Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 709746, 6 pages
http://dx.doi.org/10.1155/2014/709746

Research Article

Global Regularity for the 2D Micropolar Fluid Flows with Mixed Partial Dissipation and Angular Viscosity

Zujin Zhang

School of Mathematics and Computer Science, Gannan Normal University, Ganzhou 341000, China

Correspondence should be addressed to Zujin Zhang; zhangzujin361@163.com

Received 25 February 2014; Revised 12 April 2014; Accepted 5 May 2014; Published 12 May 2014

Academic Editor: Maurizio Grasselli

This paper establishes the global existence and uniqueness of classical solutions to the 2D micropolar fluid flows with mixed partial dissipation and angular viscosity.

1. Introduction

In this paper, we investigate the Cauchy problem for the viscous incompressible micropolar fluid flows. In three-dimensional case it can be expressed as

$$\begin{gathered}\mathbf{v}_t-(\nu+\kappa)\,\Delta\mathbf{v}-2\kappa\nabla\times\mathbf{w}+\nabla\pi+(\mathbf{v}\cdot\nabla)\,\mathbf{v}=\mathbf{0},\\ \mathbf{w}_t-\gamma\Delta\mathbf{w}-(\alpha+\beta)\,\nabla\nabla\cdot\mathbf{w}+4\kappa\mathbf{w}-2\kappa\nabla\times\mathbf{v}+(\mathbf{v}\cdot\nabla)\,\mathbf{w}=\mathbf{0},\\ \nabla\cdot\mathbf{v}=0,\\ \mathbf{v}(0)=\mathbf{v}_0,\qquad\mathbf{w}(0)=\mathbf{w}_0.\end{gathered}\tag{1}$$

Here, $\mathbf{v}=(v_1,v_2,v_3)$ is the divergence-free fluid velocity field, π is a scalar pressure, $\mathbf{w}=(w_1,w_2,w_3)$ is the microrotation field (angular velocity of the rotation of the particles of the fluid), and the constant $\nu\ \ge\ 0$ is the Newtonian kinetic viscosity, $\kappa>0$ is the dynamics microrotation viscosity, and $\alpha,\beta,\gamma\ge 0$ are the angular viscosities (see, e.g., [1, 2]).

The micropolar fluid equations (1) enable us to consider some physical phenomena that cannot be treated by the classical Navier-Stokes equations ($\mathbf{w}=\mathbf{0}$ in (1)), such as the motion of animal blood, liquid crystals, and dilute aqueous polymer solutions. Physically, $(1)_1$ represents the conservation of linear momentum, $(1)_2$ reflects the conservation of angular momentum, and $(1)_3$ is the incompressibility of the fluid, specifying the conservation of mass.

Besides their physical applications, the micropolar fluid equations (1) are also mathematically important. The existence of weak and strong solutions was established by Galdi and Rionero [3] and Yamaguchi [4], respectively.

In this paper, we study the global regularity problem of the 2D micropolar fluid equations. Assuming that the velocity component in the z-direction is zero and the axes of rotation of particles are parallel to the z-axis, that is,

$$\mathbf{v}=(v_1,v_2,0),\qquad\mathbf{w}=(0,0,w_3),\tag{2}$$

we obtain by gathering (2) into (1)

$$\begin{gathered}\mathbf{v}_t-(\nu+\kappa)\,\Delta\mathbf{v}-2\kappa\nabla\times w+\nabla\pi+(\mathbf{v}\cdot\nabla)\,\mathbf{v}=\mathbf{0},\\ w_t-\gamma\Delta w+4\kappa w-2\kappa\nabla\times\mathbf{v}+(\mathbf{v}\cdot\nabla)\,w=0,\\ \nabla\cdot\mathbf{v}=0,\\ \mathbf{v}(0)=\mathbf{v}_0,\qquad w(0)=w_0,\end{gathered}\tag{3}$$

where $\mathbf{v}=(v_1,v_2)$ is a vector and w is a scalar. Here and in what follows, we use the notations

$$\nabla\times\mathbf{v}=\partial_x v_2-\partial_y v_1,\qquad\nabla\times w=\left(\partial_y w,-\partial_x w\right).\tag{4}$$

The global regularity of (3) with full viscosity has been established by Łukaszewicz [2] (see also [5] for more explicit result). The purpose of this paper is to investigate the global regularity of the 2D micropolar fluid flows with mixed partial dissipation and angular viscosity. To be precise, we will consider the following system:

$$\begin{gathered}\mathbf{v}_t-(\nu+\kappa)\,\mathbf{v}_{xx}-2\kappa\nabla\times w+\nabla\pi+(\mathbf{v}\cdot\nabla)\,\mathbf{v}=\mathbf{0},\\ w_t-\gamma w_{yy}+4\kappa w-2\kappa\nabla\times\mathbf{v}+(\mathbf{v}\cdot\nabla)\,w=0,\\ \nabla\cdot\mathbf{v}=0,\\ \mathbf{v}(0)=\mathbf{v}_0,\qquad w(0)=w_0.\end{gathered}\tag{5}$$

Our study is partially motivated by the global well-posedness of the 2D MHD equations with partial viscosities (see [6, 7], for instance), that of the 2D Boussinesq equations with partial viscosity (see, e.g., [8, 9]), and that of the 2D micropolar fluid equations with zero angular viscosity [10].

The main result of this paper now reads.

Theorem 1. *Suppose $\nu>0$, $\kappa>0$, $(\mathbf{v}_0,w_0)\in H^2(\mathbb{R}^2)$ with $\nabla\cdot\mathbf{v}_0=0$. Then (5) with initial data $(\mathbf{v}_0,w_0)$ possesses a unique global classical solution $(\mathbf{v},w)$. In addition, for any $T>0$, $(\mathbf{v},w)$ satisfies*

$$\begin{gathered}(\mathbf{v},w)\in L^\infty\left(0,T;H^2\right),\qquad \omega_x\in L^2\left(0,T;H^1\right),\\ w_y\in L^2\left(0,T;H^2\right),\end{gathered}\tag{6}$$

where $\omega=\nabla\times\mathbf{v}$ is the vorticity.

Remark 2. Using the same method in this paper, we may also establish the global regularity for the following system:

$$\begin{gathered}\mathbf{v}_t-(\nu+\kappa)\,\mathbf{v}_{yy}-2\kappa\nabla\times w+\nabla\pi+(\mathbf{v}\cdot\nabla)\,\mathbf{v}=\mathbf{0},\\ w_t-\gamma w_{xx}+4\kappa w-2\kappa\nabla\times\mathbf{v}+(\mathbf{v}\cdot\nabla)\,w=0,\\ \nabla\cdot\mathbf{v}=0,\\ \mathbf{v}(0)=\mathbf{v}_0,\qquad w(0)=w_0.\end{gathered}\tag{7}$$

The rest of this paper is organized as follows. In Section 2, we recall an elementary lemma from [7]. Section 3 is devoted to establishing the a priori bounds for $\|\omega\|_2$ and $\|\nabla w\|_2$, while the bounds for $\|\nabla\omega\|_2$ and $\|\nabla^2 w\|_2$ are provided in Section 4. With the a priori estimates in Sections 3 and 4, we may conclude the proof of Theorem 1 as in [7]. Throughout this paper, the L^2-norm of a function f is denoted by $\|f\|_2$.

2. An Elementary Lemma

We recall in this section the following elementary lemma from [7].

Lemma 3. *Assume that f, g, g_y, h, and h_x all belong to $L^2(\mathbb{R}^2)$. Then,*

$$\iint|fgh|\,dx\,dy\le 2\|f\|_2\|g\|_2^{1/2}\left\|g_y\right\|_2^{1/2}\|h\|_2^{1/2}\left\|h_x\right\|_2^{1/2}.\tag{8}$$

Proof. We provide a proof of (8) simpler than that of [7].

Applying Hölder inequality,

$$\begin{aligned}F^2(x)&=\int 2F(x)\,F'(x)\,dx\\&\le 2\left(\int|F(x)|^2dx\right)^{1/2}\left(\int\left|F_x(x)\right|^2dx\right)^{1/2}.\end{aligned}\tag{9}$$

Thus,

$$\sup_{x\in\mathbb{R}}|F(x)|\le\sqrt{2}\left(\int|F(x)|^2dx\right)^{1/4}\left(\int\left|F_x(x)\right|^2dx\right)^{1/4}.\tag{10}$$

Consequently,

$$\begin{aligned}&\iint|fgh|\,dx\,dy\\&\le\int\left[\left(\int|f|^2dx\right)^{1/2}\left(\int|g|^2dx\right)^{1/2}\sup_{x\in\mathbb{R}}|h|\right]dy\\&\le\sqrt{2}\int\left[\left(\int|f|^2dx\right)^{1/2}\left(\int|g|^2dx\right)^{1/2}\right.\\&\qquad\left.\times\left(\int|h|^2dx\right)^{1/4}\left(\int\left|h_x\right|^2dx\right)^{1/4}\right]dy\\&\le\sqrt{2}\|f\|_2\left[\sup_{y\in\mathbb{R}}\left(\int|g|^2dx\right)^{1/2}\right]\|h\|_2^{1/2}\left\|h_x\right\|_2^{1/2}\\&\le\sqrt{2}\|f\|_2\left(\int\sup_{y\in\mathbb{R}}|g|^2dx\right)^{1/2}\|h\|_2^{1/2}\left\|h_x\right\|_2^{1/2}\\&\le 2\|f\|_2\left[\int\left(\int|g|^2dy\right)^{1/2}\left(\int\left|g_y\right|^2dy\right)^{1/2}dx\right]^{1/2}\\&\quad\times\|h\|_2^{1/2}\left\|h_x\right\|_2^{1/2}\\&\le 2\|f\|_2\|g\|_2^{1/2}\left\|g_y\right\|_2^{1/2}\|h\|_2^{1/2}\left\|h_x\right\|_2^{1/2}.\end{aligned}\tag{11}$$

□

3. A Priori Bounds for $\|\omega\|_2$ and $\|\nabla w\|_2$

In this section, we establish the a priori bounds for $\|\omega\|_2$ and $\|\nabla w\|_2$. First, we have the following energy estimates.

Proposition 4. *Assume $(\mathbf{v},w)$ solves (5) on $[0,T]$. Then,*

$$\begin{aligned}&\|\mathbf{v}(t)\|_2^2+\|w(t)\|_2^2+(\nu+\kappa)\int_0^t\left\|\mathbf{v}_x(\tau)\right\|_2^2d\tau\\&\quad+\gamma\int_0^t\left\|w_y(\tau)\right\|_2^2d\tau\le C\left\|(\mathbf{v}_0,w_0)\right\|_2^2.\end{aligned}\tag{12}$$

Here C is a constant depending only on ν, κ, γ, and T.

Proof. Taking the inner product of $(5)_1$ with $\mathbf{v}$ and $(5)_2$ with w in $L^2(\mathbb{R}^3)$, respectively, we deduce

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|(\mathbf{v},w)\|_2^2 + (\nu+\kappa)\|\mathbf{v}_x\|_2^2 + \gamma\left\|w_y\right\|_2^2 + 4\kappa\|w\|_2^2 &= 4\kappa\int(\nabla\times w)\cdot\mathbf{v}\,dx\,dy \\ &\equiv I,\end{aligned}\tag{13}$$

where we use the following facts (the first one being well-known in the mathematical theory of fluid dynamics, and its proof is provided in the appendix):

$$\begin{gathered}\nabla\cdot\mathbf{v}=0\Longrightarrow\int[(\mathbf{v}\cdot\nabla)\mathbf{v}]\cdot\Delta\mathbf{v}\,dx\,dy=0,\\ \int(\nabla\times w)\cdot\mathbf{v}\,dx\,dy=\int w\cdot(\nabla\times\mathbf{v})\,dx\,dy,\\ \int\nabla\pi\cdot\mathbf{v}\,dx\,dy=-\int\pi(\nabla\cdot\mathbf{v})\,dx\,dy=0.\end{gathered}\tag{14}$$

Now, I can be dominated as

$$\begin{aligned}I&=4\kappa\int(\nabla\times w)\cdot\mathbf{v}\,dx\,dy\\&=4\kappa\int\left(w_yv_1-w_xv_2\right)dx\,dy\\&=4\kappa\int\left(w_yv_1+w\partial_xv_2\right)dx\,dy\\&\le\frac{\gamma}{2}\left\|w_y\right\|_2^2+C\|\mathbf{v}\|_2^2+\frac{\nu+\kappa}{2}\|\mathbf{v}_x\|_2^2+C\|w\|_2^2.\end{aligned}\tag{15}$$

Substituting (15) into (13), we obtain (12) by invoking Gronwall inequality. □

Remark 5. Due to the partial dissipation and angular viscosity, we are not able to establish the uniform boundedness of $\|(\mathbf{v}(t),w(t))\|_2$ on $[0,\infty)$ but rather the exponential growth:

$$\|(\mathbf{v}(t),w(t))\|_2\le e^{Ct}\left\|(\mathbf{v}_0,w_0)\right\|_2.\tag{16}$$

Now, we are in a position to derive the bounds for $\|\omega\|_2$ and $\|\nabla w\|_2$.

Proposition 6. *Assume as in Proposition 4. Then the vorticity* $\omega=\nabla\times\mathbf{v}$ *and* w *satisfy*

$$\begin{aligned}\|(\omega(t),\nabla w(t))\|_2^2+2(\nu+\kappa)\int_0^t\left\|\omega_x(\tau)\right\|_2^2d\tau\\+2\gamma\int_0^t\left\|\nabla w_y(\tau)\right\|_2^2d\tau\le2\left\|(\omega_0,\nabla w_0)\right\|_2^2.\end{aligned}\tag{17}$$

Proof. Taking the curl of $(5)_1$, we find

$$\omega_t-(\nu+\kappa)\omega_{xx}-2\kappa\nabla\times(\nabla\times w)+(\mathbf{v}\cdot\nabla)\omega=0.\tag{18}$$

Then, taking the inner product of (18) with ω and $(5)_2$ with $-\Delta w$ in $L^2(\mathbb{R}^2)$, respectively, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|(\omega,\nabla w)\|_2^2&+(\nu+\kappa)\|\omega_x\|_2^2+\gamma\left\|\nabla w_y\right\|_2^2+4\kappa\|\nabla w\|_2^2\\&=2\kappa\int\nabla\times(\nabla\times w)\cdot\omega\,dx\,dy+2\kappa\int\nabla\times\mathbf{v}\cdot\Delta w\,dx\,dy\\&=-2\kappa\int\Delta w\cdot\omega\,dx\,dy+2\kappa\int\nabla\times\mathbf{v}\cdot\Delta w\,dx\,dy\\&=0.\end{aligned}\tag{19}$$

Applying Gronwall inequality, we may complete the proof of Proposition 6. □

4. A Priori Bounds for $\|\nabla\omega\|_2$ and $\|\nabla^2w\|_2$

This section is devoted to deriving the a priori bounds for $\|\nabla\omega\|_2$ and $\|\nabla^2w\|_2$.

Proposition 7. *Assume as in Proposition 6. Then,*

$$\begin{aligned}\|(\nabla\omega(t),\Delta w(t))\|_2^2+(\nu+\kappa)\int_0^t\left\|\nabla\omega_x(\tau)\right\|_2^2d\tau\\+\gamma\int_0^t\left\|\Delta w_y(\tau)\right\|_2^2d\tau\le C\left\|(\nabla\omega_0,\Delta w_0)\right\|_2^2.\end{aligned}\tag{20}$$

Here C *is a constant depending only on* ν, κ, γ, *and* T.

Proof. Taking the inner product of (18) with $-\Delta\omega$ and $(5)_2$ with Δ^2w in $L^2(\mathbb{R}^3)$, respectively, we find

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla\omega\|_2^2+(\nu+\kappa)\left\|\nabla\omega_x\right\|_2^2&=-2\kappa\int\nabla\times(\nabla\times w)\cdot\Delta\omega\,dx\,dy\\&\quad+\int[(\mathbf{v}\cdot\nabla)\omega]\cdot\Delta\omega\,dx\,dy,\\ \frac{1}{2}\frac{d}{dt}\|\Delta w\|_2^2+\gamma\left\|\Delta w_y\right\|_2^2&=-4\kappa\int w\cdot\Delta^2w\,dx\,dy\\&\quad+2\kappa\int\omega\cdot\Delta^2w\,dx\,dy-\int[(\mathbf{v}\cdot\nabla)w]\cdot\Delta^2w\,dx\,dy.\end{aligned}\tag{21}$$

Gathering the above equations together, noticing that

$$\begin{gathered}\nabla\times(\nabla\times w)=-\Delta w,\\ \nabla\cdot\mathbf{v}=0\Longrightarrow\int[(\mathbf{v}\cdot\nabla)\omega]\cdot\Delta\omega\,dx\,dy\\ =-\int[(\nabla\mathbf{v}\cdot\nabla)\omega]\cdot\nabla\omega\,dx\,dy,\end{gathered}\tag{22}$$

we see

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(\nabla\omega,\Delta w)\|_2^2+(\nu+\kappa)\left\|\nabla\omega_x\right\|_2^2+\gamma\left\|\Delta w_y\right\|_2^2\\
&\quad=-4\kappa\int|\Delta w|^2dx\,dy+4\kappa\int\Delta\omega\cdot\Delta w\,dx\,dy\\
&\qquad+\int[(\nabla\mathbf{v}\cdot\nabla)\,\omega]\cdot\nabla\omega\,dx\,dy\\
&\qquad-\int\Delta\left[(\mathbf{v}\cdot\nabla)\,w\right]\cdot\Delta w\,dx\,dy.
\end{aligned}\tag{23}$$

Expanding the right-hand side of (23) gives

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(\nabla\omega,\Delta w)\|_2^2+(\nu+\kappa)\left\|\nabla\omega_x\right\|_2^2+\gamma\left\|\Delta w_y\right\|_2^2+4\kappa\|\Delta w\|_2^2\\
&\quad\le4\kappa\int\omega_{xx}\cdot\Delta w\,dx\,dy+4\kappa\int\omega_{yy}\cdot\Delta w\,dx\,dy\\
&\qquad+\int\partial_xv_1\cdot\omega_x\cdot\omega_x\,dx\,dy+\int\partial_xv_2\cdot\omega_y\cdot\omega_x\,dx\,dy\\
&\qquad+\int\partial_yv_1\cdot\omega_x\cdot\omega_y\,dx\,dy+\int\partial_yv_2\cdot\omega_y\cdot\omega_y\,dx\,dy\\
&\qquad+C\int|\Delta\mathbf{v}|\cdot|\nabla w|\cdot|\Delta w|\,dx\,dy\\
&\qquad+C\int|\nabla\mathbf{v}|\cdot\left|\nabla^2w\right|\cdot|\Delta w|\,dx\,dy\\
&\quad\equiv\sum_{i=1}^{8}K_i.
\end{aligned}\tag{24}$$

For K_1, applying Hölder inequality yields

$$\begin{aligned}
K_1&=4\kappa\int\omega_{xx}\cdot\Delta w\,dx\,dy\\
&\le\int\frac{\nu+\kappa}{12}\left|\nabla\omega_x\right|^2+C|\Delta w|^2\,dx\,dy.
\end{aligned}\tag{25}$$

For K_2, integrating by parts gives

$$\begin{aligned}
K_2&=4\kappa\int\omega_{yy}\cdot\Delta w\,dx\,dy\\
&=-4\kappa\int\omega_y\cdot\Delta w_y\,dx\,dy\\
&\le\int C|\nabla\omega|^2+\frac{\gamma}{6}\left|\Delta w_y\right|^2\,dx\,dy.
\end{aligned}\tag{26}$$

For K_3, we apply Lemma 3 to deduce

$$\begin{aligned}
K_3&=\int\partial_xv_1\cdot\omega_x\cdot\omega_x dx\,dy\\
&\le C\left\|\partial_xv_1\right\|_2^{1/2}\left\|\partial_{xy}v_1\right\|_2^{1/2}\left\|\omega_x\right\|_2^{1/2}\left\|\omega_{xx}\right\|_2^{1/2}\left\|\omega_x\right\|_2\\
&\le C\|\omega\|_2^{1/2}\left\|\omega_x\right\|_2^{1/2}\left\|\omega_x\right\|_2^{1/2}\left\|\omega_{xx}\right\|_2^{1/2}\left\|\omega_x\right\|_2\\
&\le C\left\|\omega_x\right\|_2^2\left\|\nabla\omega_x\right\|_2^{1/2}\left(\text{By Proposition 6, }\ \omega\in L^\infty\left(0,T;L^2\right)\right)\\
&\le\frac{\nu+\kappa}{12}\left\|\nabla\omega_x\right\|_2^2+C\left\|\omega_x\right\|_2^{2/3}\|\nabla\omega\|_2^2.
\end{aligned}\tag{27}$$

Similarly, we have

$$\begin{aligned}
K_4&=\int\partial_xv_2\cdot\omega_y\cdot\omega_x dx\,dy\\
&\le C\left\|\partial_yv_2\right\|_2^{1/2}\left\|\partial_{xy}v_2\right\|_2^{1/2}\left\|\omega_y\right\|_2^{1/2}\left\|\omega_{yx}\right\|_2^{1/2}\left\|\omega_x\right\|_2\\
&\le C\|\omega\|_2^{1/2}\left\|\omega_x\right\|_2^{1/2}\left\|\omega_y\right\|_2^{1/2}\left\|\nabla\omega_x\right\|_2^{1/2}\left\|\omega_x\right\|_2\\
&\le C\left\|\omega_x\right\|_2^{3/2}\left\|\omega_x\right\|_2^{1/2}\left\|\nabla\omega_x\right\|_2^{1/2}\\
&\le\frac{\nu+\kappa}{12}\left\|\nabla\omega_x\right\|_2^2+C\left\|\omega_x\right\|_2^{2/3}\|\nabla\omega\|_2^2;\\
K_5&=\int\partial_yv_1\cdot\omega_x\cdot\omega_y\,dx\,dy\\
&\le C\left\|\partial_yv_1\right\|_2\left\|\omega_x\right\|_2^{1/2}\left\|\omega_{xy}\right\|_2^{1/2}\left\|\omega_y\right\|_2^{1/2}\left\|\omega_{yx}\right\|_2^{1/2}\\
&\le C\|\omega\|_2\left\|\omega_x\right\|_2^{1/2}\left\|\omega_y\right\|_2^{1/2}\left\|\nabla\omega_x\right\|_2\\
&\le\frac{\nu+\kappa}{12}\left\|\nabla\omega_x\right\|_2^2+C\|\nabla\omega\|_2^2;\\
K_6&=\int\partial_yv_2\cdot\omega_y\cdot\omega_y\,dx\,dy\\
&=-\int\partial_xv_1\cdot\omega_y\cdot\omega_y\,dx\,dy\\
&=2\int v_1\cdot\omega_y\cdot\omega_{yx}\,dx\,dy\\
&\le C\left\|v_1\right\|_2^{1/2}\left\|\partial_yv_1\right\|_2^{1/2}\left\|\omega_y\right\|_2^{1/2}\left\|\omega_{yx}\right\|_2^{1/2}\left\|\omega_{yx}\right\|_2\\
&\le C\left\|\omega_y\right\|_2^{1/2}\left\|\nabla\omega_x\right\|_2^{3/2}\\
&\le\frac{\nu+\kappa}{12}\left\|\nabla\omega_x\right\|_2^2+C\|\nabla\omega\|_2^2.
\end{aligned}\tag{28}$$

Now, for K_7, K_8, we use Lemma 3 and Young inequality to see

$$\begin{aligned}K_7 &= C\int |\Delta \mathbf{v}|\cdot|\nabla w|\cdot|\Delta w|\,dx\,dy\\ &\le C\|\Delta\mathbf{v}\|_2^{1/2}\|\Delta\mathbf{v}_x\|_2^{1/2}\|\nabla w\|_2\|\Delta w\|_2^{1/2}\|\Delta w_y\|_2^{1/2}\\ &\le C\|\nabla\omega\|_2^{1/2}\|\nabla\omega_x\|_2^{1/2}\|\nabla w\|_2\|\Delta w\|_2^{1/2}\|\Delta w_y\|_2^{1/2}\\ &\le \frac{\nu+\kappa}{12}\|\nabla\omega_x\|_2^2+\frac{\gamma}{6}\|\Delta w_y\|_2^2+C\left(\|\nabla w\|_2^2+\|\Delta w\|_2^2\right);\end{aligned} \tag{29}$$

$$\begin{aligned}K_8 &= C\int |\nabla\mathbf{v}|\cdot\left|\nabla^2 w\right|\cdot|\Delta w|\;dx\,dy\\ &\le C\|\nabla\mathbf{v}\|_2^{1/2}\|\nabla\mathbf{v}_x\|_2^{1/2}\left\|\nabla^2 w\right\|_2^{1/2}\left\|\nabla^2 w_y\right\|_2^{1/2}\|\Delta w\|_2\\ &\le C\|\omega\|_2^{1/2}\|\omega_x\|_2^{1/2}\|\Delta w\|_2^{1/2}\|\Delta w_y\|_2^{1/2}\|\Delta w\|_2\\ &\le \frac{\gamma}{6}\|\Delta w_y\|_2^2+C\|\omega_x\|_2^{2/3}\|\Delta w\|_2^2.\end{aligned} \tag{30}$$

Gathering (25)–(30) into (24) yields

$$\begin{aligned}&\frac{d}{dt}\|(\nabla\omega,\Delta w)\|_2^2+(\nu+\kappa)\|\nabla\omega_x\|_2^2+\gamma\|\Delta w_y\|_2^2\\ &\quad\le C\left(1+\|\omega_x\|_2^{2/3}\right)\|(\nabla\omega,\Delta w)\|_2^2.\end{aligned} \tag{31}$$

According to Proposition 6, we may invoke Gronwall inequality to deduce (20). □

Appendix

In this appendix, we provide the proof of (14) for reader's convenience.

Lemma A.1. *Let $\mathbf{v}=(v_1,v_2)\in H^2(\mathbb{R}^2)$ be divergence-free; that is, $\nabla\cdot\mathbf{v}=\partial_x v_1+\partial_y v_2=0$. Then*

$$\int[(\mathbf{v}\cdot\nabla)\,\mathbf{v}]\cdot\Delta\mathbf{v}\,dx\,dy=0. \tag{A.1}$$

Proof. Integration by parts formula gives

$$\begin{aligned}&\int[(\mathbf{v}\cdot\nabla)\,\mathbf{v}]\cdot\Delta\mathbf{v}\,dx\,dy\\ &=-\int[(\partial_x\mathbf{v}\cdot\nabla)\,\mathbf{v}]\cdot\partial_x\mathbf{v}\,dx\,dy\\ &\quad-\int\left[\left(\partial_y\mathbf{v}\cdot\nabla\right)\mathbf{v}\right]\cdot\partial_y\mathbf{v}\,dx\,dy\\ &=-\int\partial_x v_1\partial_x v_1\partial_x v_1\,dx\,dy-\int\partial_x v_1\partial_x v_2\partial_x v_2\,dx\,dy\\ &\quad-\int\partial_x v_2\partial_y v_1\partial_x v_1\,dx\,dy-\int\partial_x v_2\partial_y v_2\partial_x v_2\,dx\,dy\\ &\quad-\int\partial_y v_1\partial_x v_1\partial_y v_1\,dx\,dy-\int\partial_y v_1\partial_x v_2\partial_y v_2\,dx\,dy\\ &\quad-\int\partial_y v_2\partial_y v_1\partial_y v_1\,dx\,dy-\int\partial_y v_2\partial_y v_2\partial_y v_2\,dx\,dy\\ &\equiv\sum_{i=1}^{8}L_i.\end{aligned} \tag{A.2}$$

Noticing that $\partial_x v_1+\partial_y v_2=0$, we have

$$L_1+L_8=L_2+L_4=L_3+L_6=L_5+L_7=0. \tag{A.3}$$

Consequently, we have

$$\int[(\mathbf{v}\cdot\nabla)\,\mathbf{v}]\cdot\Delta\mathbf{v}\,dx\,dy=0 \tag{A.4}$$

as desired. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

Zujin Zhang was partially supported by the Youth Natural Science Foundation of Jiangxi Province (20132BAB211007) and the National Natural Science Foundation of China (11326138).

References

[1] A. C. Eringen, "Theory of micropolar fluids," *Journal of Applied Mathematics and Mechanics*, vol. 16, pp. 1–18, 1966.

[2] G. Łukaszewicz, *Micropolar Fluids, Theory and Applications*, Modeling and Simulation in Science, Engineering and Technology, Birkhäuser, Boston, Mass, USA, 1999.

[3] G. P. Galdi and S. Rionero, "A note on the existence and uniqueness of solutions of the micropolar fluid equations," *International Journal of Engineering Science*, vol. 15, no. 2, pp. 105–108, 1977.

[4] N. Yamaguchi, "Existence of global strong solution to the micropolar fluid system in a bounded domain," *Mathematical Methods in the Applied Sciences*, vol. 28, no. 13, pp. 1507–1526, 2005.

[5] B.-Q. Dong and Z.-M. Chen, "Asymptotic profiles of solutions to the 2D viscous incompressible micropolar fluid flows," *Discrete and Continuous Dynamical Systems. Series A*, vol. 23, no. 3, pp. 765–784, 2009.

[6] C. S. Cao, D. Regmi, and J. H. Wu, "The 2D MHD equations with horizontal dissipation and horizontal magnetic diffusion," *Journal of Differential Equations*, vol. 254, no. 7, pp. 2661–2681, 2013.

[7] C. S. Cao and J. H. Wu, "Global regularity for the 2D MHD equations with mixed partial dissipation and magnetic diffusion," *Advances in Mathematics*, vol. 226, no. 2, pp. 1803–1822, 2011.

[8] D. Chae, “Global regularity for the 2D Boussinesq equations with partial viscosity terms,” *Advances in Mathematics*, vol. 203, no. 2, pp. 497–513, 2006.

[9] T. Y. Hou and C. M. Li, “Global well-posedness of the viscous Boussinesq equations,” *Discrete and Continuous Dynamical Systems. Series A*, vol. 12, no. 1, pp. 1–12, 2005.

[10] B.-Q. Dong and Z. F. Zhang, “Global regularity of the 2D micropolar fluid flows with zero angular viscosity,” *Journal of Differential Equations*, vol. 249, no. 1, pp. 200–213, 2010.