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Research Article

Criterion on $L^{p_1} \times L^{p_2} \to L^q$-Boundedness for Oscillatory Bilinear Hilbert Transform

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We investigate the bilinear Hilbert transform with oscillatory factors and the truncated bilinear Hilbert transform. The main result is that the $L^{p_1} \times L^{p_2} \to L^q$-boundedness of the two operators is equivalent with $1 \leq p_1, p_2 < \infty$, and $1/q = 1/p_1 + 1/p_2$. In addition, we also discuss the boundedness of a variant operator of bilinear Hilbert transform with a nontrivial polynomial phase.

1. Introduction

In this paper, we mainly discuss the following operator defined by

$$\mathcal{T}(f, g)(x) = \text{p.v.} \int_{\mathbb{R}} e^{iP(x,t)} f(x-t)\, g(x+t) \frac{dt}{t}, \quad (1)$$

where $P(\cdot,\cdot)$ is a real-valued polynomial defined on $\mathbb{R}\times\mathbb{R}$ and f, g are smooth functions with compact support.

Clearly, when $P = 0$, the operator $\mathcal{T}$ becomes the normal bilinear Hilbert transform defined as

$$T(f, g)(x) = \text{p.v.} \int_{\mathbb{R}} f(x-t)\, g(x+t) \frac{dt}{t}. \quad (2)$$

If $P(x,t) = P_1(x-t) + P_2(x+t) + P_3(x)$ for some one-variable polynomials P_1, P_2, and P_3, then clearly the boundedness of $\mathcal{T}$ and T is equivalent. For this case, we introduce the following definitions.

Definition 1. For a polynomial $P(\cdot,\cdot)$ defined on $\mathbb{R}\times\mathbb{R}$, one calls P degenerate, if

$$P(x,t) = P_1(x-t) + P_2(x+t) + P_3(x) \quad (3)$$

holds with three one-variable polynomials P_1, P_2, and P_3.

In fact, a simple verification shows that if the degree of $P(\cdot,\cdot)$ is less than 3, then P is degenerate. However, when $d \geq 3$, there exist nondegenerate polynomials with the degree d. We will prove this in Section 2. It should be pointed out that Definition 1 may be extended to such polynomials defined on $\mathbb{R}^n \times \mathbb{R}^n$ (see [1]). We will give a characterization of nondegenerate polynomials by Ind_β in Section 2. Now we first give the definition of Ind_β.

Definition 2. Let $P(\cdot,\cdot)$ be a polynomial with the degree $d \geq 3$. For $1 \leq \beta \leq d-2$, denote Ind_β by

$$\text{Ind}_\beta(P)(x,t) = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right)^{\beta} \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial t}\right)^{d-1-\beta} \frac{\partial}{\partial t} P(x,t). \quad (4)$$

For the operator $\mathcal{T}$ defined by (1), the following result has been obtained in [1].

Theorem A. *Let $P(\cdot,\cdot)$ be a real-valued polynomial. Then the operator $\mathcal{T}$ extends as a bounded operator from $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R})$ to $L^q(\mathbb{R})$ with $1 < p_1, p_2 < \infty$, and $q > 2/3$ such that $1/q = 1/p_1 + 1/p_2$; furthermore, the operator norm $\|\mathcal{T}\|$ depends only on the degree of P.*

In the study of oscillatory singular integrals, the boundedness of the original operator without oscillatory factors may imply the boundedness of the corresponding truncated operator. For the Calderón-Zygmund singular integral operators of convolution type, one can see Ricci and Stein [2]. For the multilinear singular operators, one may refer to [1]. In [3]

Lu and Zhang established a criterion for the boundedness of the following operator:

$$\mathscr{K} f(x) = \text{p.v.} \int_{\mathbb{R}^n} e^{iP(x,t)} K(x-y) f(y)\, dy, \tag{5}$$

where $P(\cdot,\cdot)$ is a real-valued polynomial defined on $\mathbb{R}^n \times \mathbb{R}^n$, $f \in C_0^\infty(\mathbb{R}^n)$, and $K(x) = \Omega(x')/|x|^n$ satisfies

(1) Ω is homogeneous of the degree 0 on S^{n-1};

(2) Ω has mean value zero on S^{n-1};

(3) $\Omega \in L^q(S^{n-1})$ for some $1 < q \leq \infty$.

For the sake of clarity, we also present the theorem obtained by Lu and Zhang as follows.

Theorem B. *Suppose that $\mathscr{K}$ is defined by (5) and $1 < p < \infty$. Each of the following statements implies the other two.*

(i) *If $P(\cdot,\cdot)$ is not of the form $P(x,y) = P_1(x) + P_2(y)$ for some polynomials P_1 and P_2 in $\mathbb{R}^n$, then the operator*

$$\mathscr{K} f(x) = p.v. \int_{\mathbb{R}^n} e^{iP(x,y)} K(x-y) f(y)\, dy \tag{6}$$

is bounded on $L^p(\mathbb{R}^n)$.

(ii) *If $Q(\cdot,\cdot)$ satisfies $Q(x+h, y+h) = Q(x,y) + R_1(x,h) + R_2(y,h)$ for some polynomials R_1, R_2, and $h \in \mathbb{R}^n$, then the operator*

$$\mathscr{G} f(x) = p.v. \int_{\mathbb{R}^n} e^{iQ(x,y)} K(x-y) f(y)\, dy \tag{7}$$

extends as a bounded operator from $L^p(\mathbb{R}^n)$ to itself.

(iii) *The following truncated operator*

$$\mathcal{S} f(x) = p.v. \int_{|x-y|<1} K(x-y) f(y)\, dy \tag{8}$$

is bounded on $L^p(\mathbb{R}^n)$.

For the bilinear Hilbert transform and the corresponding truncated operator, we can formulate some analogous results as Theorem B.

Theorem 3. *Suppose that $0 < p_1, p_2, q < \infty$ satisfy $1/q = 1/p_1 + 1/p_2$. If the truncated operator*

$$S(f,g)(x) = p.v. \int_{|t|<1} f(x-t)\, g(x+t) \frac{dt}{t} \tag{9}$$

is bounded from $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R})$ to $L^q(\mathbb{R})$, then the bilinear Hilbert transform

$$T(f,g)(x) = p.v. \int_{\mathbb{R}} f(x-t)\, g(x+t) \frac{dt}{t} \tag{10}$$

is bounded from $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R})$ to $L^q(\mathbb{R})$. Moreover, when $1 \leq p_1, p_2 < \infty$, the converse also holds; that is, if T extends to a bounded operator from $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R})$ to $L^q(\mathbb{R})$, then so does S.

Theorem 4. *Let $P(\cdot,\cdot)$ be a real-valued polynomial and let $1 \leq p_1, p_2 < \infty$, and $q \geq 1/2$ satisfy $1/q = 1/p_1 + 1/p_2$. If $P(x+h,t) - P(x,t)$ is degenerate with respect to x, t for all $h \in \mathbb{R}$ in the sense of Definition 1 and the following operator*

$$\mathscr{T}(f,g)(x) = p.v. \int_{\mathbb{R}} e^{iP(x,t)} f(x-t)\, g(x+t) \frac{dt}{t} \tag{11}$$

is bounded from $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R})$ to $L^q(\mathbb{R})$, then the truncated operator

$$S(f,g)(x) = p.v. \int_{|t|<1} f(x-t)\, g(x+t) \frac{dt}{t} \tag{12}$$

is bounded from $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R})$ to $L^q(\mathbb{R})$.

We summarize the above two theorems as follows.

Theorem 5. *Suppose that $1 \leq p_1, p_2 < \infty$, and $q \geq 1/2$ satisfy $1/q = 1/p_1 + 1/p_2$. Each of the following three statements implies the other two.*

(i) *The bilinear Hilbert transform*

$$T(f,g)(x) = p.v. \int_{\mathbb{R}} f(x-t)\, g(x+t) \frac{dt}{t} \tag{13}$$

is bounded from $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R})$ to $L^q(\mathbb{R})$.

(ii) *The oscillatory bilinear Hilbert transform*

$$\mathscr{T}(f,g)(x) = p.v. \int_{\mathbb{R}} e^{iP(x,t)} f(x-t)\, g(x+t) \frac{dt}{t} \tag{14}$$

is bounded from $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R})$ to $L^q(\mathbb{R})$ for any real-valued polynomial P.

(iii) *The truncated operator*

$$S(f,g)(x) = p.v. \int_{|t|<1} f(x-t)\, g(x+t) \frac{dt}{t} \tag{15}$$

is bounded from $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R})$ to $L^q(\mathbb{R})$.

Next, we will consider a variant operator of bilinear Hilbert transform defined by

$$T_\sigma(f,g)(x) = \int_{\mathbb{R}} e^{iP(x,t)} f(x-t)\, g(x+t) \frac{dt}{(1+|t|)^\sigma} \tag{16}$$

for $\sigma \geq 1$. Clearly, the operator $T_\sigma(f,g)$ has no singular property at origin, so it is the key at ∞.

We will use the power decay property of a bilinear functional to obtain the following theorem.

Theorem 6. *Suppose that $P(\cdot,\cdot)$ is a real-valued nondegenerate polynomial with the degree $d \geq 3$ and $|Ind_\beta(P)| \geq 1$ for some $1 \leq \beta \leq d-2$. If $1 \leq p_1, p_2 < \infty$, and $q \geq 1/2$ satisfy $1/q = 1/p_1 + 1/p_2$, then the operator T_σ defined by (16) is bounded from $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R})$ to $L^q(\mathbb{R})$; that is,*

$$\|T_\sigma(f,g)\|_q \leq C \|f\|_{p_1} \|g\|_{p_2}, \tag{17}$$

where C is a positive constant.

2. Some Lemmas

Throughout the paper, we always assume that $1 \leq p_1, p_2 < \infty$, and $1/q = 1/p_1 + 1/p_2$, unless the contrary is explicitly stated. We first introduce some lemmas which are useful in the proof of the main theorems.

Lemma 7. *Suppose that P defined on $\mathbb{R} \times \mathbb{R}$ is a homogeneous polynomial of the degree d with $d \geq 3$. Then P is degenerate if and only if $\mathrm{Ind}_\beta(P) = 0$ for all $1 \leq \beta \leq d-2$.*

Proof. By Definition 1, if P is degenerate, then

$$P(x,t) = P_1(x-t) + P_2(x+t) + Q(x), \tag{18}$$

where P_1, P_2, and Q are homogeneous single-variable polynomials of the degree d. Observe that

$$\begin{gathered} \frac{\partial}{\partial t} Q(x) = 0, \\ \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial t}\right) P_2(x+t) = 0, \\ \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right) P_1(x-t) = 0. \end{gathered} \tag{19}$$

This implies that

$$\mathrm{Ind}_\beta(P) = 0 \tag{20}$$

for all $1 \leq \beta \leq d-2$.

Now we assume that $\mathrm{Ind}_\beta(P) = 0$ for all $1 \leq \beta \leq d-2$. Make changes of variables

$$\begin{gathered} u = x + t \\ v = x - t. \end{gathered} \tag{21}$$

By the chain rule, we have that

$$\begin{aligned} \frac{\partial^\beta}{\partial u^\beta} &= \frac{1}{2^\beta}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right)^\beta, \\ \frac{\partial^{d-1-\beta}}{\partial v^{d-1-\beta}} &= \frac{1}{2^{d-1-\beta}}\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial t}\right)^{d-1-\beta}. \end{aligned} \tag{22}$$

Since P is a homogeneous polynomial with the degree d, $(\partial/\partial t)P$ is a homogeneous polynomial of the degree $d-1$. There, therefore, exists a homogeneous polynomial R of the degree $d-1$ such that

$$R(u,v) = \frac{\partial}{\partial t} P(x,t). \tag{23}$$

It follows from (23) that

$$\mathrm{Ind}_\beta(P) = 2^{d-1} \partial_u^\beta \partial_v^{d-1-\beta} R. \tag{24}$$

The assumption $\mathrm{Ind}_\beta(P) = 0$, for all $1 \leq \beta \leq d-2$, implies that

$$R(u,v) = c_1 u^{d-1} + c_2 v^{d-1} \tag{25}$$

for some constants c_1 and c_2. Integrating both sides of the equality (25) with respect to t gives that

$$P(x,t) = \frac{c_1}{d}(x+t)^d - \frac{c_2}{d}(x-t)^d + c_3 x^d, \tag{26}$$

for some constant c_3. Hence P is degenerate. □

Remark 8. For $d \geq 3$ and $1 \leq k \leq d-1$, $(x-t)^k (x+t)^{d-k}$ and t^d are nondegenerate polynomials. Furthermore, when P is a general polynomial of the degree d, we can write P as sums of homogeneous polynomials,

$$P(x,t) = \sum_{k=0}^{d} P_k(x,t), \tag{27}$$

where P_k is a homogeneous polynomial with the degree k. If P is degenerate, then each P_k is degenerate for $3 \leq k \leq d$. In fact, if P is degenerate, then $P(x,t) = P_1(x-t) + P_2(x+t) + Q(x)$. It follows that $P_k(x,t) = P_{1,k}(x-t) + P_{2,k}(x+t) + Q_k(x)$ for each $0 \leq k \leq d$. Consequently, each P_k is degenerate.

Lemma 9. *Let P be a polynomial with the degree $d \geq 3$. For any $\tau \in \mathbb{R}$, set*

$$\begin{aligned} Q_\tau(u,v) = {} & P\left(\frac{u+v}{2}, \frac{u-v}{2} + \tau\right) \\ & - P\left(\frac{u+v}{2}, \frac{u-v}{2}\right). \end{aligned} \tag{28}$$

Then, one has

$$\partial_u^\beta \partial_v^{d-1-\beta} Q_\tau(u,v) = 2^{1-d} \tau \mathit{Ind}_\beta(P) \tag{29}$$

for each $1 \leq \beta \leq d-2$.

Proof. By the same change of variables as in the proof of Lemma 7, we obtain

$$\begin{aligned} Q_\tau(u,v) &= P(x, t+\tau) - P(x,t) \\ &= \tau \int_0^1 \partial_t P(x, t + s\tau)\, ds. \end{aligned} \tag{30}$$

We can use the chain rule as in (22) to differentiate both sides of (28) by the differential operator $\partial_u^\beta \partial_v^{d-1-\beta}$ and easily yield the desired result. □

Lemma 10. *Suppose that k and n are two positive integers and that l_j, $1 \leq j \leq n$, are linear mappings from $\mathbb{R}^{(n-1)k}$ to $\mathbb{R}^k$. Assume that $1 \leq p_j < \infty$ for $j = 1, \ldots, n$ satisfy*

$$\frac{1}{q} = \sum_{j=1}^{n} \frac{1}{p_j}. \tag{31}$$

If the linear mapping $\Lambda: \mathbb{R}^{nk} \to \mathbb{R}^{nk}$ defined by

$$\Lambda(x,y) = (x + l_1(y), \ldots, x + l_n(y)) \tag{32}$$

is onto for $x \in \mathbb{R}^k$, $y \in \mathbb{R}^{(n-1)k}$, then one has

$$\begin{aligned} &I(f_1,\dots,f_n) \\ &= \left(\int_{\mathbb{R}^k}\left(\int_{\mathbb{R}^{(n-1)k}}\prod_{j=1}^{n}\left|f_j\left(x+l_j(y)\right)\right|\left|\eta(y)\right|dy\right)^q dx\right)^{1/q} \\ &\le C\prod_{j=1}^{n}\left\|f_j\right\|_{p_j}, \end{aligned} \tag{33}$$

where η is a bounded measurable function with compact support and $f_j \in L^{p_j}(\mathbb{R}^k)$ for $1 \le j \le n$.

Proof. For the case $q \ge 1$, applying generalized Minkowski's and Hölder's inequality gives that

$$\begin{aligned} &I(f_1,\dots,f_n) \\ &\le \int_{\mathbb{R}^{(n-1)k}}\left(\int_{\mathbb{R}^k}\left(\prod_{j=1}^{n}\left|f_j\left(x+l_j(y)\right)\right|\left|\eta(y)\right|\right)^q dx\right)^{1/q} dy \\ &\le \int_{\mathbb{R}^{(n-1)k}}\left|\eta(y)\right|dy \\ &\quad\times\prod_{j=1}^{n}\left(\int_{\mathbb{R}^k}\left|f_j\left(x+l_j(y)\right)\right|^{p_j}dx\right)^{1/p_j} \\ &= \|\eta\|_1\prod_{j=1}^{n}\left\|f_j\right\|_{p_j}. \end{aligned} \tag{34}$$

Now we deal with the case $0 < q < 1$. We first show that the following estimate

$$\begin{aligned} &\int_{|x-h|\le 1}\left(\int_{\mathbb{R}^{(n-1)k}}\prod_{j=1}^{n}\left|f_j\left(x+l_j(y)\right)\right|\left|\eta(y)\right|dy\right)^q dx \\ &\quad\le C\prod_{j=1}^{n}\left\|f_j\right\|^q_{L^{p_j}(|x-h|\le N)} \end{aligned} \tag{35}$$

holds uniformly for $h \in \mathbb{R}^k$, where N will be decided by the $\operatorname{supp}(\eta)$ and the norm of l_j, $1 \le j \le n$.

Since $0 < q < 1$, by Hölder's inequality, it is clear that we have that

$$\begin{aligned} &\int_{|x-h|\le 1}\left(\int_{\mathbb{R}^{(n-1)k}}\prod_{j=1}^{n}\left|f_j\left(x+l_j(y)\right)\right|\left|\eta(y)\right|dy\right)^q dx \\ &\quad\le C\left(\int_{|x-h|\le 1}\int_{\mathbb{R}^{(n-1)k}}\prod_{j=1}^{n}\left|f_j\left(x+l_j(y)\right)\right|\left|\eta(y)\right|dy\,dx\right)^q. \end{aligned} \tag{36}$$

Since the linear mapping $\Lambda: \mathbb{R}^{nk} \to \mathbb{R}^{nk}$ is onto, we can make change of variables $u_j = x+l_j(y)$, $1 \le j \le n$, and then $(x, y) = \Lambda^{-1}(u_1,\dots,u_n)$. We also have

$$\begin{aligned} &\int_{|x-h|\le 1}\int_{\mathbb{R}^{(n-1)k}}\prod_{j=1}^{n}\left|f_j\left(x+l_j(y)\right)\right|\left|\eta(y)\right|dy\,dx \\ &\quad\le C\int_{|u_1-h|\le N,\dots,|u_n-h|\le N}\prod_{j=1}^{n}\left|f_j\left(u_j\right)\right|du_1\cdots du_n \\ &\quad\le C\left(N^k\omega_k\right)^{(n-1/q)}\prod_{j=1}^{n}\left\|f_j\right\|^q_{L^{p_j}(|x-h|\le N)}, \end{aligned} \tag{37}$$

where ω_k is the volume of the unit ball in $\mathbb{R}^k$ and N is a fixed number such that $|l_j(y)| + 1 \le N$ for all $y \in \operatorname{supp}(\eta)$, $1 \le j \le n$.

Note the following fact. For any $F \in L^1(\mathbb{R}^k)$ and $r > 0$, it follows that

$$\begin{aligned} \int_{\mathbb{R}^k}\left(\int_{|x-h|<r}F(x)\,dx\right)dh &= \int_{\mathbb{R}^k}\left(F * \chi_{B_r}\right)(h)\,dh \\ &= r^k\omega_k\int_{\mathbb{R}^k}F(x)\,dx, \end{aligned} \tag{38}$$

where χ_{B_r} is the characteristic function of the ball centered at 0 with radius r.

In view of the equality (38), taking integration on both sides of (35) with respect to h over $\mathbb{R}^k$, we conclude that

$$\begin{aligned} &\omega_k\int_{\mathbb{R}^k}\left(\int_{\mathbb{R}^{(n-1)k}}\prod_{j=1}^{n}\left|f_j\left(x+l_j(y)\right)\right|\left|\eta(y)\right|dy\right)^q dx \\ &\quad\le C\int_{\mathbb{R}^k}\prod_{j=1}^{n}\left\|f_j\right\|^q_{L^{p_j}(|x-h|\le N)}dh \\ &\quad\le C\prod_{j=1}^{n}\left(\int_{\mathbb{R}^k}\int_{|x-h|\le N}\left|f_j(x)\right|^{p_j}dx\,dh\right)^{q/p_j} \\ &\quad= CN^k\omega_k\prod_{j=1}^{n}\|f\|^q_{p_j}. \end{aligned} \tag{39}$$

The proof is therefore concluded. □

As a special case of Lemma 10, we immediately obtain Corollary 11.

Corollary 11. *Let $f \in L^{p_1}(\mathbb{R})$ and $g \in L^{p_2}(\mathbb{R})$. If $1 \le p_1$, $p_2 < \infty$ satisfy $1/q = 1/p_1 + 1/p_2$, then there exists a constant $C = C(p_1, p_2, q)$ such that*

$$\begin{aligned} &\int_{\mathbb{R}}\left(\int_{|t|<1}\left|f(x-t)\,g(x+t)\right|dt\right)^q dx \\ &\quad\le C\|f\|^q_{p_1}\|g\|^q_{p_2}. \end{aligned} \tag{40}$$

Actually, the estimate of Corollary 11 may also be found in [4, 5].

Generally speaking, the power decay estimates of oscillatory integrals are necessary to investigate the boundedness of oscillatory integral operators. The following lemma to appear in [5] is just an oscillatory integral and will be used to prove Lemma 14.

Lemma 12. *Suppose that $P(\cdot,\cdot)$ is a real-valued polynomial with the degree $d \geq 2$ and that there exists a positive integer number $1 \leq \beta \leq d-1$ such that*

$$\left|\partial_x^{\beta}\partial_y^{d-\beta}P(x,y)\right| \geq 1. \tag{41}$$

Define the functional

$$\Lambda_\lambda(f,g) = \iint_{\mathbb{R}} e^{i\lambda P(x,y)} f(x)\, g(y)\, \eta(x,y)\, dx\, dy, \tag{42}$$

where η is a smooth function with compact support, $f, g \in L^2(\mathbb{R})$, and $\lambda \in \mathbb{R}$. Then there exists some constant $\delta > 0$ such that the following power decay estimate

$$\left|\Lambda_\lambda(f,g)\right| \leq C(1+|\lambda|)^{-\delta}\|f\|_2\|g\|_2 \tag{43}$$

holds, where the constant C is independent of f, g and λ.

Remark 13. It should be pointed out how the constant C depends on the cut-off function η, provided that other conditions are not changed. Indeed, if, for the collection $\Theta = \{\eta\}$, every function η is uniformly supported in a bounded set and $\|\eta\|_{C^m}$ has an upper bound independent of $\eta \in \Theta$ for each positive integer m, then the constant C can be chosen such that the estimate (43) holds uniformly for $\eta \in \Theta$. The decay estimate (43) and its other variants have been systematically investigated by many scholars. For certain polynomials with some other conditions, the power exponent δ can be explicitly given. One may refer to [6] and other references appearing in this paper.

Many oscillatory integral operators have power decay estimates which are indispensable to study mapping properties. The following lemma is significant and will be useful in the next section.

Lemma 14. *Suppose that $P(\cdot,\cdot)$ is a real-valued nondegenerate polynomial with the degree d and that there is some β, $1 \leq \beta \leq d-2$, such that $|Ind_\beta(P)| \geq 1$. Let*

$$\mathcal{S}_\lambda(f,g)(x) = \int_{\mathbb{R}} e^{i\lambda P(x,t)} f(x-t)\, g(x+t)\, \eta(t)\, dt, \tag{44}$$

where $\eta \in C_0^\infty$ has support contained in $[-1,1]$. If $1 \leq p_1$, $p_2 < \infty$, and $q \geq 1/2$ satisfy

$$\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2}, \tag{45}$$

then there exists some $\varepsilon > 0$ such that

$$\left\|\mathcal{S}_\lambda(f,g)\right\|_q \leq C(1+|\lambda|)^{-\varepsilon}\|f\|_{p_1}\|g\|_{p_2}, \tag{46}$$

for $f \in L^{p_1}(\mathbb{R})$, $g \in L^{p_2}(\mathbb{R})$, and $\lambda \in \mathbb{R}$.

Proof. Choose a nonnegative function $\varphi \in C_0^\infty$ such that

$$\operatorname{supp}\varphi \subset [-2,2], \quad \sum_{k\in\mathbb{Z}} \varphi(x-k) = 1 \tag{47}$$

for every $x \in \mathbb{R}$.

To obtain the decay estimate (46), it suffices to show that the following estimate

$$\int_{\mathbb{R}} \left|\mathcal{S}_\lambda(f,g)(x)\,\varphi(x-k)\right|^q dx \leq C(1+|\lambda|)^{-q\varepsilon}\|f\|^q_{L^{p_1}(B_5(k))}\|g\|^q_{L^{p_2}(B_5(k))} \tag{48}$$

holds uniformly in $k \in \mathbb{Z}$, where $B_5(k)$ denotes an interval with the center at k and the radius 5.

If inequality (48) has been established, we may use Hölder's inequality to obtain (46).

In fact, since $\operatorname{supp}\varphi \subset [-2,2]$, at most five terms are not zero in the summation for any fixed x. Thus, if $0 < q < 1$, then we have

$$\left(\sum_k |\varphi(x-k)|\right)^q \leq \sum_k |\varphi(x-k)|^q, \tag{49}$$

and if $q \geq 1$, then it follows that

$$\left(\sum_k |\varphi(x-k)|\right)^q \leq 5^{q-1}\sum_k |\varphi(x-k)|^q. \tag{50}$$

Taking the summations of both sides of the inequality (48) with respect to k over $\mathbb{Z}$ and using the two inequalities (49) and (50), we notice that the summation of the left side of (48) is greater than $C\|\mathcal{S}_\lambda(f,g)\|_q^q$.

For the summation of the right side of (48), using Hölder's inequality, we obtain that

$$\begin{aligned}\sum_k \|f\|^q_{L^{p_1}(B_5(k))}\|g\|^q_{L^{p_2}(B_5(k))} &\leq \left(\sum_k \int_{|x-k|\leq 5} |f(x)|^{p_1} dx\right)^{q/p_1} \\ &\quad\times \left(\sum_k \int_{|x-k|\leq 5} |g(x)|^{p_2} dx\right)^{q/p_2} \\ &\leq 10\|f\|^q_{p_1}\|g\|^q_{p_2}.\end{aligned} \tag{51}$$

Next let us turn to the proof of the inequality (48). By Lemma 10 or Corollary 11, we know that $\mathcal{S}_\lambda$ is bounded from $L^{p_1} \times L^{p_2}$ to L^q and its norm is independent of λ. By the multilinear interpolation of Riesz-Thörin, it is enough to show that the estimate (46) holds for $p_1 = p_2 = 2$ and $q = 1$. Thus we merely need to show that the inequality (48) holds with the case $p_1 = p_2 = 2$ and $q = 1$. Furthermore, since φ has a compact support, it suffices to prove that (48) holds for $p_1 = p_2 = q = 2$.

This is equivalent to proving the following inequality:

$$\begin{aligned}&\left\|\mathcal{S}_\lambda(f,g)\varphi(\cdot-k)\right\|_2\\&\quad\le C(1+|\lambda|)^{-\varepsilon}\|f\|_{L^2(B_5(k))}\|g\|_{L^2(B_5(k))}.\end{aligned}\tag{52}$$

By the change of variables, we can easily deduce that

$$\begin{aligned}&\int_{\mathbb{R}}\left|\mathcal{S}_\lambda(f,g)(x)\varphi(x-k)\right|^2dx\\&=\iiint_{\mathbb{R}}e^{i\lambda(P(x,t_1)-P(x,t_2))}f(x-t_1)\overline{f(x-t_2)}\\&\qquad\times g(x+t_1)\overline{g(x+t_2)}\\&\qquad\times\eta(t_1)\overline{\eta(t_2)}\varphi(x-k)\overline{\varphi(x-k)}dt_1dt_2dx\\&=\iiint_{\mathbb{R}}e^{i\lambda(P(x,t+\tau)-P(x,t))}\\&\qquad\times f(x-t-\tau)\overline{f(x-t)}\\&\qquad\times g(x+t+\tau)\overline{g(x+t)}\eta(t+\tau)\overline{\eta(t)}\\&\qquad\times\varphi(x-k)\overline{\varphi(x-k)}dt\,d\tau\,dx\\&=\frac{1}{2}\int_{D_\tau}\int_{D_u}\int_{D_v}e^{i\lambda Q_\tau(u,v)}F_\tau(v)G_\tau(u)\eta_\tau(u,v)\\&\qquad\times\psi(u,v)\,du\,dv\,d\tau,\end{aligned}\tag{53}$$

where

$$\begin{aligned}Q_\tau(u,v)&=P\left(k+\frac{u+v}{2},\frac{u-v}{2}+\tau\right)\\&\quad-P\left(k+\frac{u+v}{2},\frac{u-v}{2}\right),\\F_\tau(v)&=f(k+v-\tau)\overline{f(k+v)},\\G_\tau(u)&=g(k+u+\tau)\overline{g(k+u)},\\\eta_\tau(u,v)&=\eta\left(\frac{u-v}{2}+\tau\right)\overline{\eta\left(\frac{u-v}{2}\right)},\\\psi(u,v)&=\varphi\left(\frac{u+v}{2}\right)\overline{\varphi\left(\frac{u+v}{2}\right)}.\end{aligned}\tag{54}$$

Consequently, for some $\rho>0$, we can rewrite the above integral as

$$\begin{aligned}&\int_{\mathbb{R}}\left|\mathcal{S}_\lambda(f,g)(x)\varphi(x-k)\right|^2dx\\&=\frac{1}{2}\int_{|\tau|<\rho}\int_{D_u}\int_{D_v}e^{i\lambda Q_\tau(u,v)}F_\tau(v)G_\tau(u)\\&\qquad\times\eta_\tau(u,v)\psi(u,v)\,du\,dv\,d\tau\\&\quad+\frac{1}{2}\int_{|\tau|\ge\rho}\int_{D_u}\int_{D_v}e^{i\lambda Q_\tau(u,v)}F_\tau(v)G_\tau(u)\\&\qquad\times\eta_\tau(u,v)\psi(u,v)\,du\,dv\,d\tau\\&=I_1+I_2.\end{aligned}\tag{55}$$

Observe that integrals I_1 and I_2 are taken in the domain

$$\{(u,v,\tau):|u|\le3,|v|\le3,|\tau|\le2\}.\tag{56}$$

Recall $\operatorname{supp}(\varphi)\subset[-2,2]$ and $\operatorname{supp}(\eta)\subset[-1,1]$, respectively.

We clearly conclude that

$$\begin{aligned}&|I_1|\\&\le C\int_{|\tau|\le\min\{\rho,2\}}\int_{|u|\le3}\int_{|v|\le3}\left|F_\tau(v)G_\tau(u)\right|du\,dv\,d\tau\\&=C\int_{|\tau|\le\min\{\rho,2\}}\int_{|u|\le3}\int_{|v|\le3}\Big|f(k+v-\tau)\overline{f(k+v)}g\\&\qquad\times(k+u+\tau)\overline{g(k+u)}\Big|du\,dv\,d\tau\\&\le C\int_{|\tau|<\min\{\rho,2\}}\left(\int_{|v|\le3}\left|f(v+k-\tau)\right|^2dv\right)^{1/2}\\&\qquad\times\left(\int_{|v|\le3}\left|f(v+k)\right|^2dv\right)^{1/2}\\&\qquad\times\left(\int_{|u|\le3}\left|g(u+k+\tau)\right|^2du\right)^{1/2}\\&\qquad\times\left(\int_{|u|\le3}\left|g(u+k)\right|^2du\right)^{1/2}d\tau\\&\le C\min\{\rho,2\}\|f\|^2_{L^2(B_5(k))}\|g\|^2_{L^2(B_5(k))}.\end{aligned}\tag{57}$$

For $|\tau|\ge\rho$, by Lemma 9, it is easily verified that

$$\partial_u^\beta\partial_v^{d-1-\beta}Q_\tau(u,v)=2^{1-d}\tau\operatorname{Ind}_\beta(P)\tag{58}$$

holds for each $1\le\beta\le d-2$.

Thus we have that

$$\left|\partial_u^\beta\partial_v^{d-1-\beta}Q_\tau(u,v)\right|=2^{1-d}|\tau|\left|\operatorname{Ind}_\beta(P)\right|\ge c|\rho|,\tag{59}$$

for some $1\le\beta\le d-2$.

Invoking Lemma 12, we obtain, for $|\tau|\ge\rho$, that

$$\begin{aligned}&\left|\int_u\int_v e^{i\lambda Q_\tau(u,v)}F_\tau(v)G_\tau(u)\eta_\tau(u,v)\varphi(u,v)\,du\,dv\right|\\&\quad\le C(1+|\lambda\rho|)^{-\delta}\left(\int_{|v|\le3}\left|F_\tau(v)\right|^2dv\right)^{1/2}\\&\qquad\times\left(\int_{|u|\le3}\left|G_\tau(u)\right|^2du\right)^{1/2},\end{aligned}\tag{60}$$

since the cut-off function $\eta_\tau\psi$ is uniformly supported in $\{(u,v):|u|\le3,|v|\le3\}$ and its C^m norms have bounds independent of τ for each positive m, and the constant C is independent of τ. Therefore, the Cauchy-Schwartz inequality implies that

$$\begin{aligned}
|I_2| &\le C(1+|\lambda\rho|)^{-\delta} \\
&\quad \times \int_{|\tau|\le 2}\left(\int_{|v|\le 3}|F_\tau(v)|^2 dv\right)^{1/2}\left(\int_{|u|\le 3}|G_\tau(u)|^2 du\right)^{1/2} d\tau \\
&\le C(1+|\lambda\rho|)^{-\delta}\left(\int_{|\tau|\le 2}\int_{|v|\le 3}|F_\tau(v)|^2 dv\, d\tau\right)^{1/2} \\
&\quad \times \left(\int_{|\tau|\le 2}\int_{|u|\le 3}|G_\tau(u)|^2 du\, d\tau\right)^{1/2} \\
&= C(1+|\lambda\rho|)^{-\delta} \\
&\quad \times \left(\int_{|\tau|\le 2}\int_{|v|\le 3}|f(v+k-\tau)|^2|f(v+k)|^2 dv\, d\tau\right)^{1/2} \\
&\quad \times \left(\int_{|\tau|\le 2}\int_{|u|\le 3}|g(u+k+\tau)|^2|g(u+k)|^2 du\, d\tau\right)^{1/2} \\
&\le C(1+|\lambda\rho|)^{-\delta} \\
&\quad \times \left(\int_{|x-k|\le 5}|f(x)|^2 dx\right)\left(\int_{|x-k|\le 5}|g(x)|^2 dx\right),
\end{aligned} \tag{61}$$

where we use the Fubini theorem in the proof of the above inequality.

Consequently, choosing $\rho = |\lambda|^{-\delta/(1+\delta)} < 2$ for $|\lambda| \ge 1$ and combining the above estimates for I_1 and I_2, we immediately obtain that

$$\begin{aligned}
&\|\mathcal{S}_\lambda(f,g)\varphi(\cdot - k)\|_2 \\
&\quad \le C(1+|\lambda|)^{-\varepsilon}\|f\|_{L^2(B_5(k))}\|g\|_{L^2(B_5(k))},
\end{aligned} \tag{62}$$

where $\varepsilon = (-\delta/2(1+\delta))$. This is just the inequality (52). □

Lemma 15. *Suppose that $P(\cdot,\cdot)$ is a real-valued polynomial and $1 \le p_1, p_2 < \infty$, and $q \ge 1/2$ satisfy*

$$\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2}. \tag{63}$$

If the operator $\mathcal{T}$ defined by (1) *satisfies*

$$\|\mathcal{T}(f,g)\|_q \le C\|f\|_{p_1}\|g\|_{p_2}, \tag{64}$$

for all $f, g \in C_c^\infty(\mathbb{R})$, then the truncated operator

$$\mathcal{S}(f,g)(x) = p.v.\int_{|t|<1} e^{iP(x,t)} f(x-t)\, g(x+t)\,\frac{dt}{t} \tag{65}$$

is also bounded from $L^{p_1}\times L^{p_2}$ to L^q.

Proof. To prove Lemma 15, we will show that

$$\begin{aligned}
&\int_{|x-h|<1/4}|\mathcal{S}(f,g)(x)|^q dx \\
&\quad \le C\|f\|^q_{L^{p_1}([h-5/4,h+5/4])}\|g\|^q_{L^{p_2}([h-5/4,h+5/4])}
\end{aligned} \tag{66}$$

holds uniformly with respect to $h \in \mathbb{R}$.

Fix $h \in \mathbb{R}$ and set

$$\begin{gathered}
f_1 = f\chi_{\{|\cdot-h|\le 1/2\}}, \qquad f_2 = f\chi_{\{1/2<|\cdot-h|\le 5/4\}}, \\
f_3 = f\chi_{\{|\cdot-h|>5/4\}}.
\end{gathered} \tag{67}$$

It is clear that $f = f_1 + f_2 + f_3$.

Now for fixed $h \in \mathbb{R}$, set

$$x \in \left(h-\frac{1}{4}, h+\frac{1}{4}\right). \tag{68}$$

If $|x-h-t| < 1/2$, then $|t| < 3/4$. We hence obtain that

$$\mathcal{S}(f_1,g)(x) = \mathcal{T}(f_1,g)(x). \tag{69}$$

For $|x-h| < 1/4$, when $1/2 \le |x-h-t| \le 5/4$, we have $1/4 \le |t| \le 3/2$. It immediately follows that

$$|\mathcal{S}(f_2,g)(x)| \le C\int_{|t|<1}|f_2(x-t)\,g(x+t)|\,dt. \tag{70}$$

Finally, both $|x-h| < 1/4$ and $|x-h-t| > 5/4$ imply $|t| > 1$. We clearly have

$$\mathcal{S}(f_3,g)(x) = 0. \tag{71}$$

Similarly, we also have that

$$\mathcal{S}(f_1,g) = \mathcal{S}(f_1, g\chi_{\{|\cdot-h|\le 5/4\}}) = \mathcal{T}(f_1, g\chi_{\{|\cdot-h|\le 5/4\}}), \tag{72}$$

provided that $|x-h| < 1/4$ and $|t| < 1$.

Therefore, if $|x-h| < 1/4$, then the equalities (69), (71), and (72) together with (70) imply that

$$\begin{aligned}
&|\mathcal{S}(f,g)(x)| \\
&\le |\mathcal{S}(f_1,g)(x)| + C\int_{|t|<1}|f_2(x-t)\,g(x+t)|\,dt \\
&= |\mathcal{T}(f_1, g\chi_{\{|\cdot-h|\le 5/4\}})(x)| \\
&\quad + C\int_{|t|<1}|f_2(x-t)\,(g\chi_{\{|\cdot-h|\le 5/4\}})(x+t)|\,dt.
\end{aligned} \tag{73}$$

Clearly the inequality (66) is easily obtained by combing (73) together with Corollary 11. We conclude the proof by integrating both sides of (66) with respect to h. □

Lemma 16. *Suppose that $1 \le p_1, p_2 < \infty$, and $q \ge 1/2$ satisfy*

$$\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2}. \tag{74}$$

Define the truncated operator as

$$S_r(f,g)(x) = p.v.\int_{|t|<r} f(x-t)\,g(x+t)\,\frac{dt}{t}. \tag{75}$$

If the operator S defined by (9) *satisfies*

$$\|S(f,g)\|_q \le C\|f\|_{p_1}\|g\|_{p_2}, \tag{76}$$

for all $f, g \in C_c^\infty(\mathbb{R})$, then the truncated operator S_r is also bounded from $L^{p_1}\times L^{p_2}$ to L^q for every $r > 0$; moreover, the operator norm of S_r is independent of r.

Proof. Clearly it implies from the definitions of S_r and S that $S_1 = S$. By a simple dilation argument, we will prove that $\|S_r\|$, the norm of the operator S_r defined by (75), is independent of $r > 0$ and equals $\|S_1\|$.

Indeed, a simple computation implies that

$$\begin{aligned} S_r(f,g)(rx) &= \text{p.v.} \int_{|t|<r} f(rx-t)\, g(rx+t)\, \frac{dt}{t} \\ &= \text{p.v.} \int_{|t|<1} f(rx-rt)\, g(rx+rt)\, \frac{dt}{t} \\ &= \text{p.v.} \int_{|t|<1} (\delta_r f)(x-t)\, (\delta_r g)(x+t)\, \frac{dt}{t} \\ &= S(\delta_r f, \delta_r g)(x). \end{aligned} \tag{77}$$

It follows that

$$\begin{aligned} \frac{\|S_r(f,g)(r\cdot)\|_q}{\|f\|_{p_1}\|g\|_{p_2}} &= \frac{\|S(\delta_r f,\delta_r g)\|_q}{\|f\|_{p_1}\|g\|_{p_2}} \\ &= \left(\frac{1}{r}\right)^{1/q} \frac{\|S(\delta_r f,\delta_r g)\|_q}{\|\delta_r f\|_{p_1}\|\delta_r g\|_{p_2}}, \end{aligned} \tag{78}$$

where the dilation operator is defined by

$$(\delta_\tau f)(x) = f(\tau x) \tag{79}$$

for all $\tau > 0$ and $x \in \mathbb{R}$.

Denote the norm of the operator S by $\|S\|$ as follows:

$$\|S\| = \sup_{\|f\|_{p_1}\neq 0, \|g\|_{p_2}\neq 0} \frac{\|S(f,g)\|_q}{\|f\|_{p_1}\|g\|_{p_2}}. \tag{80}$$

Taking supremum over all $f, g \in C_c^\infty$ for (78), we have

$$\left(\frac{1}{r}\right)^{1/q} \|S_r\| = \left(\frac{1}{r}\right)^{1/q} \|S\|, \tag{81}$$

and naturally $\|S_r\| = \|S\|$. □

Lemma 17. *Suppose that $P(\cdot,\cdot)$ is a real-valued polynomial and $1 \le p_1, p_2 < \infty$, and $q \ge 1/2$ satisfy*

$$\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2}. \tag{82}$$

If the operator $\mathcal{T}$ defined by (1) *satisfies*

$$\|\mathcal{T}(f,g)\|_q \le C\|f\|_{p_1}\|g\|_{p_2}, \tag{83}$$

for all $f, g \in C_c^\infty(\mathbb{R})$, then the following operator

$$\mathcal{T}_r(f,g)(x) = \int_{\mathbb{R}} e^{iP(rx,rt)} f(x-t)\, g(x+t)\, \frac{dt}{t} \tag{84}$$

is also bounded from $L^{p_1} \times L^{p_2}$ to L^q for every $r > 0$; moreover, the operator norm of $\mathcal{T}_r$ is independent of r.

Proof. By a simple dilation argument, we will prove that $\|\mathcal{T}_r\|$, the norm of the operator $\mathcal{T}_r$, is independent of $r > 0$ and equals $\|\mathcal{T}_1\|$ and naturally equals $\|\mathcal{T}\|$.

Indeed, a simple computation implies that

$$\mathcal{T}_r(f,g)\left(\frac{x}{r}\right) = \mathcal{T}\left(\delta_{r^{-1}}(f), \delta_{r^{-1}}(g)\right)(x), \tag{85}$$

$$\begin{aligned} &\frac{\|\mathcal{T}_r(f,g)(\cdot/r)\|_q}{\|f\|_{p_1}\|g\|_{p_2}} \\ &\quad = \frac{\|\mathcal{T}(\delta_{r^{-1}}(f), \delta_{r^{-1}}(g))\|_q}{\|f\|_{p_1}\|g\|_{p_2}} \\ &\quad = r^{1/q} \frac{\|\mathcal{T}(\delta_{r^{-1}}(f), \delta_{r^{-1}}(g))\|_q}{\|\delta_{r^{-1}}(f)\|_{p_1}\|\delta_{r^{-1}}(g)\|_{p_2}}. \end{aligned} \tag{86}$$

Taking supremum over all $f, g \in C_c^\infty$ for (86), we immediately have

$$r^{1/q}\|\mathcal{T}_r\| = r^{1/q}\|\mathcal{T}\|, \tag{87}$$

and hence $\|\mathcal{T}_r\| = \|\mathcal{T}\|$. □

Remark 18. Lemma 17 shows that, for the oscillatory integral operator, the operator norm only depends on the degree of the polynomial but is independent of the coefficient of the polynomial.

3. Proof of the Main Results

Proof of Theorem 3. Associated with each positive r, denote the operator S_r by

$$S_r(f,g)(x) = \text{p.v.} \int_{|t|<r} f(x-t)\, g(x+t)\, \frac{dt}{t}, \tag{88}$$

where $f, g \in C_c^\infty(\mathbb{R})$; hence, we have $S = S_1$ by the definition of S in Theorem 3. By the dilation argument as in the proof of Lemma 16, we can easily conclude that if S can be extended to a bounded operator from $L^{p_1} \times L^{p_2}$ to L^q, then so is S_r and the operator norm is independent of $r > 0$. Observe that $C_c^\infty(\mathbb{R})$ is dense in all $L^p(\mathbb{R})$ for $0 < p < \infty$. For $0 < p < 1$, we regard $L^p(\mathbb{R})$ as a complete metric space. Since we can approximate $f \in L^p(\mathbb{R})$ by finite linear combinations of characteristic function of intervals, we may assume that $f = \chi_I$ for some finite open interval I. By the Lusin theorem, for any closed interval $O \subset I$ there exists a function $\eta \in C_c^\infty$ such that $\text{supp}(\eta) \subset I$, $0 \le \eta \le 1$, and $\eta(x) = 1$ for $x \in O$. Hence we obviously have

$$\|f - \eta\|_p \le |I - O|^{1/p}. \tag{89}$$

Fix $f, g \in C_c^\infty(\mathbb{R})$ temporarily and choose a large $N > 0$ such that both $\text{supp}(f)$ and $\text{supp}(g)$ are contained in $[-N,$

$N]$. If $|t| > N$, we see that either $x - t \notin \operatorname{supp}(f)$ or $x + t \notin \operatorname{supp}(g)$ must hold for every x. Hence we conclude that

$$\begin{aligned} S_N(f,g)(x) &= T(f,g)(x), \\ \|T(f,g)\|_q = \|S_N(f,g)\|_q &\le \|S_N\| \|f\|_{p_1} \|g\|_{p_2} \\ &= \|S\| \|f\|_{p_1} \|g\|_{p_2}, \end{aligned} \tag{90}$$

where $\|S_N\|$ and $\|S\|$ are operator norms of S_N and S from $L^{p_1} \times L^{p_2}$ to L^q, respectively.

Obviously it follows that

$$\|T\| \le \|S\|. \tag{91}$$

The validity of the converse follows from Lemma 15.

This finishes the proof of Theorem 3. □

Proof of Theorem 4. We assume that

$$\|\mathcal{T}(f,g)\|_q \le C\|f\|_{p_1} \|g\|_{p_2} \tag{92}$$

holds for some polynomial $P(x,t)$ with $P(x+h,t) - P(x,t)$ being degenerate for all $h \in \mathbb{R}$. We wish to obtain

$$\begin{aligned} &\int_{|x-h|<1} |S(f,g)(x)|^q dx \\ &\quad \le C\|f\|^q_{L^{p_1}(B_2(h))} \|g\|^q_{L^{p_2}(B_2(h))}, \end{aligned} \tag{93}$$

where the constant C is independent of h and $B_2(h) = (h-2, h+2)$.

Since $P(x+h,t) - P(x,t)$ is degenerate with respect to x, t, we can rewrite P as

$$\begin{aligned} P(x,t) = {} & P(x-h,t) \\ & + Q_1(x-t,h) + Q_2(x+t,h) + R(x,h), \end{aligned} \tag{94}$$

for some polynomials Q_1, Q_2, and R.

For $h \in \mathbb{R}$, it follows that

$$\begin{aligned} & S(f,g)(x) \\ &\quad = \int_{|t|<1} f(x-t)\, g(x+t) \frac{dt}{t} \\ &\quad = e^{-iR(x,h)} \int_{|t|<1} e^{iP(x,t)} f_h(x-t)\, g_h(x+t) \\ &\qquad\qquad \times \left(e^{-iP(x-h,t)} - e^{-iP(x-h,0)}\right) \frac{dt}{t} \\ &\qquad + e^{-iR(x,h)-iP(x-h,0)} \\ &\qquad \times \int_{|t|<1} e^{iP(x,t)} f_h(x-t)\, g_h(x+t) \frac{dt}{t} \\ &\quad = S_h^1(f,g)(x) + S_h^2(f,g)(x), \end{aligned} \tag{95}$$

where

$$\begin{aligned} f_h(x) &= e^{-iQ_1(x,h)} f(x), \\ g_h(x) &= e^{-iQ_2(x,h)} g(x). \end{aligned} \tag{96}$$

For the polynomial P, when $|x-h| < 1$ and $|t| < 1$, we immediately have

$$\left|e^{-iP(x-h,t)} - e^{-iP(x-h,0)}\right| \le C|t|. \tag{97}$$

It is easy to drive that

$$\left|S_h^1(f,g)(x)\right| \le C \int_{|t|<1} \left|f(x-t)\, g(x+t)\right| dt, \tag{98}$$

for $|x-h| < 1$.

It follows from Corollary 11 and the inequality (98) that

$$\begin{aligned} &\int_{|x-h|<1} \left|S_h^1(f,g)(x)\right|^q dx \\ &\quad \le C\|f\|^q_{L^{p_1}(B_2(h))} \|g\|^q_{L^{p_2}(B_2(h))}. \end{aligned} \tag{99}$$

If $|x-h| < 1$ and $|t| < 1$, then we have

$$\begin{aligned} f(x-t) &= \left(f\chi_{(h-2,h+2)}\right)(x-t), \\ g(x+t) &= \left(f\chi_{(h-2,h+2)}\right)(x+t). \end{aligned} \tag{100}$$

By the definition of S_h^2, we thus obtain

$$S_h^2(f,g)(x) = S_h^2\left(f\chi_{(h-2,h+2)}, g\chi_{(h-2,h+2)}\right)(x) \tag{101}$$

for $|x-h| < 1$.

On the other hand, by Lemma 15, S_h^2 is bounded from $L^{p_1} \times L^{p_2}$ to L^q; that is,

$$\begin{aligned} & \left\|S_h^2(f,g)\right\|^q_{L^q(B_1(h))} \\ &\quad = \int_{|x-h|<1} \left|S_h^2(f\chi_{(h-2,h+2)}, g\chi_{(h-2,h+2)})(x)\right|^q dx \\ &\quad \le \int_{\mathbb{R}} \left|S_h^2(f\chi_{(h-2,h+2)}, g\chi_{(h-2,h+2)})(x)\right|^q dx \\ &\quad \le C\|f\|^q_{L^{p_1}(B_2(h))} \|g\|^q_{L^{p_2}(B_2(h))}. \end{aligned} \tag{102}$$

Hence the inequality (93) is also valid for S_h^2. The theorem is a consequence of (93) by taking integration with respect to h over $\mathbb{R}$ and an application of Hölder's inequality. □

Proof of Theorem 5. The implication (i) $\Rightarrow$ (ii) is contained in [1]. The implication (ii) $\Rightarrow$ (iii) is a direct consequence of Theorem 4, and (iii) $\Rightarrow$ (i) can attribute to Theorem 3. □

Proof of Theorem 6. We first prove Theorem 6 for the case $\sigma > 1$. One will see that the oscillatory factor is not necessary in this case.

It follows that

$$\begin{aligned} & \left|T_\sigma(f,g)(x)\right| \\ &\quad \le \int_{|t|<1} \left|f(x-t)\, g(x+t)\right| dt \\ &\qquad + \sum_{j=0}^{\infty} \int_{2^j \le |t| < 2^{j+1}} \left|f(x-t)\, g(x+t)\right| \frac{dt}{|t|^\sigma} \\ &\quad = S(|f|, |g|)(x) + \sum_{j=0}^{\infty} T_\sigma^j(f,g)(x). \end{aligned} \tag{103}$$

If we can show that

$$\left\|T_\sigma^j(f,g)\right\|_q \le C2^{-j\varepsilon}\|f\|_{p_1}\|g\|_{p_2} \tag{104}$$

holds for some $\varepsilon > 0$ and each $j = 1, 2, \ldots$, then our desired result follows immediately.

Now let us consider the operator $T_\sigma^j(f,g)$.

We conclude that

$$\begin{aligned} T_\sigma^j(f,g)(x) &= \int_{2^j \le |t| < 2^{j+1}} |f(x-t)\,g(x+t)| \frac{dt}{|t|^\sigma} \\ &\le 2^{-j(\sigma-1)+1} \\ &\quad \times \int_{1/2\le|t|<1} \left|f\left(x-2^{j+1}t\right) g\left(x+2^{j+1}t\right)\right| dt \\ &\le 2^{-j(\sigma-1)+1} \int_{|t|<1} \left|f\left(x-2^{j+1}t\right) g\left(x+2^{j+1}t\right)\right| dt. \end{aligned} \tag{105}$$

Therefore, it follows that

$$\begin{aligned} &\left\|T_\sigma^j(f,g)\right\|_q^q \\ &\le 2^{(-j(\sigma-1)+1)q} \\ &\quad \times \int_{\mathbb{R}} \left(\int_{|t|<1} \left|f\left(x-2^{j+1}t\right) g\left(x+2^{j+1}t\right)\right| dt\right)^q dx \\ &\le 2^{(-j(\sigma-1)+1)q} 2^{j+1} \\ &\quad \times \int_{\mathbb{R}} \left(\int_{|t|<1} \left|f\left(2^{j+1}(x-t)\right) g\left(2^{j+1}(x+t)\right)\right| dt\right)^q dx \\ &= 2^{(-j(\sigma-1)+1)q} 2^{j+1} \int_{\mathbb{R}} \left(\int_{|t|<1} \left|\tilde{f}(x-t)\tilde{g}(x+t)\right| dt\right)^q dx, \end{aligned} \tag{106}$$

where $\tilde{f}(\cdot) = f(2^{j+1}(\cdot))$ and $\tilde{g}(\cdot) = g(2^{j+1}(\cdot))$.

Clearly, $f \in L^p(\mathbb{R})$ implies $\tilde{f} \in L^p(\mathbb{R})$.

It implies from Corollary 11 that

$$\begin{aligned} &\int_{\mathbb{R}} \left(\int_{|t|<1} \left|\tilde{f}(x-t)\tilde{g}(x+t)\right| dt\right)^q dx \\ &\quad \le C\left(\int_{\mathbb{R}} \left|\tilde{f}(x)\right|^{p_1} dx\right)^{q/p_1} \left(\int_{\mathbb{R}} \left|\tilde{g}(x)\right|^{p_2} dx\right)^{q/p_2} \\ &\quad = C2^{-(j+1)}\|f\|_{p_1}^q\|g\|_{p_2}^q. \end{aligned} \tag{107}$$

Thus we have

$$\left\|T_\sigma^j(f,g)\right\|_q \le C2^{-j(\sigma-1)}\|f\|_{p_1}\|g\|_{p_2}. \tag{108}$$

It remains to consider the case $\sigma = 1$. By Lemma 14, we can reduce the theorem to boundedness of the following operator defined as

$$\begin{aligned} T_\varphi(f,g)(x) = \int_{\mathbb{R}} & e^{iP(x,t)} f(x-t)\,g(x+t) \\ & \times (1-\varphi(t)) \frac{dt}{1+|t|}, \end{aligned} \tag{109}$$

where φ is a smooth function with compact support contained in $[-1,1]$ and equals 1 near the origin. Hence it suffices to show that T_φ is bounded from $L^{p_1} \times L^{p_2}$ to L^q.

Since φ is smooth with compact support and equals 1 near the origin, it is not hard to have that

$$\sum_{j=0}^{\infty} \left(\varphi\left(\frac{x}{2^{j+1}}\right) - \varphi\left(\frac{x}{2^j}\right)\right) = 1 - \varphi(x). \tag{110}$$

We rewrite the integral (109) as

$$\begin{aligned} &T_\varphi(f,g)(x) \\ &\quad = \sum_{j=0}^{\infty} \int_{\mathbb{R}} e^{iP(x,t)} f(x-t)\,g(x+t)\,\psi_j(t) \frac{dt}{|t|} \\ &\quad + W(f,g)(x), \\ &\quad = \sum_{j=0}^{\infty} T_\varphi^j(f,g)(x) + W(f,g)(x), \end{aligned} \tag{111}$$

where

$$\begin{aligned} \psi_j(t) &= \varphi\left(\frac{x}{2^{j+1}}\right) - \varphi\left(\frac{x}{2^j}\right), \\ W(f,g)(x) &= -\int_{\mathbb{R}} e^{iP(x,t)} f(x-t)\,g(x+t)\,(1-\varphi(t)) \frac{dt}{(1+|t|)\,|t|}, \\ T_\varphi^j(f,g)(x) &= \int_{\mathbb{R}} e^{iP(x,t)} f(x-t)\,g(x+t)\,\psi_j(t) \frac{dt}{|t|}, \end{aligned} \tag{112}$$

for $j = 0, 1, 2, \ldots$.

It is clear that we have

$$|W(f,g)(x)| \le C \int_{\mathbb{R}} |f(x-t)\,g(x+t)| \frac{dt}{(1+|t|)^2}, \tag{113}$$

which implies that W is bounded from $L^{p_1} \times L^{p_2}$ to L^q as shown for $\sigma > 1$. Hence it is enough to prove that

$$\left\|T_\varphi^j(f,g)\right\|_q \le C2^{-j\varepsilon}\|f\|_{p_1}\|g\|_{p_2} \tag{114}$$

for some $\varepsilon > 0$.

Let

$$\mathcal{T}_{\varphi}^{j}(f,g)(x)=\int_{\mathbb{R}} e^{iP(2^{j}x,2^{j}t)} f(x-t)\,g(x+t)\,\psi_{0}(t)\,\frac{dt}{|t|}, \tag{115}$$

for $j=0,1,2,\ldots$.

By the dilation argument as in the proof of Lemma 17, we easily conclude that the operator norms, $\|T_{\varphi}^{j}\|$ and $\|\mathcal{T}_{\varphi}^{j}\|$, from $L^{p_1}\times L^{p_2}$ to L^{q}, are equal.

Observe that

$$\mathrm{Ind}_{\beta}\left(P\left(2^{j}(\cdot),2^{j}(\cdot)\right)\right)=2^{jd}\mathrm{Ind}_{\beta}(P). \tag{116}$$

We can now apply Lemma 14 to obtain that the decay estimate (114) holds, since $|\mathrm{Ind}_{\beta}(P)|\geq 1$ for some $1\leq\beta\leq d-2$.

This finishes the proof of Theorem 6. □

Remark 19. It should be pointed out that if the polynomial $P(\cdot,\cdot)$ is degenerate, then the operator T_{σ} defined by (16) may not be bounded from $L^{p_1}(\mathbb{R})\times L^{p_2}(\mathbb{R})$ to $L^{q}(\mathbb{R})$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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