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Research Article

Unified Common Fixed Point Theorems for a Hybrid Pair of Mappings via an Implicit Relation Involving Altering Distance Function

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The object of this paper is to emphasize the role of a suitable implicit relation involving altering distance function which covers a multitude of contraction conditions in one go. By using this implicit relation, we prove a new coincidence and common fixed point theorem for a hybrid pair of occasionally coincidentally idempotent mappings in a metric space employing the common limit range property. Our main result improves and generalizes a host of previously known results. We also utilize suitable illustrative examples to substantiate the realized improvements in our results.

1. Introduction and Preliminaries

Fixed point theory is one of the most rapidly growing research areas in nonlinear functional analysis. Apart from numerous extensions of Banach Contraction Principle for single valued mappings, it was also naturally extended to multivalued mappings by Nadler Jr. [1] in 1969 which is also sometimes referred to as Nadler Contraction Principle. Since then, there has been continuous and intense research activity in multimap fixed point theory (including hybrid fixed point results) and by now there exists an extensive literature on this specific theme (see, e.g., [2–7] and the references therein). The study of common fixed points of mappings satisfying hybrid contraction conditions has been at the center of vigorous research activity. Here, it can be pointed out that hybrid fixed theorems have numerous applications in science and engineering.

In the following lines, we present some definitions and their implications which will be utilized throughout this paper.

Let (X, d) be a metric space. Then, on the lines of Nadler Jr. [1], we adopt that

(1) $CL(X) = \{A : A \text{ is a nonempty closed subset of } X\}$,

(2) $CB(X) = \{A : A \text{ is a nonempty closed and bounded subset of } X\}$,

(3) for nonempty closed and bounded subsets A, B of X and $x \in X$,

$$d(x, A) = \inf \{d(x, a) : a \in A\},$$

$$H(A, B) = \max \{\sup \{d(a, B) : a \in A\}, \sup \{d(b, A) : b \in B\}\}. \quad (1)$$

It is well known that $CB(X)$ is a metric space with the distance H which is known as the Hausdorff-Pompeiu metric on $CB(X)$. The following terminology is also standard.

Let (X, d) be a metric space with $f : X \to X$ and $T : X \to CB(X)$. Then

(1) a point $x \in X$ is a fixed point of f (resp., T) if $x = fx$ (resp., $x \in Tx$). The set of all fixed points of f (resp., T) is denoted by $F(f)$ (resp., $F(T)$);

(2) a point $x \in X$ is a coincidence point of f and T if $fx \in Tx$. The set of all coincidence points of f and T is denoted by $C(f,T)$;

(3) a point $x \in X$ is a common fixed point of f and T if $x = fx \in Tx$. The set of all common fixed points of f and T is denoted by $F(f,T)$.

In 1984, Khan et al. [8] utilized the idea of altering distance function in metric fixed point theory which is indeed a control function that alters distance between two points in a metric space. Thereafter, this idea has further been utilized by several mathematicians (see, e.g., [9–13]).

Definition 1 (see [8]). An altering distance function is a mapping $\psi : [0,\infty) \to [0,\infty)$ which satisfies that

(ψ_1) $\psi(t)$ is increasing and continuous and

(ψ_2) $\psi(t) = 0$ if and only if $t = 0$.

Certain ideas on commutativity and weak commutativity for a pair of hybrid mappings on metric spaces were utilized by Kaneko [14, 15]. In 1989, Singh et al. [16] extended the notion of compatible mappings to hybrid pair of mappings and proved some common fixed point theorems for nonlinear hybrid contractions. Such ways of proving new results continue to attract the attention of many researchers of this domain where it can be observed that under compatibility the fixed point results often require continuity of one of the underlying mappings.

Kamran [17] extended the property (E.A) (due to Aamri and El Moutawakil [18]) to a hybrid pair of mappings. Most recently, Imdad et al. [19] established common limit range property (essentially motivated by Sintunavarat and Kumam [20]) for a hybrid pair of mappings and proved some fixed point theorems in symmetric (or semimetric) spaces.

The notions of coincidentally idempotent and occasionally coincidentally idempotent hybrid pairs of mappings were, respectively, introduced and used by Imdad et al. [21] and Pathak and Rodríguez-López [22]. An easy and natural example is available in Kadelburg et al. [23, Example 1] exhibiting the importance of the occasionally coincidentally idempotent property over coincidentally idempotent property.

The technical definitions of the earlier mentioned notions are described in the following lines.

Definition 2. Let (X,d) be a metric space with $f : X \to X$ and $T : X \to CB(X)$. A hybrid pair of mappings (f,T) is said to be

(1) commuting on X [14] if $fTx \subseteq Tfx$ for all $x \in X$,

(2) weakly commuting on X [15] if $H(fTx, Tfx) \leq d(fx, Tx)$ for all $x \in X$,

(3) compatible [16] if $fTx \in CB(X)$ for all $x \in X$ and $\lim_{n\to\infty} H(Tfx_n, fTx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $Tx_n \to A \in CB(X)$ and $fx_n \to t \in A$, as $n \to \infty$,

(4) noncompatible [24] if there exists at least one sequence $\{x_n\}$ in X such that $Tx_n \to A \in CB(X)$ and $fx_n \to t \in A$, as $n \to \infty$, but $\lim_{n\to\infty} H(Tfx_n, fTx_n)$ is either nonzero or nonexistent,

(5) with the property (E.A) [17] if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = t \in A = \lim_{n\to\infty} Tx_n, \tag{2}$$

for some $t \in X$ and $A \in CB(X)$,

(6) with common limit range property with respect to the mapping f [19] if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = fu \in A = \lim_{n\to\infty} Tx_n, \tag{3}$$

for some $u \in X$ and $A \in CB(X)$,

(7) coincidentally idempotent [21] if $ffv = fv$ for every $v \in C(f,T)$; that is, f is idempotent at the coincidence points of f and, T, and

(8) occasionally coincidentally idempotent [22] if $ffv = fv$ for some $v \in C(f,T)$.

Some relations between the introduced notions can be seen in [25, 26].

Remark 3. Note that if a pair (f,T) satisfies the property (E.A) along with the closedness of $f(X)$, then the pair also satisfies the common limit range property with respect to the mapping f (see Theorem 7). However, common limit range property may be satisfied without the closedness of $f(X)$ (e.g., Example 6).

In this paper, an attempt has been made to derive common fixed point theorems for a hybrid pair of mappings using the notion of common limit range property with occasionally coincidentally idempotent property involving implicit relations and altering distance. The presented theorems extend and unify various known fixed point results [21, 23, 24, 27–37]. Some examples are also furnished to exhibit that our results are proper extensions of the known ones.

2. Implicit Relations

In the recent past, implicit relations have been utilized to prove unified common fixed points results covering various kinds of contraction mappings in one go. In fact, this idea was initiated by Popa [38, 39], where he introduced an implicit function which covers a variety of contraction classes. In [33], Popa and Patriciu introduced the following implicit function and utilized the same to prove some coincidence and common fixed point results for hybrid pair of mappings covering several contraction conditions.

In what follows, Φ will be the set of all continuous functions $\phi : \mathbb{R}_+^6 \to \mathbb{R}$ satisfying the following conditions:

(ϕ_1) ϕ is nondecreasing in its first variable;

(ϕ_2) $\phi(t,0,0,t,t,0) \leq 0$ implies $t = 0$.

Example 4 (see [33]). The following functions are the examples of implicit function belonging to the set Φ:

(1)

$$\phi(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - \max\left\{t_2, \frac{t_3+t_4}{2}, \frac{t_5+t_6}{2}\right\}, \quad (4)$$

(2)

$$\phi(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - k\max\{t_2,t_3,t_4,t_5,t_6\}, \quad (5)$$

where $0 < k < 1$,

(3)

$$\phi(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - at_2 - b(t_3+t_4) - c(t_5+t_6), \quad (6)$$

where $a, b, c \geq 0$ and $b + c < 1$,

(4)

$$\phi(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - at_2 - b(t_3+t_4) - c\min\{t_5,t_6\}, \quad (7)$$

where $a, c \geq 0$ and $0 < b < 1$,

(5)

$$\phi(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - at_2 - b(t_3+t_4) - c\sqrt{t_5t_6}, \quad (8)$$

where $a, c \geq 0$ and $0 < b < 1$,

(6)

$$\phi(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - at_2 - b\max\{t_3,t_4\} - c\max\{t_5,t_6\}, \quad (9)$$

where $a, b, c \geq 0$ and $0 < b + c < 1$,

(7)

$$\phi(t_1,t_2,t_3,t_4,t_5,t_6) = t_1^2 - t_2^2 - a\frac{t_3^2+t_4^2}{1+\min\{t_5,t_6\}}, \quad (10)$$

where $0 < a < 1$,

(8)

$$\phi(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - \max\{t_2,t_3,t_4\} - (1-\alpha)\max\{at_5+bt_6\}, \quad (11)$$

where $0 \leq \alpha < 1$, $0 < a < 1$ and $b \geq 0$,

(9)

$$\phi(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - \max\{ct_2,ct_3,ct_4,at_5+bt_6\}, \quad (12)$$

where $a, b, c \geq 0$ and $\max\{a, c\} < 1$,

(10)

$$\phi(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - \max\left\{t_2, k\left(\frac{t_3+t_4}{2}\right), \frac{t_5+t_6}{2}\right\}, \quad (13)$$

where $0 < k < 1$,

(11)

$$\phi(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - \max\{k_1(t_2+t_3+t_4), k_2(t_5+t_6)\}, \quad (14)$$

where $k_1, k_2 \geq 0$ and $\max\{k_1, k_2\} < 1$, and

(12)

$$\phi(t_1,t_2,t_3,t_4,t_5,t_6) = t_1^2 - \frac{t_3^2t_4^2+t_5^2t_6^2}{1+t_2}. \quad (15)$$

Certainly, apart from foregoing examples, there are many other functions that meet the requirements (ϕ_1) and (ϕ_2).

3. Main Results

Now we prove our main result.

Theorem 5. *Let f be a self-mapping of a metric space (X, d) and T a mapping from X into $CB(X)$ satisfying*

$$\phi(\psi(H(Tx,Ty)), \psi(d(fx,fy)), \psi(d(fx,Tx)), \psi(d(fy,Ty)), \psi(d(fx,Ty)), \psi(d(fy,Tx))) \leq 0, \quad (16)$$

for all $x, y \in X$, where $\phi \in \Phi$ and $\psi(t)$ is an altering distance function. Suppose that the pair (f, T) satisfies the common limit range property with respect to the mapping f. Then the mappings f and T have a coincidence point (i.e., $C(f, T) \neq \emptyset$).

Moreover, the mappings f and T have a common fixed point in X provided that the pair (f, T) is occasionally coincidentally idempotent.

Proof. Suppose that the pair (f, T) enjoys the common limit range property with respect to the mapping f. Then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = fu \in A = \lim_{n\to\infty} Tx_n, \quad (17)$$

for some $u \in X$ and $A \in CB(X)$. First we show that $fu \in Tu$. To accomplish this, using inequality (16) with $x = x_n$ and $y = u$, we obtain

$$\phi(\psi(H(Tx_n,Tu)), \psi(d(fx_n,fu)), \psi(d(fx_n,Tx_n)), \psi(d(fu,Tu)), \psi(d(fx_n,Tu)), \psi(d(fu,Tx_n))) \leq 0. \quad (18)$$

Taking the limit as $n \to \infty$, we have

$$\phi(\psi(H(A,Tu)), \psi(d(fu,fu)), \psi(d(fu,A)), \psi(d(fu,Tu)), \psi(d(fu,Tu)), \psi(d(fu,A))) \leq 0. \quad (19)$$

Since $fu \in A$, we have $d(fu, Tu) \le H(A, Tu)$. Using (ϕ_1) in inequality (19), we have

$$\phi(\psi(d(fu,Tu)), \psi(0), \psi(0), \psi(d(fu,Tu)), \psi(d(fu,Tu)), \psi(0)) \le 0, \quad (20)$$

or, equivalently,

$$\phi(\psi(d(fu,Tu)), 0, 0, \psi(d(fu,Tu)), \psi(d(fu,Tu)), 0) \le 0. \quad (21)$$

From the condition (ϕ_2), we have $\psi(d(fu,Tu)) = 0$ which implies $d(fu,Tu) = 0$. Hence $fu \in Tu$ which shows that the pair (f,T) has a coincidence point (i.e., $C(f,T) \ne \emptyset$) so that the set of coincidence points is nonempty.

In the case that the mappings f and T are occasionally coincidentally idempotent; then $ffv = fv$ for some $v \in C(f,T)$, which implies $ffv = fv \in Tv$. Now we assert that $fv \in Tfv$. On using inequality (16) with $x = v$ and $y = fv$, we get

$$\phi(\psi(H(Tv,Tfv)), \psi(d(fv,ffv)), \psi(d(fv,Tv)), \psi(d(ffv,Tfv)), \psi(d(fv,Tfv)), \psi(d(ffv,Tv))) \le 0. \quad (22)$$

Since $fv \in Tv$, we have $d(Tfv, fv) \le H(Tfv, Tv)$. Using (ϕ_1) in inequality (22), we get

$$\phi(\psi(d(fv,Tfv)), \psi(0), \psi(0), \psi(d(fv,Tfv)), \psi(d(fv,Tfv)), \psi(0)) \le 0, \quad (23)$$

or, equivalently,

$$\phi(\psi(d(fv,Tfv)), 0, 0, \psi(d(fv,Tfv)), \psi(d(fv,Tfv)), 0) \le 0. \quad (24)$$

In view of (ϕ_2), we have $\psi(d(fv,Tfv)) = 0$ which implies $d(fv,Tfv) = 0$; that is, $fv \in Tfv$. Thus in all, we have $fv = ffv \in Tfv$ which shows that fv is a common fixed point of the mappings f and T. □

Example 6. Consider $X = [0,+\infty)$ equipped with the standard metric. Define mappings $f : X \to X$ and $T : X \to CB(X)$ as follows:

$$fx = \begin{cases} 2x^2, & 0 \le x < 1; \\ x^2 + x + 3, & 1 \le x < 2; \\ 2x^2 - 3, & x \ge 2, \end{cases} \qquad Tx = [0, x^2 + 1]. \quad (25)$$

Then

(i) $f(X) = [0,2) \cup [5,+\infty)$ is not closed in X;

(ii) $C(f,T) = [0,1) \cup \{2\}$;

(iii) for $x_n = 1/n$, it is $\lim_{n\to\infty} fx_n = 0 = f0 \in [0,1] = T0 = \lim_{n\to\infty} Tx_n$; hence, (f,T) enjoys the common limit range property with respect to the mapping f;

(iv) $ff0 = f0 = 0$; hence, (f,T) is occasionally coincidentally idempotent;

(v) $ff2 = f5 = 47 \ne f2$; hence, (f,T) is not coincidentally idempotent.

Define $\psi(t) = t/2$ (which is an altering distance function), while $\phi \in \Phi$ is given by

$$\phi(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - \max\left\{t_2, \frac{1}{2}\left(\frac{t_3+t_4}{2}\right), \frac{t_5+t_6}{2}\right\} \quad (26)$$

(see Example 4(10) with $k = 1/2$). Write

$$\begin{aligned} L &= \phi(\psi(H(Tx,Ty)), \psi(d(fx,fy)), \psi(d(fx,Tx)), \psi(d(fy,Ty)), \psi(d(fx,Ty)), \psi(d(fy,Tx))) \\ &= \psi(H(Tx,Ty)) \\ &\quad - \max\left\{\psi(d(fx,fy)), \frac{1}{2}\left(\frac{\psi(d(fx,Tx)) + \psi(d(fy,Ty))}{2}\right), \frac{\psi(d(fx,Ty)) + \psi(d(fy,Tx))}{2}\right\}. \end{aligned} \quad (27)$$

In order to check the contractive condition (16) of Theorem 5, without loss of generality, we can suppose that $0 \le x < y < \infty$. Then $\psi(H(Tx,Ty)) = (1/2)(y^2 - x^2)$. Consider the following possible cases.

(1) For the case $0 \le x < y < 1$, we have

$$\begin{aligned} L &\le \psi(H(Tx,Ty)) - \psi(d(fx,fy)) \\ &= \frac{1}{2}(y^2 - x^2) - \frac{1}{2}\cdot 2(y^2 - x^2) \\ &= -\frac{1}{2}(y^2 - x^2) < 0. \end{aligned} \quad (28)$$

(2) In the case $1 \le x < y < 2$, one finds

$$\begin{aligned} L &\le \psi(H(Tx,Ty)) - \psi(d(fx,fy)) \\ &= \frac{1}{2}(y^2 - x^2) - \frac{1}{2}(y^2 + y - x^2 - x) \\ &= -\frac{1}{2}(y - x) < 0. \end{aligned} \quad (29)$$

(3) If $2 \leq x < y$, then one can show on the lines of case (1) that $L < 0$.

(4) If $0 \leq x < 1 \leq y < 2$, then $d(fx, Ty) = 0$, $d(fy, Tx) = y^2 + y + 3 - (x^2 + 1) = y^2 + y + 2 - x^2$, and

$$\begin{aligned} L &\leq \psi(H(Tx, Ty)) - \frac{\psi(d(fx, Ty)) + \psi(d(fy, Tx))}{2} \\ &= \frac{y^2 - x^2}{2} - \frac{y^2 + y + 2 - x^2}{4} \\ &= \frac{y^2 - y - 2 - x^2}{4} < 0. \end{aligned} \tag{30}$$

(5) If $0 \leq x < 1$, $y \geq 2$, then

$$\begin{aligned} L &\leq \psi(H(Tx, Ty)) - \psi(d(fx, fy)) \\ &= \frac{1}{2}(y^2 - x^2) - \frac{1}{2}(2y^2 - 3 - 2x^2) \\ &= \frac{1}{2}(3 - y^2 + x^2) < 0. \end{aligned} \tag{31}$$

(6) The case $1 \leq x < 2 \leq y$ is similar to the case (2).

Thus, all the conditions of Theorem 5 are satisfied and the pair (f, T) has common fixed points (which are 0 and 1/2).

The same conclusion cannot be reached using [35, Theorem 3.1] or [33, Theorem 4.1], as $f(X)$ is not closed and (f, T) is not coincidentally idempotent.

Theorem 7. *Let f be a self-mapping of a metric space (X, d) and T a mapping from X into $CB(X)$ satisfying inequality (16) of Theorem 5. Suppose that the pair (f, T) satisfies the property (E.A) and $f(X)$ is a closed subset of X; then the mappings f and T have a coincidence point (i.e., $C(f, T) \neq \emptyset$).*

Moreover, the mappings f and T have a common fixed point in X provided that the pair (f, T) is occasionally coincidentally idempotent.

Proof. If the pair (f, T) enjoys the property (E.A), then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n \to \infty} fx_n = z \in A = \lim_{n \to \infty} Tx_n, \tag{32}$$

for some $z \in X$ and $A \in CB(X)$. Since $f(X)$ is a closed subset of X, there exists some $u \in X$ such that $z = fu$. Hence condition (32) implies

$$\lim_{n \to \infty} fx_n = fu \in A = \lim_{n \to \infty} Tx_n, \tag{33}$$

for some $u \in X$ and $A \in CB(X)$ which shows that the pair (f, T) also satisfies the common limit range property with respect to the mapping f. Now, the conclusions follow from Theorem 5. □

Example 8. Consider $X = [0, 2]$ equipped with the standard metric. Define mappings $f : X \to X$ and $T : X \to CB(X)$ as

$$fx = \begin{cases} 2 - x, & 0 \leq x \leq 1; \\ 2, & 1 < x \leq 2, \end{cases} \qquad Tx = \begin{cases} \left[\dfrac{x+1}{2}, \dfrac{5}{4}\right], & 0 \leq x \leq 1; \\ \left[\dfrac{3}{4}, 1\right], & 1 < x \leq 2. \end{cases} \tag{34}$$

Then

(i) $f(X) = [1, 2]$ is closed in X;

(ii) $C(f, T) = [3/4, 1]$;

(iii) for $x_n = 1 - 1/n$, it is $\lim_{n \to \infty} fx_n = 1 \in [1, 5/4] = \lim_{n \to \infty} Tx_n$; hence, (f, T) enjoys the property (E.A);

(iv) $ff1 = f1 = 1$; hence, (f, T) is occasionally coincidentally idempotent;

(v) $ff(3/4) = f(5/4) = 2 \neq f(3/4)$; hence, (f, T) is not coincidentally idempotent.

Take $\psi(t) = 2t$ (which is an altering distance function) and $\phi \in \Phi$ given by

$$\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \frac{1}{2}t_2 - \frac{1}{2}(t_3 + t_4) \tag{35}$$

(see Example 4(3) with $a = b = 1/2$, $c = 0$). Denote

$$\begin{aligned} L &= \phi(\psi(H(Tx, Ty)), \psi(d(fx, fy)), \psi(d(fx, Tx)), \\ &\qquad \psi(d(fy, Ty)), \psi(d(fx, Ty)), \psi(d(fy, Tx))) \\ &= \psi(H(Tx, Ty)) - \frac{1}{2}\psi(d(fx, fy)) \\ &\quad - \frac{1}{2}(\psi(d(fx, Tx)) + \psi(d(fy, Ty))). \end{aligned} \tag{36}$$

In order to check the contractive condition (16) of Theorem 5, without loss of generality, we can suppose that $0 \leq x < y \leq 2$. Consider the following possible cases.

(1) Consider $0 \leq x < y \leq 1$. Then

$$\begin{aligned} L &\leq \psi(H(Tx, Ty)) - \frac{1}{2}\psi(d(fx, fy)) \\ &= 2 \cdot \frac{y - x}{2} - \frac{1}{2} \cdot 2(y - x) = 0. \end{aligned} \tag{37}$$

(2) Consider $1 < x < y \leq 2$. Then $L \leq \psi(H(Tx, Ty)) - (1/2)\psi(d(fx, fy)) = 0 - 0 = 0$.

(3) Consider $0 \leq x \leq 1 < y \leq 2$. Then $H(Tx, Ty) = 1/4$, $d(fx, Tx) \geq 0$, $d(fy, Ty) = 1$, and

$$\begin{aligned} L &\leq \psi(H(Tx, Ty)) - \frac{1}{2}[\psi(d(fx, Tx)) + \psi(d(fy, Ty))] \\ &\leq 2 \cdot \frac{1}{4} - \frac{1}{2}[0 + 2 \cdot 1] = -\frac{1}{2} < 0. \end{aligned} \tag{38}$$

Hence, all the conditions of Theorem 7 are fulfilled and the pair (f, T) has a common fixed point (which is 1).

The same conclusion cannot be obtained using, for example, [35, Theorem 3.1] or [33, Theorem 4.1], since (f, T) is not coincidentally idempotent.

Notice that a noncompatible hybrid pair always satisfies the property (E.A). Hence, we get the following corollary.

Corollary 9. *Let f be a self-mapping of a metric space (X, d) and T a mapping from X into $CB(X)$ satisfying inequality* (16) *of Theorem 5. Suppose that the pair (f, T) is noncompatible and $f(X)$ is a closed subset of X. Then the mappings f and T have a coincidence point (i.e., $C(f, T) \neq \emptyset$).*

Moreover, the mappings f and T have a common fixed point in X provided that the pair (f, T) is occasionally coincidentally idempotent.

Corollary 10. *The conclusions of Theorems 5 and 7 and Corollary 9 remain true if inequality* (16) *is replaced by one of the following contraction conditions. For all $x, y \in X$ and some $\psi \in \Psi$,*

(1)

$$\psi(H(Tx, Ty)) \leq \max\left\{\psi(d(fx, fy)), \frac{\psi(d(fx, Tx)) + \psi(d(fy, Ty))}{2}, \frac{\psi(d(fx, Ty)) + \psi(d(fy, Tx))}{2}\right\}, \quad (39)$$

(2)

$$\begin{aligned}\psi(H(Tx, Ty)) &\leq k \max\{\psi(d(fx, fy)), \psi(d(fx, Tx)), \\ &\quad \psi(d(fy, Ty)), \\ &\quad \psi(d(fx, Ty)), \psi(d(fy, Tx))\},\end{aligned} \quad (40)$$

where $0 < k < 1$,

(3)

$$\begin{aligned}\psi(H(Tx, Ty)) &\leq a\psi(d(fx, fy)) \\ &\quad + b[\psi(d(fx, Tx)) + \psi(d(fy, Ty))] \\ &\quad + c[\psi(d(fx, Ty)) + \psi(d(fy, Tx))],\end{aligned} \quad (41)$$

where $a, b, c \geq 0$ and $b + c < 1$,

(4)

$$\begin{aligned}\psi(H(Tx, Ty)) &\leq a\psi(d(fx, fy)) \\ &\quad + b[\psi(d(fx, Tx)) + \psi(d(fy, Ty))] \\ &\quad + c\min\{\psi(d(fx, Ty)), \psi(d(fy, Tx))\},\end{aligned} \quad (42)$$

where $a, c \geq 0$ and $0 < b < 1$,

(5)

$$\begin{aligned}\psi(H(Tx, Ty)) &\leq a\psi(d(fx, fy)) \\ &\quad + b[\psi(d(fx, Tx)) + \psi(d(fy, Ty))] \\ &\quad + c\sqrt{\psi(d(fx, Ty))\psi(d(fy, Tx))},\end{aligned} \quad (43)$$

where $a, c \geq 0$ and $0 < b < 1$,

(6)

$$\begin{aligned}\psi(H(Tx, Ty)) &\leq a\psi(d(fx, fy)) \\ &\quad + b\max\{\psi(d(fx, Tx)), \psi(d(fy, Ty))\} \\ &\quad + c\max\{\psi(d(fx, Ty)), \psi(d(fy, Tx))\},\end{aligned} \quad (44)$$

where $a, b, c \geq 0$ and $0 < b + c < 1$,

(7)

$$\begin{aligned}\psi(H^2(Tx, Ty)) &\leq \psi(d^2(fx, fy)) \\ &\quad + a\frac{\psi(d^2(fx, Tx)) + \psi(d^2(fy, Ty))}{1 + \min\{\psi(d(fx, Ty)), \psi(d(fy, Tx))\}},\end{aligned} \quad (45)$$

where $0 < a < 1$,

(8)

$$\begin{aligned}\psi(H(Tx, Ty)) &\leq \max\{\psi(d(fx, fy)), \psi(d(fx, Tx)), \psi(d(fy, Ty))\} \\ &\quad + (1 - \alpha)(a\psi(d(fx, Ty)) + b\psi(d(fy, Tx))),\end{aligned} \quad (46)$$

where $0 \leq \alpha < 1$, $0 < a < 1$, and $b \geq 0$,

(9)

$$\begin{aligned}\psi(H(Tx, Ty)) &\leq \max\{c\psi(d(fx, fy)), \\ &\quad c\psi(d(fx, Tx)), c\psi(d(fy, Ty)), \\ &\quad a\psi(d(fx, Ty)) + b\psi(d(fy, Tx))\},\end{aligned} \quad (47)$$

where $a, b, c \geq 0$ and $\max\{a, c\} < 1$,

(10)

$$\psi(H(Tx,Ty)) \leq \max\left\{\psi(d(fx,fy)), k\frac{\psi(d(fx,Tx))+\psi(d(fy,Ty))}{2}, \frac{\psi(d(fx,Ty))+\psi(d(fy,Tx))}{2}\right\}, \tag{48}$$

where $0 < k < 1$,

(11)

$$\psi(H(Tx,Ty)) \leq \max\{k_1(\psi(d(fx,fy)) + \psi(d(fx,Tx)) + \psi(d(fy,Ty))), k_2(\psi(d(fx,Ty)) + \psi(d(fy,Tx)))\}, \tag{49}$$

where $k_1, k_2 \geq 0$ *and* $\max\{k_1, k_2\} < 1$,

(12)

$$\psi(H^2(Tx,Ty)) \leq (\psi(d^2(fx,Tx))\psi(d^2(fy,Ty)) + \psi(d^2(fx,Ty))\psi(d^2(fy,Tx))) \times (1+\psi(d(fx,fy)))^{-1}. \tag{50}$$

Proof. The conclusion follows from Theorem 5 in view of Example 4, (1)–(12). □

On setting $\psi(t) = t$ in the earlier defined theorems involving altering distance, we can get some natural results which improve hybrid type fixed point results given in the literature. For the sake of simplicity, we only derive the following corollary by putting $\psi(t) = t$ in Theorem 5.

Corollary 11. *Let f be a self-mapping of a metric space (X,d) and T a mapping from X into $CB(X)$ satisfying*

$$\phi(H(Tx,Ty), d(fx,fy), d(fx,Tx), d(fy,Ty), d(fx,Ty), d(fy,Tx)) \leq 0, \tag{51}$$

for all $x, y \in X$ and some $\phi \in \Phi$. Suppose that the pair (f,T) satisfies the common limit range property with respect to the mapping f. Then the mappings f and T have a coincidence point (i.e., $C(f,T) \neq \emptyset$).

Moreover, the mappings f and T have a common fixed point in X provided that the pair (f,T) is occasionally coincidentally idempotent.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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