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Research Article

Global Stability of a Discrete Mutualism Model

Kun Yang,[1] Xiangdong Xie,[2] and Fengde Chen[1]

[1] *College of Mathematics and Computer Science, Fuzhou University, Fuzhou, Fujian 350116, China*
[2] *Department of Mathematics, Ningde Normal University, Fujian 352100, China*

Correspondence should be addressed to Xiangdong Xie; ndsyxxd@163.com

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A discrete mutualism model is studied in this paper. By using the linear approximation method, the local stability of the interior equilibrium of the system is investigated. By using the iterative method and the comparison principle of difference equations, sufficient conditions which ensure the global asymptotical stability of the interior equilibrium of the system are obtained. The conditions which ensure the local stability of the positive equilibrium is enough to ensure the global attractivity are proved.

1. Introduction

There are many examples where the interaction of two or more species is to the advantage of all; we call such a situation the mutualism. For example, cellulose of white ants' gut provides nutrients for flagellates, while flagellates provide nutrients for white ants through the decomposition of cellulose to glucose. As was pointed out by Chen et al. [1] "the mutual advantage of mutualism or symbiosis can be very important. As a topic of theoretical ecology, even for two species, this area has not been as widely studied as the others even though its importance is comparable to that of predator-prey and competition interactions." Thus, it seems interesting to study some relevant topics on the symbiosis system.

The following model was proposed by Chen et al. [1] to describe the mutualism mechanism:

$$\begin{aligned} \frac{dN_1}{dt} &= r_1 N_1(t)\left[\frac{K_1+\alpha_1 N_2(t)}{1+N_2(t)} - N_1(t)\right], \\ \frac{dN_2}{dt} &= r_2 N_2(t)\left[\frac{K_2+\alpha_2 N_1(t)}{1+N_1(t)} - N_2(t)\right], \end{aligned} \tag{1}$$

where r_i refers to the intrinsic rate of population N_i and $\alpha_i > K_i$ $(i = 1, 2)$. In the absence of other species, the carrying capacity of the species N_i is K_i. Thanks to the cooperation of the other species, the carrying capacity of the species N_i becomes $(K_i + \alpha_i N_{3-i})/(1 + N_{3-i})$.

Li [2] argued that the nonautonomous one is more appropriate, and he proposed the following two-species cooperative model:

$$\begin{aligned} \frac{dN_1}{dt} &= r_1(t) N_1(t) \\ &\quad\times\left[\frac{K_1(t)+\alpha_1(t) N_2(t-\tau_2(t))}{1+N_2(t-\tau_2(t))} - N_1(t-\sigma_1(t))\right], \\ \frac{dN_2}{dt} &= r_2(t) N_2(t) \\ &\quad\times\left[\frac{K_2(t)+\alpha_2(t) N_1(t-\tau_1(t))}{1+N_1(t-\tau_1(t))} - N_2(t-\sigma_2(t))\right], \end{aligned} \tag{2}$$

where $r_i, K_i, \alpha_i, \tau_i, \sigma_i \in C(R, R+)$, $\alpha_i > K_i$, $r_i, K_i, \alpha_i, \tau_i, \sigma_i$ $(i = 1, 2)$ are periodic functions of period $\omega > 0$. Here the author incorporates the time delays to the model, which means that the cooperation effect needs to spend some time to realize, but not immediately realize. By applying the coincidence degree theory, Li showed that the system has at least one positive periodic solution. For more works related to the system (1) and (2), one could refer to [1–9] and the references cited therein.

It is well known that the discrete time models governed by difference equations are more appropriate than the continuous ones when the populations have nonoverlapping

generations, and discrete time models can also provide efficient computational models of continuous models for numerical simulations. Corresponding to system (2), Li [10] proposed the following delayed discrete model of mutualism:

$$\begin{aligned} x_1(k+1) &= x_1(k)\exp\left\{r_1(k)\left[\frac{K_1(k)+\alpha_1(k)x_2(k-\tau_2(k))}{1+x_2(k-\tau_2(k))} - x_1(k-\sigma_1(k))\right]\right\}, \\ x_1(k+1) &= x_1(k)\exp\left\{r_1(k)\left[\frac{K_2(k)+\alpha_2(k)x_1(k-\tau_1(k))}{1+x_1(k-\tau_1(k))} - x_2(k-\sigma_2(k))\right]\right\}. \end{aligned} \tag{3}$$

Under the assumption that $r_i, K_i, \alpha_i, \tau_i, \sigma_i$ $(i = 1,2)$ are positive periodic sequences with a common cycle ω, and $\alpha_i > K_i$ holds, by applying coincidence degree theory, he showed that system (3) has at least one positive ω-periodic solution, where ω is a positive integer. Chen [11] argued that the general nonautonomous nonperiodic system is more appropriate, and he showed that the system (3) is permanent. For more work about cooperative system, we can refer to [12–19].

It brings to our attention that neither Li [10] nor Chen [11] investigated the stability property of the system (3), which is one of the most important topics on the study of population dynamics. We mention here that, with $\sigma_i(k) \neq 0$, system (3) is a pure-delay system, and it is not an easy thing to investigate the stability property of the system. This motivated us to discuss a simple system, that is, the following autonomous cooperative system:

$$\begin{aligned} x_1(k+1) &= x_1(k)\exp\left\{r_1\left[\frac{K_1+\alpha_1 x_2(k)}{1+x_2(k)} - x_1(k)\right]\right\}, \\ x_2(k+1) &= x_2(k)\exp\left\{r_2\left[\frac{K_2+\alpha_2 x_1(k)}{1+x_1(k)} - x_2(k)\right]\right\}, \end{aligned} \tag{4}$$

where $x_i(k)$ $(i = 1,2)$ are the population density of the ith species at k-generation. r_i, K_i, $i = 1,2$, are all positive constants.

Throughout this paper, we assume that the coefficients of system (4) satisfy

(H_1) r_i, K_i, α_i $(i = 1,2)$ are all positive constants and $\alpha_i > K_i$ $(i = 1,2)$.

The aim of this paper is, by further developing the analysis technique of [15], to obtain a set of sufficient conditions to ensure the global asymptotical stability of the interior equilibrium of system (4). More precisely, we will prove the following result.

Theorem 1. *In addition to (H_1), further assume that*

(H_2) $r_i\alpha_i \leq 1$, $(i = 1,2)$

holds; then the unique positive equilibrium (x_1^*, x_2^*) of the system (4) is globally asymptotically stable.

The rest of the paper is arranged as follows. In Section 2 we will introduce a useful lemma and investigate the local stability property of the positive equilibrium. With the help of several useful lemmas, the global attractivity of positive equilibrium of the system (4) is investigated in Section 3. An example together with its numeric simulation is presented in Section 4 to show the feasibility of our results. We end this paper by a brief discussion.

2. Local Stability

In view of the actual ecological implications of system (4), we assume that the initial value $x_i(0) > 0$ $(i = 1,2)$. Obviously, any solution of system (4) with positive initial condition is well defined on Z_+, where $Z_+ = \{0,1,2,\ldots\}$ and remains positive for all $n \geq 0$.

We determine the positive equilibrium of the system (4) through solving the following equations:

$$\begin{aligned} x_1 &= x_1\exp\left[r_1\left(\frac{K_1+\alpha_1 x_2}{1+x_2} - x_1\right)\right], \\ x_2 &= x_2\exp\left[r_2\left(\frac{K_2+\alpha_2 x_1}{1+x_1} - x_2\right)\right], \end{aligned} \tag{5}$$

which is equivalent to

$$x_1 = \frac{K_1+\alpha_1 x_2}{1+x_2}, \qquad x_2 = \frac{K_2+\alpha_2 x_1}{1+x_1}, \tag{6}$$

and so

$$\begin{aligned} &(\alpha_1+1)x_2^2 + (1-\alpha_1\alpha_2+K_1-K_2)x_2 \\ &\quad - (K_2+\alpha_2 K_1) = 0. \end{aligned} \tag{7}$$

Since $\alpha_1 + 1 > 0$, $K_2 + \alpha_2 K_1 > 0$, (7) admits a unique positive solution $\overline{x}_2$. From the first equation of system (6), one could obtain $\overline{x}_1$; therefore, system (4) admits a unique positive equilibrium $E_+(\overline{x}_1, \overline{x}_2)$.

Following we will discuss the local stability of equilibrium $E_+(\overline{x}_1, \overline{x}_2)$. The Jacobian matrix of system (4) at $E_+(\overline{x}_1, \overline{x}_2)$ is as follows:

$$J(E_+) = \begin{pmatrix} 1 - r_1\overline{x}_1 & r_2\overline{x}_2\dfrac{\alpha_2 - K_2}{(1+\overline{x}_1)^2} \\ r_1\overline{x}_1\dfrac{\alpha_1 - K_1}{(1+\overline{x}_2)^2} & 1 - r_2\overline{x}_2 \end{pmatrix}. \tag{8}$$

The characteristic equation of $J(E_+)$ is

$$F(\lambda) = \lambda^2 + B\lambda + C = 0, \tag{9}$$

where

$$\begin{aligned} B &= -(2 - r_1\overline{x}_1 - r_2\overline{x}_2), \\ C &= (1 - r_1\overline{x}_1)(1 - r_2\overline{x}_2) \\ &\quad - r_1 r_2 \overline{x}_1\overline{x}_2 \frac{(\alpha_1 - K_1)(\alpha_2 - K_2)}{(1+\overline{x}_1)^2(1+\overline{x}_2)^2}. \end{aligned} \tag{10}$$

Lemma 2 (see [20]). *Let $F(\lambda) = \lambda^2 + B\lambda + C = 0$, where B and C are constants. Suppose $F(1) > 0$ and λ_1, λ_2 are two roots of $F(\lambda) = 0$. Then $|\lambda_1| < 1$ and $|\lambda_2| < 1$ if and only if $F(-1) > 0$ and $C < 1$.*

Let λ_1 and λ_2 be the two roots of (9), which are called eigenvalues of equilibrium $E_+(\overline{x}_1, \overline{x}_2)$. From [20] we know that if $|\lambda_1| < 1$ and $|\lambda_2| < 1$, then $E_+(\overline{x}_1, \overline{x}_2)$ is locally asymptotically stable.

Theorem 3. *Assume that (H_1) and (H_2) hold; then $E_+(\overline{x}_1, \overline{x}_2)$ is locally asymptotically stable.*

Proof. Since (7) has two real solutions, it follows that its discriminant is positive; that is,

$$\begin{aligned}\Delta &= (1 - \alpha_1\alpha_2 + K_1 - K_2)^2 + 4(\alpha_1 + 1)(K_2 + \alpha_2 K_1)\\ &= (1 + \alpha_1\alpha_2 + K_1 + K_2)^2 - 4(\alpha_1 - K_1)(\alpha_2 - K_2) > 0,\end{aligned} \tag{11}$$

and so

$$(1 + \alpha_1\alpha_2 + K_1 + K_2)^2 > 4(\alpha_1 - K_1)(\alpha_2 - K_2). \tag{12}$$

Equation (6) combined with the above inequality implies that

$$\begin{aligned}&(1 + \overline{x}_1)^2(1 + \overline{x}_2)^2\\ &\quad = \left(1 + \frac{K_1 + \alpha_1\overline{x}_2}{1 + \overline{x}_2}\right)^2 (1 + \overline{x}_2)^2\\ &\quad = ((\alpha_1 + 1)\overline{x}_2 + (K_1 + 1))^2\\ &\quad = \left(\frac{1 + \alpha_1\alpha_2 + K_1 + K_2 + \sqrt{\Delta}}{2}\right)^2\\ &\quad > \frac{(1 + \alpha_1\alpha_2 + K_1 + K_2)^2}{4}\\ &\quad > \frac{4(\alpha_1 - K_1)(\alpha_2 - K_2)}{4} = (\alpha_1 - K_1)(\alpha_2 - K_2).\end{aligned} \tag{13}$$

The above inequality leads to

$$1 - \frac{(\alpha_1 - K_1)(\alpha_2 - K_2)}{(1 + \overline{x}_1)^2(1 + \overline{x}_2)^2} > 0. \tag{14}$$

From (9) and (14) it follows that

$$\begin{aligned}F(1) &= 1 + B + C\\ &= r_1 r_2 \overline{x}_1 \overline{x}_2 \left(1 - \frac{(\alpha_1 - K_1)(\alpha_2 - K_2)}{(1 + \overline{x}_1)^2(1 + \overline{x}_2)^2}\right) > 0.\end{aligned} \tag{15}$$

Also, from (6) and (H_2), one has

$$\overline{x}_i < \alpha_i, \quad r_i\overline{x}_i < 1 \quad (i = 1, 2). \tag{16}$$

According to inequality (14) and (16), we have

$$\begin{aligned}F(-1) &= 1 - B + C\\ &= r_1 r_2 \overline{x}_1 \overline{x}_2 \left(1 - \frac{(\alpha_1 - K_1)(\alpha_2 - K_2)}{(1 + \overline{x}_1)^2(1 + \overline{x}_2)^2}\right)\\ &\quad + (4 - 2r_1\overline{x}_1 - 2r_2\overline{x}_2) > 0,\\ C - 1 &= -r_1\overline{x}_1 - r_2\overline{x}_2\\ &\quad + r_1 r_2 \overline{x}_1 \overline{x}_2 \left(1 - \frac{(\alpha_1 - K_1)(\alpha_2 - K_2)}{(1 + \overline{x}_1)^2(1 + \overline{x}_2)^2}\right)\\ &< -r_1\overline{x}_1(1 - r_2\overline{x}_2) - r_2\overline{x}_2 < 0.\end{aligned} \tag{17}$$

From Lemma 2 we can obtain that $E_+(\overline{x}_1, \overline{x}_2)$ is locally asymptotically stable. This completes the proof of Theorem 3. □

3. Global Stability

We will give a strict proof of Theorem 1 in this section. To achieve this objective, we introduce several useful lemmas.

Lemma 4 (see [20]). *Let $f(u) = u\exp(\alpha - \beta u)$, where α and β are positive constants; then $f(u)$ is nondecreasing for $u \in (0, (1/\beta)]$.*

Lemma 5 (see [20]). *Assume that sequence $\{u(k)\}$ satisfies*

$$u(k+1) = u(k)\exp(\alpha - \beta u(k)), \quad k = 1, 2, \ldots, \tag{18}$$

where α and β are positive constants and $u(0) > 0$. Then

(i) *if $\alpha < 2$, then $\lim_{k\to+\infty} u(k) = \alpha/\beta$;*

(ii) *if $\alpha < 1$, then $u(k) \le (1/\beta)$, $k = 2, 3, \ldots$.*

Lemma 6 (see [21]). *Suppose that functions $f, g : Z_+ \times [0, \infty) \to [0, \infty)$ satisfy $f(k, x) \le g(k, x)$ $(f(k, x) \ge g(k, x))$ for $k \in Z_+$ and $x \in [0, \infty)$ and $g(k, x)$ is nondecreasing with respect to x. If $\{x(k)\}$ and $\{u(k)\}$ are the nonnegative solutions of the difference equations*

$$x(k+1) = f(k, x(k)), \qquad u(k+1) = g(k, u(k)), \tag{19}$$

respectively, and $x(0) \le u(0)$ $(x(0) \ge u(0))$, then

$$x(k) \le u(k) \; (x(k) \ge u(k)) \quad \forall k \ge 0. \tag{20}$$

Proof of Theorem 1. Let $(x_1(k), x_2(k))$ be arbitrary solution of system (4) with $x_1(0) > 0$ and $x_2(0) > 0$. Denote

$$\begin{aligned}U_1 &= \limsup_{k\to+\infty} x_1(k), \qquad V_1 = \liminf_{k\to+\infty} x_1(k),\\ U_2 &= \limsup_{k\to+\infty} x_2(k), \qquad V_2 = \liminf_{k\to+\infty} x_2(k).\end{aligned} \tag{21}$$

We claim that $U_1 = V_1 = \overline{x}_1$ and $U_2 = V_2 = \overline{x}_2$.

From the first equation of system (4), we obtain

$$x_1(k+1) \le x_1(k)\exp\{r_1\alpha_1 - r_1x_1(k)\}, \quad k = 0,1,2,\ldots. \tag{22}$$

Consider the auxiliary equation as follows:

$$u(k+1) = u(k)\exp\{r_1\alpha_1 - r_1u(k)\}, \quad k = 0,1,2,\ldots. \tag{23}$$

Because of $0 < r_1\alpha_1 \le 1$, according to (ii) of Lemma 5, we can obtain $u(k) \le (1/r_1)$ for all $k \ge 2$, where $u(k)$ is arbitrary positive solution of (23) with initial value $u(0) > 0$. From Lemma 4, $f(u) = u\exp(r_1\alpha_1 - r_1u)$ is nondecreasing for $u \in (0,(1/r_1)]$. According to Lemma 6 we can obtain $x(k) \le u(k)$ for all $k \ge 2$, where $u(k)$ is the solution of (23) with the initial value $u(2) = x(2)$. According to (i) of Lemma 5, we can obtain

$$U_1 = \limsup_{k\to+\infty} x_1(k) \le \lim_{k\to+\infty} u(k) = \alpha_1 \overset{\text{def}}{=} M_1^{x_1}. \tag{24}$$

From the second equation of system (4), we obtain

$$x_2(k+1) \le x_2(k)\exp\{r_2\alpha_2 - r_2x_2(k)\}, \quad k = 0,1,2,\ldots. \tag{25}$$

Similar to the above analysis, we have

$$U_2 = \limsup_{k\to+\infty} x_2(k) \le \lim_{k\to+\infty} u(k) = \alpha_2 \overset{\text{def}}{=} M_1^{x_2}. \tag{26}$$

Then, for sufficiently small constant $\varepsilon > 0$, there is an integer $k_1 > 2$ such that

$$x_1(k) \le M_1^{x_1} + \varepsilon, \qquad x_2(k) \le M_1^{x_2} + \varepsilon \quad \forall k > k_1. \tag{27}$$

According to the first equation of system (4) we can obtain

$$x_1(k+1) \ge x_1(k)\exp\{r_1K_1 - r_1x_1(k)\}. \tag{28}$$

Consider the auxiliary equation as follows:

$$u(k+1) = u(k)\exp\{r_1K_1 - r_1u(k)\}. \tag{29}$$

Since $0 < r_1K_1 \le 1$, according to (ii) of Lemma 5, we can obtain $u(k) \le (1/K_1)$ for all $k \ge 2$, where $u(k)$ is arbitrary positive solution of (23) with initial value $u(0) > 0$. From Lemma 4, $f(u) = u\exp(r_1K_1 - r_1u)$ is nondecreasing for $u \in (0,(1/K_1)]$. According to Lemma 6 we can obtain $x(k) \ge u(k)$ for all $k \ge 2$, where $u(k)$ is the solution of (23) with the initial value $u(k_1) = x(k_1)$. According to (i) of Lemma 5, we have

$$V_1 = \liminf_{k\to+\infty} x_1(k) \ge \lim_{k\to+\infty} u(k) = K_1 \overset{\text{def}}{=} N_1^{x_1}. \tag{30}$$

From the second equation of system (4), we obtain

$$x_2(k+1) \ge x_2(k)\exp\{r_2K_2 - r_2x_2(k)\}. \tag{31}$$

Similar to the above analysis, we have

$$V_2 = \liminf_{k\to+\infty} x_2(k) \ge \lim_{k\to+\infty} u(k) = K_2 \overset{\text{def}}{=} N_1^{x_2}. \tag{32}$$

Then, for sufficiently small constant $\varepsilon > 0$, there is an integer $k_2 > k_1$ such that

$$x_1(k) \ge N_1^{x_1} - \varepsilon, \qquad x_2(k) \ge N_1^{x_2} - \varepsilon \quad \forall k > k_2. \tag{33}$$

Equation (24) combined with the first equation of system (4) leads to

$$x_1(k+1) \le x_1(k)\exp\left\{r_1\left[\frac{K_1 + \alpha_1\left(M_1^{x_2} + \varepsilon\right)}{1 + \left(M_1^{x_2} + \varepsilon\right)} - x_1(k)\right]\right\}, \quad k > k_2. \tag{34}$$

Similar to the analysis of (23) and (24), we have

$$U_1 = \limsup_{k\to+\infty} x_1(k) \le \frac{K_1 + \alpha_1\left(M_1^{x_2} + \varepsilon\right)}{1 + \left(M_1^{x_2} + \varepsilon\right)}. \tag{35}$$

Because of arbitrariness of $\varepsilon > 0$, we have $U_1 \le M_2^{x_1}$, where

$$M_2^{x_1} = \frac{K_1 + \alpha_1 M_1^{x_2}}{1 + M_1^{x_2}} < M_1^{x_1}. \tag{36}$$

Equation (24) combined with the second equation of system (4) leads to

$$x_2(k+1) \le x_2(k)\exp\left\{r_2\left[\frac{K_2 + \alpha_2\left(M_1^{x_1} + \varepsilon\right)}{1 + \left(M_1^{x_1} + \varepsilon\right)} - x_2(k)\right]\right\}, \quad k > k_2. \tag{37}$$

Similar to the analysis of (23) and (24), we can obtain

$$U_2 = \limsup_{k\to+\infty} x_2(k) \le \frac{K_2 + \alpha_2\left(M_1^{x_1} + \varepsilon\right)}{1 + \left(M_1^{x_1} + \varepsilon\right)}. \tag{38}$$

Because of arbitrariness of $\varepsilon > 0$, we have $U_2 \le M_2^{x_2}$, where

$$M_2^{x_2} = \frac{K_2 + \alpha_2 M_1^{x_1}}{1 + M_1^{x_1}} < M_1^{x_2}. \tag{39}$$

Then, for sufficiently small constant $\varepsilon > 0$, there is an integer $k_3 > k_2$ such that

$$x_1(k) \le M_2^{x_1} - \varepsilon, \qquad x_2(k) \le M_2^{x_2} - \varepsilon, \quad \forall k > k_3. \tag{40}$$

Equation (27) combined with the first equation of system (4) leads to

$$x_1(k+1) \ge x_1(k)\exp\left\{r_1\left[\frac{K_1 + \alpha_1\left(N_1^{x_2} - \varepsilon\right)}{1 + \left(N_1^{x_2} - \varepsilon\right)} - x_1(k)\right]\right\}, \quad k > k_3. \tag{41}$$

From this, we can finally obtain

$$V_1 = \liminf_{k\to+\infty} x_1(k) \geq \frac{K_1 + \alpha_1\left(N_1^{x_2} - \varepsilon\right)}{1 + \left(N_1^{x_2} - \varepsilon\right)}. \tag{42}$$

Because of the arbitrariness of ε, we have $V_1 \geq N_2^{x_1}$, where

$$N_2^{x_1} = \frac{K_1 + \alpha_1 N_1^{x_2}}{1 + N_1^{x_2}} > K_1 = N_1^{x_1}. \tag{43}$$

Equation (27) combined with the first equation of system (4) leads to

$$x_2(k+1) \geq x_2(k)\exp\left\{r_1\left[\frac{K_2 + \alpha_2\left(N_1^{x_1} - \varepsilon\right)}{1 + \left(N_1^{x_1} - \varepsilon\right)} - x_1(k)\right]\right\}, \quad k > k_3. \tag{44}$$

From the above inequality we can obtain

$$V_2 = \liminf_{k\to+\infty} x_2(k) \geq \frac{K_2 + \alpha_2\left(N_1^{x_1} - \varepsilon\right)}{1 + \left(N_1^{x_1} - \varepsilon\right)}. \tag{45}$$

Because of the arbitrariness of ε, we have $V_2 \geq N_2^{x_2}$, where

$$N_2^{x_2} = \frac{K_2 + \alpha_2 N_1^{x_1}}{1 + N_1^{x_1}} > K_2 = N_1^{x_2}. \tag{46}$$

Then, for sufficiently small constant $\varepsilon > 0$, there is an integer $k_4 > k_3$ such that

$$x_1(k) \geq N_2^{x_1} - \varepsilon, \qquad x_2(k) \geq N_2^{x_2} - \varepsilon \qquad \forall k > k_4. \tag{47}$$

Continuing the above steps, we can get four sequences $\{M_k^{x_1}\}$, $\{M_k^{x_2}\}$, $\{N_k^{x_1}\}$, and $\{N_k^{x_1}\}$ such that

$$M_k^{x_1} = \frac{K_1 + \alpha_1 M_{k-1}^{x_2}}{1 + M_{k-1}^{x_2}}, \qquad M_k^{x_2} = \frac{K_2 + \alpha_2 M_{k-1}^{x_1}}{1 + M_{k-1}^{x_1}}; \tag{48}$$

$$N_k^{x_1} = \frac{K_1 + \alpha_1 N_{k-1}^{x_2}}{1 + N_{k-1}^{x_2}}, \qquad N_k^{x_2} = \frac{K_2 + \alpha_2 N_{k-1}^{x_1}}{1 + N_{k-1}^{x_1}}. \tag{49}$$

Clearly, we have

$$\begin{gathered} N_k^{x_1} \leq V_1 \leq U_1 \leq M_k^{x_1}, \\ N_k^{x_2} \leq V_2 \leq U_2 \leq M_k^{x_2}, \\ k = 0, 1, 2, \ldots. \end{gathered} \tag{50}$$

Now, we will prove $\{M_k^{x_i}\}$ $(i = 1, 2)$ is monotonically decreasing and $\{N_k^{x_i}\}$ $(i = 1, 2)$ is monotonically increasing by means of inductive method.

First of all, it is clear that $M_2^{x_i} \leq M_1^{x_i}$, $N_2^{x_i} \geq N_1^{x_i}$ $(i = 1, 2)$. For $i \geq 2$, we assume that $M_i^{x_1} \leq M_{i-1}^{x_1}$ and $N_i^{x_1} \geq N_{i-1}^{x_1}$ hold; then

$$M_{i+1}^{x_2} = \frac{K_2 + \alpha_2 M_i^{x_1}}{1 + M_i^{x_1}} \leq \frac{K_2 + \alpha_2 M_{i-1}^{x_1}}{1 + M_{i-1}^{x_1}} = M_i^{x_2}, \tag{51}$$

$$N_{i+1}^{x_2} = \frac{K_2 + \alpha_2 N_i^{x_1}}{1 + N_i^{x_1}} \geq \frac{K_2 + \alpha_2 N_{i-1}^{x_1}}{1 + N_{i-1}^{x_1}} = N_i^{x_2}, \tag{52}$$

$$M_{i+1}^{x_1} = \frac{K_1 + \alpha_1 M_i^{x_2}}{1 + M_i^{x_2}} \leq \frac{K_1 + \alpha_1 M_{i-1}^{x_2}}{1 + M_{i-1}^{x_2}} = M_i^{x_1}, \tag{53}$$

$$N_{i+1}^{x_1} = \frac{K_1 + \alpha_1 N_i^{x_2}}{1 + N_i^{x_2}} \geq \frac{K_1 + \alpha_1 N_{i-1}^{x_2}}{1 + N_{i-1}^{x_2}} = N_i^{x_1}. \tag{54}$$

Equations (51)–(54) show that $\{M_k^{x_1}\}$ and $\{M_k^{x_2}\}$ are monotonically decreasing and $\{N_k^{x_1}\}$ and $\{N_k^{x_2}\}$ are monotonically increasing. Consequently, $\lim_{k\to+\infty}\{M_k^{x_i}\}$ and $\lim_{k\to+\infty}\{N_k^{x_i}\}$ $(i = 1, 2)$ both exist. Let

$$\lim_{k\to+\infty} M_k^{x_i} = X_i^*, \quad \lim_{k\to+\infty} N_k^{x_i} = x_i^*, \quad i = 1, 2. \tag{55}$$

From (48) and (55), we have

$$\begin{gathered} X_1^* = \frac{K_1 + \alpha_1 X_2^*}{1 + X_2^*}, \\ X_2^* = \frac{K_2 + \alpha_2 X_1^*}{1 + X_1^*}. \end{gathered} \tag{56}$$

From (49) and (55), we get

$$\begin{gathered} x_1^* = \frac{K_1 + \alpha_1 x_2^*}{1 + x_2^*}, \\ x_2^* = \frac{K_2 + \alpha_2 x_1^*}{1 + x_1^*}. \end{gathered} \tag{57}$$

Equations (56) and (57) show that (X_1^*, X_2^*) and (x_1^*, x_2^*) are all solutions of system (6). However, system (6) has unique positive solution $(\overline{x}_1, \overline{x}_2)$. Therefore

$$U_i = V_i = \lim_{k\to+\infty} x_i(k) = \overline{x}_i, \quad i = 1, 2. \tag{58}$$

That is, $E_+(\overline{x}_1, \overline{x}_2)$ is globally attractive.

From Theorem 3, we get that equilibrium $E_+(\overline{x}_1, \overline{x}_2)$ is locally asymptotically stable. And so, $E_+(\overline{x}_1, \overline{x}_2)$ is globally asymptotically stable. This ends the proof of Theorem 1. □

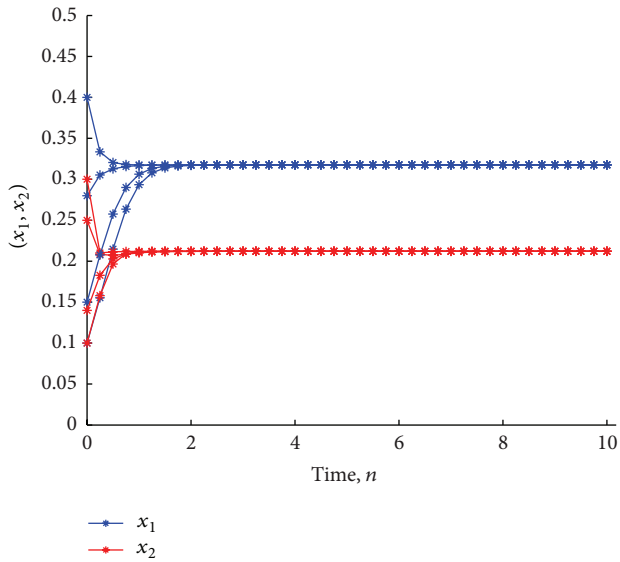

FIGURE 1: Dynamic behaviors of the solution $(x_1(t), x_2(t))$ of system (59), with the initial conditions $(x_1(0), x_2(0)) = (0.1, 0.25)$, $(0.28, 0.3)$, $(0.15, 0.14)$, and $(0.4, 0.1)$, respectively.

4. Example

In this section, we will give an example to illustrate the feasibility of the main result.

Example. Consider the following cooperative system:

$$x_1(k+1) = x_1(k) \exp\left\{2\left[\frac{0.3 + 0.4x_2(k)}{1 + x_2(k)} - x_1(k)\right]\right\},$$
$$x_2(k+1) = x_2(k) \exp\left\{4\left[\frac{0.2 + 0.25x_1(k)}{1 + x_1(k)} - x_2(k)\right]\right\}. \tag{59}$$

By calculating, we have that positive equilibrium $E_+(\overline{x}_1, \overline{x}_2) = (0.317495, 0.212049)$, $r_1\alpha_1 = 0.8 < 1$, $r_2\alpha_2 = 1$, $K_i < \alpha_i$ $(i = 1, 2)$ and the coefficients of system (59) satisfy (H_1) and (H_2). From Theorem 1, positive equilibrium $E_+(\overline{x}_1, \overline{x}_2)$ is globally asymptotically stable. Numeric simulation also supports our finding (see Figure 1).

5. Discussion

It is well known [6] that, for autonomous two-species Lotka-Volterra mutualism model, the conditions which ensure the existence of positive equilibrium are enough to ensure that the equilibrium is globally stable. However, for the two-species discrete Lotka-Volterra mutualism model, Lu and Wang [22] proved that a cooperative system cannot be permanent. That is, the dynamic behaviors of discrete Lotka-Volterra mutualism model are very different to the continuous ones.

Recently, by using the iterative method, Xie et al. [8] showed that, for a mutualism model with infinite delay, conditions which ensure the permanence of the system are enough to ensure the global stability of the system. As a corollary of their result, one could draw the conclusion that system (1) admits a unique positive equilibrium, which is globally stable. One interesting issue is proposed. For the discrete type mutualism model (4), is there any relationship between the existence of positive equilibrium and the stability property of the positive equilibrium?

In this paper, by using the linear approximation, comparison principle of difference equations, and method of iteration scheme, we showed that the conditions which ensure the local stability property of the positive equilibrium ((H_2) $0 < r_i\alpha_i \leq 1$ $(i = 1, 2)$) are also enough to guarantee the global stability of the positive equilibrium $E_+(\overline{x}_1, \overline{x}_2)$.

At the end of this paper, we would like to mention here that, for the Lotka-Volterra type mutualism system with time delay, delay is one of the most important factors to influence the dynamic behaviors of the system [23–25]. It seems interesting to incorporate the time delay to the system (4) and investigate the dynamic behaviors of the system; we leave this for future study.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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