_n - \mathbb{A}x_n \rangle - \left\langle \nu - u_n, \frac{u_n - x_n}{\alpha} \right\rangle.\end{aligned} \tag{60}$$

Noting that $\|u_n - x_n\| \to 0$ and $\mathbb{A}$ is Lipschitz continuous, we obtain $\langle \nu - \nu, w \rangle \ge 0$. Since ϕ is maximal monotone, we have $\nu \in \phi^{-1}(0)$ and hence $\nu \in \mathrm{VI}(\mathbb{C}, \mathbb{A})$.

Proof of Step 6. Since $x_{n+1} = \mathrm{proj}_{\mathbb{C}_{n+1}} \psi(x_n)$, we have

$$\langle \psi(x_n) - x_{n+1}, x_{n+1} - y \rangle \ge 0 \tag{61}$$

for all $y \in \mathbb{C}_{n+1}$. Since $\Lambda \subset \mathbb{C}_{n+1}$, we get

$$\langle \psi(x_n) - x_{n+1}, x_{n+1} - y \rangle \ge 0 \tag{62}$$

for all $y \in \Lambda$. Noting that $x_n \to \nu \in \Lambda$, we deduce

$$\langle \psi(\nu) - \nu, \nu - y \rangle \ge 0 \tag{63}$$

for all $y \in \Lambda$. Thus $\nu = \mathrm{proj}_{\Lambda} \psi(\nu) = x^{\dagger}$. This completes the proof. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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