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Research Article

The Iteration Solution of Matrix Equation $AXB = C$ Subject to a Linear Matrix Inequality Constraint

Na Huang and Changfeng Ma

School of Mathematics and Computer Science, Fujian Normal University, Fuzhou 350007, China

Correspondence should be addressed to Changfeng Ma; macf@fjnu.edu.cn

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We propose a feasible and effective iteration method to find solutions to the matrix equation $AXB = C$ subject to a matrix inequality constraint $DXE \geq F$, where $DXE \geq F$ means that the matrix $DXE - F$ is nonnegative. And the global convergence results are obtained. Some numerical results are reported to illustrate the applicability of the method.

1. Introduction

In this paper, we consider the following problem:

$$\begin{aligned} AXB &= C, \\ DXE &\geq F, \end{aligned} \tag{1}$$

where $A \in R^{p\times m}$, $B \in R^{n\times q}$, $C \in R^{p\times q}$, $D \in R^{s\times m}$, $E \in R^{n\times t}$, and $F \in R^{s\times t}$ are known constant matrixes and $X \in R^{m\times n}$ is unknown matrix.

The solutions X to the linear matrix equation with special structures have been widely studied, for example, symmetric solutions (see [1–5]), R-symmetric solutions (see [6]), (R, S)-symmetric solutions (see [7, 8]), bisymmetric solutions (see [9–12]), centrosymmetric solutions (see [13]), and other general solutions (see [14–23]). Some iterative methods to solve a pair of linear matrix equations have been studied as well (see [24–31]).

However, very little research has been done on the solutions to a matrix equation subjected to a matrix inequality constraint. In 2012, Peng et al. (see [32]) proposed a feasible and effective algorithm to find solutions to the matrix equation $AX = B$ subjected to a matrix inequality constraint $CXD \geq E$ based on the polar decomposition in Hilbert space. Next year, Li et al. (see [33]) used a similar way to study the bisymmetric solutions to the same problem. Motivated and inspired by the work mentioned above, in this paper, we consider the solutions of the matrix equation $AXB = C$ over linear inequality $DXE \geq F$ constraint. We use the theory on the analytical solution of matrix equation $AXB = C$ to transform the problem into a matrix inequality smallest nonnegative deviation problem. And then, combined with the polar decomposition theory, an iterative method for solving this transformed problem is proposed. Meanwhile, the global convergence results are obtained. Some numerical results are reported and indicate that the proposed method is quite effective.

Throughout this paper, we use the following notation: for $A \in R^{m\times n}$, we write A^T, A^+, and $\|A\|$ to denote the transpose, the Moore-Penrose generalized inverse, and the Frobenius norm of the matrix A, respectively. For any $A = (a_{ij})$, $B = (b_{ij})$, we write $A \geq B$ if $a_{ij} \geq b_{ij}$. $A \otimes B$ denotes the Kronecker product defined as $A \otimes B = (a_{ij}B)$. For the matrix $X = (x_1, x_2, \ldots, x_n) \in R^{m\times n}$, $\mathrm{vec}(X)$ denotes the vec operator defined as $\mathrm{vec}(X) = (x_1^T, x_2^T, \ldots, x_n^T)^T$. For $A = (a_{ij}) \in R^{m\times n}$, $[A]_+$ is a matrix with ijth entry equal to $\max\{0, a_{ij}\}$. Obviously, $A = [A]_+ - [-A]_+$. The inner product in space $R^{m\times n}$ is defined as

$$\langle A, B\rangle = \mathrm{tr}\left(B^T A\right), \quad \forall A,\ B \in R^{m\times n}. \tag{2}$$

Hence $R^{m\times n}$ is a Hilbert inner product space, and the norm of a matrix generated by this inner product space is the Frobenius norm.

This paper is organized as follows. In Section 2, we transform problem (1) into a matrix inequality smallest nonnegative deviation problem. Then we study the existence of the solutions for problem (1) in Section 3. The iterative method to the transformed problem and convergence analysis are presented in Section 4. Section 5 shows some numerical experiments. Finally, we conclude this paper in Section 6.

2. Transforming the Original Problem

In this section, we use the theory on the analytical solution of matrix equation $AXB = C$ to transform problem (1) into a matrix inequality smallest nonnegative deviation problem. Firstly, we present the following lemma about the analytical solution of matrix equation $AXB = C$.

Lemma 1 (see Theorem 1.21 in [34]). *Given $A \in R^{p\times m}$, $B \in R^{n\times q}$, and $C \in R^{p\times q}$, the matrix equation $AXB = C$ is solvable for X in $R^{m\times n}$ if and only if $AA^+CB^+B = C$. Moreover, if the matrix equation $AXB = C$ is solvable, then the general solutions can be expressed as*

$$X = A^+CB^+ + G - A^+AGBB^+, \tag{3}$$

where G is an arbitrary $m \times n$ matrix.

Assume that $AA^+CB^+B = C$; that is, assume that the matrix equation $AXB = C$ is solvable. Substituting (3) into the second inequality of (1), we get

$$D\left(A^+CB^+ + G - A^+AGBB^+\right)E \geq F. \tag{4}$$

By simple calculation, we have

$$D\left(G - A^+AGBB^+\right)E \geq F - DA^+CB^+E. \tag{5}$$

Hence (3) is a solution of (1) if and only if the matrix G in (1) satisfies (5). However, inequality (5) may be unsolvable; namely,

$$\left\{G \in R^{m\times n} \mid D\left(G - A^+AGBB^+\right)E \geq \widetilde{F}\right\} = \emptyset, \tag{6}$$

where $\widetilde{F} = F - DA^+CB^+E$. In this case, we can find $Y \in R_+^{s\times t}$ such that

$$D\left(G - A^+AGBB^+\right)E + Y \geq \widetilde{F} \tag{7}$$

is solvable (if (5) is solvable, then $Y = 0$). Obviously, there exist many $Y \in R_+^{s\times t}$ satisfied (7). Here we need to find a Y such that $\|Y\| \leq \|\overline{Y}\|$, where Y, $\overline{Y}$ satisfied (7). Thus we consider the following smallest nonnegative deviation of the matrix inequality, which is also a quadratic programming problem:

$$\begin{aligned} &\text{minimize} \quad P(G,Y) = \|Y\|^2 \\ &\text{subject to} \quad D\left(G - A^+AGBB^+\right)E + Y \geq \widetilde{F}, \\ &\qquad\qquad Y \geq 0, \quad G \in R^{m\times n}. \end{aligned} \tag{8}$$

If G and Y solve (8) with $Y = 0$, then G solves (7), and (3) is a solution of (1). If G and Y solve (8) $Y \neq 0$, then G solves (7) in the smallest nonnegative deviation sense, and (3) is a solution of the matrix equation $AXB = C$ over the nonnegative smallest deviation constraint of the inequality $DXE \geq F$. Conversely, if G solves (7), then G and $Y = 0$ solve (8). So, to find $X \in R^{m\times n}$ satisfied (1), we only need to solve the smallest nonnegative deviation problem (8).

Suppose that G and Y solve (8); then

$$Y = \left[\widetilde{F} - D\left(G - A^+AGBB^+\right)E\right]_+. \tag{9}$$

On the other hand, if a pair of matrices G and Y solves the smallest nonnegative deviation problem (8), then there exists a nonnegative matrix Z satisfying

$$D\left(G - A^+AGBB^+\right)E + Y - Z = \widetilde{F}. \tag{10}$$

Consequently, the smallest nonnegative deviation problem (8) is equivalent to the following optimization problem:

$$\begin{aligned} &\text{minimize} \quad P(G,Y,Z) = \|Y\|^2 \\ &\text{subject to} \quad D\left(G - A^+AGBB^+\right)E + Y - Z = \widetilde{F}, \\ &\qquad\qquad Y \geq 0, \quad Z \geq 0, \quad G \in R^{m\times n}. \end{aligned} \tag{11}$$

Eliminating Y from (11) yields the following optimization problem:

$$\begin{aligned} &\text{minimize} \quad F(G,Z) = \left\|D(G - A^+AGBB^+)E - Z - \widetilde{F}\right\|^2 \\ &\text{subject to} \quad Z \geq 0, \quad G \in R^{m\times n}. \end{aligned} \tag{12}$$

Suppose that a pair (G, Z) solves (12); then

$$Z = \left[D\left(G - A^+AGBB^+\right)E - \widetilde{F}\right]_+, \tag{13}$$

which together with the matrices $Y = [\widetilde{F} - D(G - A^+AGBB^+)E]_+$ and G solves (11), and hence G and Y solve (8). This allows one to determine whether or not (3) is a solution of (1). Therefore, to solve (1), we first solve optimization problem (12). Our iteration method proposed below will take advantage of these equivalent forms of (1).

3. The Solution of the Problem

To illustrate the existence of the solutions G^*, Y^*, and Z^* of (8), (11), and (12), we give the following theorem in the first place.

Theorem 2 (see [35, 36]). *Let $\mathscr{M} \subseteq R^{s\times t}$ be a closed convex cone (i.e., $\mathscr{M}$ is closed convex set and $\alpha u \in \mathscr{M}$, for all $u \in \mathscr{M}$ and $\alpha \geq 0$). Let $\mathscr{M}^*$ be the polar cone of $\mathscr{M}$; that is,*

$$\mathscr{M}^* = \left\{y \in R^{s\times t} \mid \langle u, y\rangle \leq 0, \forall u \in M, y \geq 0\right\}. \tag{14}$$

Then for all $f \in R^{s\times t}$, f has unique polar decomposition of the form

$$f = \hat{u} + \hat{y}, \quad \hat{u} \in \mathscr{M}, \quad \hat{y} \in \mathscr{M}^*, \quad \langle \hat{u}, \hat{y}\rangle = 0. \tag{15}$$

Theorem 2 implies that $\hat{u}$ is the projection of f onto $\mathscr{M}$ and $\hat{y}$ is the projection of f onto $\mathscr{M}^*$.

For problem (12), we give the following two matrix sets:

$$\mathscr{M} = \left\{Q \in R^{s\times t} \mid Q = D\left(G - A^{+}AGBB^{+}\right)E - Z, \quad Z \geq 0, G \in R^{m\times n}\right\},$$
$$\mathscr{N} = \left\{Y \in R^{s\times t} \mid D^TYE^T - A^{+}AD^TYE^TBB^{+} = 0, Y \geq 0\right\}. \tag{16}$$

Now we will prove that $\mathscr{M}$ is a closed convex cone and $\mathscr{N} = \mathscr{M}^*$.

Lemma 3. *The matrix set $\mathscr{M}$ is a closed convex cone in the Hilbert space $R^{s\times t}$.*

Proof. For all $Q \in \mathscr{M}$, there exists $Z \geq 0$ and $G \in R^{m\times n}$ such that $Q = D(G - A^{+}AGBB^{+})E - Z$. By the definition of Kronecker product, we have

$$\begin{aligned}
&\operatorname{vec}(Q) \\
&= \left[E^T \otimes D - \left(BB^{+}E\right)^T \otimes \left(DA^{+}A\right)\right]\operatorname{vec}(G) \\
&\quad - (I \otimes I)\operatorname{vec}(Z) \\
&= \left[E^T \otimes D - \left(E^TBB^{+}\right) \otimes \left(DA^{+}A\right)\right] \\
&\quad \times \left([\operatorname{vec}(G)]_{+} - [\operatorname{vec}(-G)]_{+}\right) - (I \otimes I)\operatorname{vec}(Z) \\
&= \left(E^T \otimes D - \left(E^TBB^{+}\right) \otimes \left(DA^{+}A\right),\right. \\
&\quad \left. - E^T \otimes D + \left(E^TBB^{+}\right) \otimes \left(DA^{+}A\right), -I \otimes I\right) \\
&\quad \cdot \left([\operatorname{vec}(G)]_{+}^T, [\operatorname{vec}(-G)]_{+}^T, \operatorname{vec}(Z)^T\right)^T \\
&= H\beta,
\end{aligned} \tag{17}$$

where the second equality follows from the definition of Moore-Penrose generalized inverse, and

$$\begin{aligned}
H &= \left(E^T \otimes D - \left(E^TBB^{+}\right) \otimes \left(DA^{+}A\right),\right. \\
&\qquad \left. -E^T \otimes D + \left(E^TBB^{+}\right) \otimes \left(DA^{+}A\right), -I \otimes I\right),
\end{aligned} \tag{18}$$
$$\beta = \left([\operatorname{vec}(G)]_{+}^T, [\operatorname{vec}(-G)]_{+}^T, \operatorname{vec}(Z)^T\right)^T.$$

It is easy to see that $\beta \geq 0$. As G and $Z \geq 0$ are arbitrary, β is arbitrary as well. Let $H = (h_1, h_2, \ldots, h_l)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_l)^T$, where $l = 2mn + st$; then $\operatorname{vec}(Q) = \sum_{i=1}^{l} \beta_i h_i$. Thus $\mathscr{M}$ is equivalent to the following set:

$$\mathscr{K} = \left\{q \in R^{st} \mid q = \sum_{i=1}^{l} \beta_i h_i, \beta_i \geq 0\right\}. \tag{19}$$

By the result in [37], we know that the set K is a closed convex cone in the Hilbert space R^{st}. Hence $\mathscr{M}$ is a closed convex cone in the Hilbert space $R^{s\times t}$. □

Lemma 4. *The matrix set $\mathscr{N}$ is the polar cone of the matrix set $\mathscr{M}$.*

Proof. By the definition of the polar cone, we get

$$\mathscr{M}^* = \left\{Y \in R^{s\times t} \mid \langle Q, Y\rangle \leq 0, \forall Q \in \mathscr{M}, Y \geq 0\right\}. \tag{20}$$

So we just need to prove $N = M^*$. Firstly, we prove $\mathscr{N} \subseteq \mathscr{M}^*$.

For all $Y \in \mathscr{N}$ and $Q = D(G - A^{+}AGBB^{+})E - Z \in \mathscr{M}$, we have

$$\begin{aligned}
\langle Q, Y\rangle &= \langle D\left(G - A^{+}AGBB^{+}\right)E - Z, Y\rangle \\
&= \langle DGE, Y\rangle - \langle DA^{+}AGBB^{+}E, Y\rangle - \langle Z, Y\rangle \\
&= \langle G, D^TYE^T\rangle - \langle G, A^{+}AD^TYE^TBB^{+}\rangle - \langle Z, Y\rangle \\
&= \langle G, D^TYE^T - A^{+}AD^TYE^TBB^{+}\rangle - \langle Z, Y\rangle \\
&= -\langle Z, Y\rangle \leq 0.
\end{aligned} \tag{21}$$

Thus $Y \in \mathscr{M}^*$. Then $\mathscr{N} \subseteq \mathscr{M}^*$.

Now we prove $\mathscr{M}^* \subseteq \mathscr{N}$. For all $Y \in \mathscr{M}^*$, if $Y \notin \mathscr{N}$, then $D^TYE^T - A^{+}AD^TYE^TBB^{+} \neq 0$. So there exists a positive real number α and $Z \geq 0$ such that

$$\begin{aligned}
&\alpha\langle D^TYE^T - A^{+}AD^TYE^TBB^{+}, D^TYE^T - A^{+}AD^TYE^TBB^{+}\rangle \\
&\quad > \langle Y, Z\rangle.
\end{aligned} \tag{22}$$

Let $Q = \alpha D(G - A^{+}AGBB^{+})E - Z$, where $G = D^TYE^T - A^{+}AD^TYE^TBB^{+}$:

$$\begin{aligned}
\langle Q, Y\rangle &= \langle \alpha D\left(G - A^{+}AGBB^{+}\right)E - Z, Y\rangle \\
&= \alpha\langle G, D^TYE^T - A^{+}AD^TYE^TBB^{+}\rangle - \langle Z, Y\rangle \\
&= \alpha \left\langle D^TYE^T - A^{+}AD^TYE^TBB^{+},\right. \\
&\qquad \left. D^TYE^T - A^{+}AD^TYE^TBB^{+}\right\rangle - \langle Z, Y\rangle \\
&> 0.
\end{aligned} \tag{23}$$

This contradicts the assumption $Y \in \mathscr{M}^*$. □

Theorem 5. *Assume that the matrices G^* and Z^* solve* (12). *Define matrices $Q^* \in R^{s\times t}$ and $Y^* \in R^{s\times t}$ as*

$$\begin{aligned}
Q^* &= D\left(G^* - A^{+}AG^*BB^{+}\right)E - Z^*, \\
Y^* &= \left[\widetilde{F} - D\left(G^* - A^{+}AG^*BB^{+}\right)E\right]_{+}.
\end{aligned} \tag{24}$$

Then $\widetilde{F} = Q^* + Y^*$, $Q^* \in \mathscr{M}$, $Y^* \in \mathscr{N}$, and $\langle Q^*, Y^*\rangle = 0$; namely, Q^* and Y^* are the polar decomposition of $\widetilde{F}$.

Proof. As G^* and Z^* solve (12), we have $Z^* = [D(G^* - A^+AG^*BB^+)E - \widetilde{F}]_+$. Then

$$\begin{aligned} Q^* &= D(G^* - A^+AG^*BB^+)E \\ &\quad - \left[D(G^* - A^+AG^*BB^+)E - \widetilde{F}\right]_+ \\ &= D(G^* - A^+AG^*BB^+)E - \widetilde{F} \\ &\quad - \left[D(G^* - A^+AG^*BB^+)E - \widetilde{F}\right]_+ + \widetilde{F} \\ &= -\left[\widetilde{F} - D(G^* - A^+AG^*BB^+)E\right]_+ + \widetilde{F} \\ &= \widetilde{F} - Y^*. \end{aligned} \tag{25}$$

Thus $\widetilde{F} = Q^* + Y^*$.

By Lemmas 3 and 4 and Theorem 2, we get that $\widetilde{F}$ has unique polar decomposition with $\mathscr{M}$ and $\mathscr{N}$; that is, there exists unique $\widehat{Q} \in \mathscr{M}$ and $\widehat{Y} \in \mathscr{N}$ such that $\widetilde{F} = \widehat{Q} + \widehat{Y}$ and $\langle \widehat{Q}, \widehat{Y}\rangle = 0$.

Consider optimization problem (12). The objective function in (12) is

$$\begin{aligned} F(G,Z) &= \left\| D(G - A^+AGBB^+)E - Z - \widetilde{F}\right\|^2 \\ &= \left\| D(G - A^+AGBB^+)E - Z - \widehat{Q} - \widehat{Y}\right\|^2 \\ &= \left\| D(G - A^+AGBB^+)E - Z - \widehat{Q}\right\|^2 + \left\|\widehat{Y}\right\|^2 \\ &\quad - 2\langle D(G - A^+AGBB^+)E - Z - \widehat{Q}, \widehat{Y}\rangle \\ &= \left\| D(G - A^+AGBB^+)E - Z - \widehat{Q}\right\|^2 + \left\|\widehat{Y}\right\|^2 \\ &\quad - 2\langle D(G - A^+AGBB^+)E, \widehat{Y}\rangle + 2\langle Z, \widehat{Y}\rangle \\ &= \left\| D(G - A^+AGBB^+)E - Z - \widehat{Q}\right\|^2 + \left\|\widehat{Y}\right\|^2 \\ &\quad - 2\langle G, D^T\widehat{Y}E^T - A^+AD^T\widehat{Y}E^TBB^+\rangle + 2\langle Z, \widehat{Y}\rangle \\ &= \left\| D(G - A^+AGBB^+)E - Z - \widehat{Q}\right\|^2 \\ &\quad + \left\|\widehat{Y}\right\|^2 + 2\langle Z, \widehat{Y}\rangle \\ &\geq \left\|\widehat{Y}\right\|^2. \end{aligned} \tag{26}$$

Since G^*, Z^* solve (12), $D(G^* - A^+AG^*BB^+)E - Z^* \in \mathscr{M}$, and $\widehat{Q} \in \mathscr{M}$, we have $\widehat{Q} = D(G^* - A^+AG^*BB^+)E - Z^* = Q^* \in \mathscr{M}$. Thus $Y^* = \widetilde{F} - Q^* = \widetilde{F} - \widehat{Q} = \widehat{Y} \in \mathscr{N}$. Then $\langle Q^*, Y^*\rangle = \langle \widehat{Q}, \widehat{Y}\rangle = 0$. This completes the proof. □

Remark 6. Theorem 5 implies that if a pair of matrices G^* and Z^* solves optimization problem (12), then $D(G^* - A^+AG^*BB^+)E - Z^*$ and $[\widetilde{F} - D(G^* - A^+AG^*BB^+)E]_+$ are the projections of $\widetilde{F}$ onto $\mathscr{M}$ and $\mathscr{N}$, respectively. Conversely, by Theorem 2 we get that $\widetilde{F}$ has unique polar decomposition of the form $\widetilde{F} = Q^* + Y^*$, $Q^* \in \mathscr{M}$, and $Y^* \in \mathscr{N}$. By the definition of $\mathscr{M}$, there exists G^* and $Z^* \geq 0$ such that $Q^* = D(G^* - A^+AG^*BB^+)E - Z^*$ and $Y^* = [\widetilde{F} - D(G^* - A^+AG^*BB^+)E]_+$. Moreover, G^*, Z^*, and Y^* solve optimization problem (11). Thus problem (11) is solvable.

By the above analysis, we get the following theorem immediately.

Theorem 7. *Problem* (1) *is solvable if and only if* $AA^+CB^+B = C$ *and* $Y^* = [\widetilde{F} - D(G^* - A^+AG^*BB^+)E]_+ = 0$, *where* G^* *and* Z^* *are the solutions of optimization problem* (12).

4. Iterative Method and Convergence Analysis

In this section, we present an iteration method to solve (1) and give the convergence analysis. We are now in a position to give our algorithm to compute the solutions G^*, Y^*, and Z^* of (8), (11), and (12).

Algorithm 8 (an iteration method for (1)).

Step 0. Input matrices A, B, C, D, E, and F. Choose the initial matrix G_0. Compute $\widetilde{F} = F - DA^+CB^+E$, $Y_0 = [\widetilde{F} - D(G_0 - A^+AG_0BB^+)E]_+$, and $Z_0 = [D(G_0 - A^+AG_0BB^+)E - \widetilde{F}]_+$. Take the stopping criterion $\varepsilon > 0$. Set $k := 0$.

Step 1. Find a solution W_k of the least squares problem

$$\text{minimize} \left\| D(W - A^+AWBB^+)E - Y_k\right\|. \tag{27}$$

Step 2. Update the sequences

$$\begin{aligned} G_{k+1} &= G_k + W_k, \\ Y_{k+1} &= \left[\widetilde{F} - D(G_{k+1} - A^+AG_{k+1}BB^+)E\right]_+, \\ Z_{k+1} &= \left[D(G_{k+1} - A^+AG_{k+1}BB^+)E - \widetilde{F}\right]_+. \end{aligned} \tag{28}$$

Step 3. If $\|Y_{k+1} - Y_k\| \leq \varepsilon$ or $\|Z_{k+1} - Z_k\| \leq \varepsilon$, then stop; otherwise, set $k := k + 1$ and go to Step 1.

Next we give the following lemma.

Lemma 9. $R^{s\times t}$ *is the direct sum of* $\overline{\mathscr{M}}$ *and* $\overline{\mathscr{N}}$; *that is,*

$$R^{s\times t} = \overline{\mathscr{M}} \oplus \overline{\mathscr{N}}, \tag{29}$$

where

$$\begin{aligned} \overline{\mathscr{M}} &= \left\{D(X - A^+AXBB^+)E \mid X \in R^{m\times n}\right\}, \\ \overline{\mathscr{N}} &= \left\{Y \in R^{s\times t} \mid D^TYE^T - A^+AD^TYE^TBB^+ = 0\right\}. \end{aligned} \tag{30}$$

Proof. Obviously, $\overline{\mathscr{M}}$ and $\overline{\mathscr{N}}$ are linear subspaces of $R^{s\times t}$. By orthogonal decomposition theorem in the Hilbert space, we obtain $R^{s\times t} = \overline{\mathscr{M}} \oplus \overline{\mathscr{M}}^{\perp}$, where $\overline{\mathscr{M}}^{\perp}$ is the orthogonal complement space of $\overline{\mathscr{M}}$. So we just need to prove $\overline{\mathscr{N}} = \overline{\mathscr{M}}^{\perp}$.

We prove $\overline{\mathcal{N}} \subseteq \overline{\mathcal{M}}^{\perp}$ firstly. For all $Y \in \overline{\mathcal{N}}$ and $D(X - A^{+}AXBB^{+})E \in \overline{\mathcal{M}}$, we have

$$\begin{aligned}
&\langle D(X - A^{+}AXBB^{+})E, Y\rangle \\
&= \langle DXE, Y\rangle - \langle DA^{+}AXBB^{+}E, Y\rangle \\
&= \langle X, D^{T}YE^{T}\rangle - \langle X, (DA^{+}A)^{T}Y(BB^{+}E)^{T}\rangle \\
&= \langle X, D^{T}YE^{T}\rangle - \langle X, A^{+}AD^{T}YE^{T}BB^{+}\rangle \\
&= \langle X, D^{T}YE^{T} - A^{+}AD^{T}YE^{T}BB^{+}\rangle \\
&= 0,
\end{aligned} \tag{31}$$

where the third equality follows from the definition of Moore-Penrose generalized inverse. Hence $Y \in \overline{\mathcal{M}}^{\perp}$; namely, $\overline{\mathcal{N}} \subseteq \overline{\mathcal{M}}^{\perp}$.

Then we prove $\overline{\mathcal{M}}^{\perp} \subseteq \overline{\mathcal{N}}$. For all $Y \in \overline{\mathcal{M}}^{\perp}$ and $D(X - A^{+}AXBB^{+})E \in \overline{\mathcal{M}}$, we have $\langle D(X - A^{+}AXBB^{+})E, Y\rangle = 0$. By the same way, we get $\langle X, D^{T}YE^{T} - A^{+}AD^{T}YE^{T}BB^{+}\rangle = 0$. As $X \in R^{m\times n}$ is arbitrary, we take $X = D^{T}YE^{T} - A^{+}AD^{T}YE^{T}BB^{+}$. Then

$$\begin{aligned}
&\langle D^{T}YE^{T} - A^{+}AD^{T}YE^{T}BB^{+}, D^{T}YE^{T} - A^{+}AD^{T}YE^{T}BB^{+}\rangle \\
&= 0.
\end{aligned} \tag{32}$$

So $D^{T}YE^{T} - A^{+}AD^{T}YE^{T}BB^{+} = 0$; that is, $Y \in \overline{\mathcal{N}}$. Thus $\overline{\mathcal{M}}^{\perp} \subseteq \overline{\mathcal{N}}$.

From the above, we get $\overline{\mathcal{N}} = \overline{\mathcal{M}}^{\perp}$. Therefore, $R^{s\times t} = \overline{\mathcal{M}} \oplus \overline{\mathcal{N}}$. □

Now we present the convergence theorem.

Theorem 10. *Let $\widetilde{F} = Q^{*} + Y^{*}$ be the unique polar decomposition of $\widetilde{F}$. Then*

$$\begin{aligned}
&\lim_{k\to\infty} Y_{k} = Y^{*}, \\
&\lim_{k\to\infty} \left(D(G_{k} - A^{+}AG_{k}BB^{+})E - Z_{k}\right) = Q^{*}.
\end{aligned} \tag{33}$$

Proof. Since matrix W_k solves (27), we have $\|D(W_k - A^{+}AW_{k}BB^{+})E - Y_k)\| \le \|Y_k\|$. This together with Algorithm 8 yields

$$\begin{aligned}
\|Y_{k+1}\| &= \left\|\left[\widetilde{F} - D(G_{k+1} - A^{+}AG_{k+1}BB^{+})E\right]_{+}\right\| \\
&= \left\|\left[\widetilde{F} - D(G_{k} - A^{+}AG_{k}BB^{+})E \right.\right.\\
&\qquad \left.\left. - D(W_{k} - A^{+}AW_{k}BB^{+})E\right]_{+}\right\| \\
&= \left\|[Y_{k} - Z_{k} - D(W_{k} - A^{+}AW_{k}BB^{+})E]_{+}\right\| \\
&\le \left\|[Y_{k} - D(W_{k} - A^{+}AW_{k}BB^{+})E]_{+}\right\| \\
&\le \left\|Y_{k} - D(W_{k} - A^{+}AW_{k}BB^{+})E\right\| \le \|Y_{k}\|,
\end{aligned} \tag{34}$$

where the first inequality follows from $Z_k \ge 0$ and the second inequality follows from nonexpansive property of the projection. This implies that the sequence $\{\|Y_k\|\}$ is monotonically decreasing and is bounded from below. So there exists a constant $\alpha \ge 0$ such that $\lim_{k\to\infty}\|Y_k\| = \alpha$. Furthermore, $\{Y_k\}$ is bounded. Then $\{Y_k\}$ has at least one cluster point. Next we show that any cluster point of the sequence $\{Y_k\}$ is equal to Y^{*}. Consequently, $\{Y_k\}$ converges toward to Y^{*}.

Let $\widetilde{Y}$ be any cluster point of the sequence $\{Y_k\}$. Without loss of generality, we suppose $\lim_{k\to\infty}Y_k = \widetilde{Y}$. Obviously, $\widetilde{Y} \ge 0$. It follows from Lemma 9 that Y_k has unique orthogonal decomposition of the form

$$Y_{k} = \widehat{Y}_{k} + \widetilde{Y}_{k}, \quad \widehat{Y}_{k} \in \overline{\mathcal{M}},\ \widetilde{Y}_{k} \in \overline{\mathcal{N}}. \tag{35}$$

Moreover, by (27), W_k satisfies $D(W_k - A^{+}AW_{k}BB^{+})E = \widehat{Y}_k$. Thus $\|D(W_k - A^{+}AW_{k}BB^{+})E - Y_k)\|$:

$$\|Y_{k+1}\| \le \left\|Y_{k} - D(W_{k} - A^{+}AW_{k}BB^{+})E\right\| = \left\|\widetilde{Y}_{k}\right\| \le \|Y_{k}\|. \tag{36}$$

Let $k \to \infty$; we get $\lim_{k\to\infty}\|Y_k\| = \lim_{k\to\infty}\|\widetilde{Y}_k\| = \alpha$. This together with $\|Y_k\|^2 = \|\widehat{Y}_k\|^2 + \|\widetilde{Y}_k\|^2$ yields that

$$\lim_{k\to\infty}\left\|\widehat{Y}_{k}\right\| = 0. \tag{37}$$

So $\lim_{k\to\infty}\widehat{Y}_k = 0$. Therefore

$$\lim_{k\to\infty}\widetilde{Y}_{k} = \lim_{k\to\infty}\left(Y_{k} - \widehat{Y}_{k}\right) = \widetilde{Y}. \tag{38}$$

Since $\widetilde{Y}_k \in \overline{\mathcal{N}}$, $\widetilde{Y} \in \overline{\mathcal{N}}$ as well. This together with $\widetilde{Y} \ge 0$ follows that $\widetilde{Y} \in \mathcal{N}$.

Since

$$\begin{aligned}
&D(G_{k} - A^{+}AG_{k}BB^{+})E - Z_{k} \\
&= D(G_{k} - A^{+}AG_{k}BB^{+})E \\
&\quad - \left[D(G_{k} - A^{+}AG_{k}BB^{+})E - \widetilde{F}\right]_{+} \\
&= \widetilde{F} - Y_{k},
\end{aligned} \tag{39}$$

we get $\lim_{k\to\infty}(D(G_k - A^{+}AG_{k}BB^{+})E - Z_k) = \lim_{k\to\infty}(\widetilde{F} - Y_k) = \widetilde{F} - \widetilde{Y}$. By Lemma 3 and $D(G_k - A^{+}AG_{k}BB^{+})E - Z_k \in \mathcal{M}$, we gain $\widetilde{F} - \widetilde{Y} \in \mathcal{M}$. By the definition of $\mathcal{M}$, there exists $\widetilde{G} \in R^{m\times n}$ and $\widetilde{Z} \ge 0$ such that $\widetilde{F} - \widetilde{Y} = D(\widetilde{G} - A^{+}A\widetilde{G}BB^{+})E - \widetilde{Z}$. Let $\widetilde{Q} = D(\widetilde{G} - A^{+}A\widetilde{G}BB^{+})E - \widetilde{Z}$. Then $\widetilde{F} = \widetilde{Q} + \widetilde{Y}$ is the unique polar decomposition of $\widetilde{F}$. Hence $\widetilde{Q} = Q^{*}$ and $\widetilde{Y} = Y^{*}$. Furthermore,

$$\begin{aligned}
&\lim_{k\to\infty}\left(D(G_{k} - A^{+}AG_{k}BB^{+})E - Z_{k}\right) \\
&= \widetilde{F} - \widetilde{Y} = \widetilde{F} - Y^{*} = Q^{*}.
\end{aligned} \tag{40}$$

This completes the proof of the theorem. □

5. Numerical Experiments

In this section, we present two numerical examples to illustrate the efficiency and the performance of Algorithm 8. Firstly, we consider least squares problem (27) in Algorithm 8.

By the definition of Kronecker product, least squares problem (27) in Algorithm 8 is equivalent to

$$\text{minmize} \left\| \left(E^T \otimes D - E_1^T \otimes D_1\right) \text{vec}(W) - \text{vec}(Y_k) \right\|, \quad (41)$$

where $D_1 = DA^+A$ and $E_1 = BB^+E$. It is well known that the normal equation of problem (41) is

$$\left(E^T \otimes D - E_1^T \otimes D_1\right)^T \times \left[\left(E^T \otimes D - E_1^T \otimes D_1\right) \text{vec}(W) - \text{vec}(Y_k)\right] = 0. \quad (42)$$

Notice that

$$\begin{aligned} &\left(E^T \otimes D - E_1^T \otimes D_1\right)^T \left(E^T \otimes D - E_1^T \otimes D_1\right) \\ &= \left(E \otimes D^T - E_1 \otimes D_1^T\right)\left(E^T \otimes D - E_1^T \otimes D_1\right) \\ &= \left(E \otimes D^T\right)\left(E^T \otimes D\right) - \left(E \otimes D^T\right)\left(E_1^T \otimes D_1\right) \\ &\quad - \left(E_1 \otimes D_1^T\right)\left(E^T \otimes D\right) + \left(E_1 \otimes D_1^T\right)\left(E_1^T \otimes D_1\right) \\ &= \left(EE^T\right) \otimes \left(D^T D\right) - \left(EE_1^T\right) \otimes \left(D^T D_1\right) \\ &\quad - \left(E_1 E^T\right) \otimes \left(D_1^T D\right) + \left(E_1 E_1^T\right) \otimes \left(D_1^T D_1\right), \end{aligned} \quad (43)$$

and therefore (42) is equal to

$$\begin{aligned} &D^T DWEE^T - D^T D_1 WE_1 E^T - D_1^T DWEE_1^T + D_1^T D_1 WE_1 E_1^T \\ &= D^T Y_k E^T - D_1^T Y_k E_1^T. \end{aligned} \quad (44)$$

Taking the definition of D_1 and E_1 into the above equation, we get

$$\begin{aligned} &D^T DWEE^T - D^T DA^+ AWBB^+ EE^T \\ &\quad - A^+ AD^T DWEE^T BB^+ \\ &\quad + A^+ AD^T DA^+ AWBB^+ EE^T BB^+ \\ &= D^T Y_k E^T - A^+ AD^T Y_k E^T BB^+, \end{aligned} \quad (45)$$

which is the normal equation of problem (27). Let

$$\begin{aligned} \mathscr{L}(W) = &D^T DWEE^T - D^T DA^+ AWBB^+ EE^T \\ &- A^+ AD^T DWEE^T BB^+ \\ &+ A^+ AD^T DA^+ AWBB^+ EE^T BB^+ \end{aligned} \quad (46)$$

and $Q = D^T Y_k E^T - A^+ AD^T Y_k E^T BB^+$. Then problem (27) is equivalent to $\mathscr{L}(W) = Q$, which can be solved by the modified conjugate gradient method (see [33]).

Algorithm 11 (modified conjugate gradient method).

Step 0. Input matrices A, B, C, D, E, F, and Y_k. Choose the initial matrix $W^{(0)}$. Compute $\mathscr{L}(W^{(0)})$, $R^{(0)} = Q - \mathscr{L}(W^{(0)})$, $\widetilde{R}^{(0)} = \mathscr{L}(R^{(0)})$, and $P^{(0)} = \widetilde{R}^{(0)}$. Take the stopping criterion $\epsilon_2 > 0$. Set $j := 0$.

Step 1. If $\|R^{(j)}\| \le \epsilon_2$, then stop; otherwise, set $j := j + 1$ and go to Step 2.

Step 2. Update the sequences

$$\begin{aligned} W^{(j)} &= W^{(j-1)} + \frac{\|R^{(j-1)}\|^2}{\|P^{(j-1)}\|^2} P^{(j-1)}, \\ R^{(j)} &= Q - \mathscr{L}\left(W^{(j)}\right), \\ \widetilde{R}^{(j)} &= \mathscr{L}\left(R^{(j)}\right), \\ P^{(j)} &= \widetilde{R}^{(j)} + \frac{\|R^{(j)}\|^2}{\|R^{(j-1)}\|^2} P^{(j-1)}. \end{aligned} \quad (47)$$

Step 3. Return to Step 1.

In our experiment, all computations were done using the PC with Pentium Dual-Core CPU E5800 @2.40 GHz. All the programming is implemented in MATLAB R2011b. The initial matrix G_0 in Algorithm 8 is taken as the null matrix and the termination criterion is $\|Y_{k+1} - Y_k\|_F \le \varepsilon = 2.22 \times 10^{-16}$ or $\|Z_{k+1} - Z_k\|_F \le 2.22 \times 10^{-16}$.

Example 12. Matrices $A, B, C, D, E,$ and F are given as follows:

$$A = \begin{pmatrix} 1.7 & 1.2 & -1.7 & 1.9 \\ 3.4 & 2.4 & -3.4 & 3.8 \\ -2.1 & -1.2 & 2.3 & -2.7 \\ 2.1 & 1.2 & -2.3 & 2.7 \end{pmatrix},$$

$$B = \begin{pmatrix} 1.2 & 0.3 & -0.3 & 0.1 & -0.1 \\ 0.1 & 1.3 & 0.2 & -0.1 & 0.2 \\ 0.2 & 0.2 & 1.2 & 0.2 & -0.2 \\ -0.1 & -0.2 & 0.2 & 1.1 & 0.2 \\ -0.2 & -0.3 & 0.3 & -0.2 & 1.2 \end{pmatrix},$$

$$C = \begin{pmatrix} 7.22 & 6.47 & 5.35 & 5.44 & 5.84 \\ 14.44 & 12.94 & 10.70 & 10.88 & 11.68 \\ -8.92 & -7.93 & -6.51 & -6.84 & -6.70 \\ 8.92 & 7.93 & 6.51 & 6.84 & 6.70 \end{pmatrix},$$

$$D = \begin{pmatrix} -2.9 & 1.3 & -2.4 & 2.1 \\ 2.9 & -1.3 & 2.4 & -2.1 \\ 2.2 & 2.2 & 2.3 & 2.9 \\ 2.2 & 2.2 & 1.4 & 1.7 \end{pmatrix},$$

$$E = \begin{pmatrix} -1.9 & 2.3 & -3.4 & 2.1 & -2.6 \\ 1.9 & -2.3 & 3.4 & -2.1 & 2.6 \\ 2.2 & 2.4 & -4.6 & 3.8 & -3.4 \\ -1.1 & -1.2 & 2.3 & -1.9 & 1.7 \\ -3.2 & -1.3 & 5.3 & -3.7 & 1.8 \end{pmatrix},$$

$$F = \begin{pmatrix} -5.8 & 3.9 & -8.2 & 2.7 & -8.1 \\ 4.8 & -6.9 & 6.2 & -5.7 & 6.1 \\ -20.0 & 2.2 & 9.0 & -12.0 & -3.9 \\ -21.0 & -0.8 & 2.6 & -15.0 & 0.12 \end{pmatrix}. \tag{48}$$

Computing by Algorithm 8, we have $Y^* = 0$,

$$Z^* = \begin{pmatrix} 0.9936 & 1.3564 & 2.0000 & 1.3722 & 1.9997 \\ 0.0064 & 1.6436 & 0.0000 & 1.6278 & 0.0003 \\ 1.7826 & 0.0000 & 9.8265 & 0.9252 & 1.2106 \\ 2.6781 & 0.3420 & 19.0882 & 1.2350 & 0.8386 \end{pmatrix} \tag{49}$$

and a solution $\widetilde{G}$ to inequality (5) as follows:

$$\widetilde{G} = \begin{pmatrix} -0.1584 & 0.1584 & -0.1071 & 0.0536 & -0.1952 \\ 0.0960 & -0.0960 & 0.1873 & -0.0936 & 0.2442 \\ -0.0132 & 0.0132 & 0.5250 & -0.2625 & 0.5331 \\ 0.0693 & -0.0693 & 0.4474 & -0.2237 & 0.4974 \end{pmatrix}. \tag{50}$$

It follows from $AA^+CB^+B = C$ and $Y^* = [\widetilde{F} - D(\widetilde{G} - A^+A\widetilde{G}BB^+)E]_+ = 0$ that problem (1) is solvable. By substituting $\widetilde{G}$ into (3), we obtain a solution $\widetilde{X}$ to problem (1) as follows:

$$\widetilde{X} = A^+CB^+ + \widetilde{G} - A^+A\widetilde{G}BB^+$$
$$= \begin{pmatrix} 1.0456 & 1.1341 & 0.5051 & 0.9876 & 0.8328 \\ 1.5637 & 1.1880 & 0.8719 & 0.9651 & 1.9153 \\ -0.7795 & -0.5409 & 0.0916 & -0.9136 & 0.1747 \\ 0.6426 & 0.2772 & 0.8161 & 0.3210 & 0.4646 \end{pmatrix}. \tag{51}$$

Furthermore, denote

$$\delta_k = \|Y_{k+1} - Y_k\|_F, \qquad \rho_k = \|Z_{k+1} - Z_k\|_F. \tag{52}$$

We have the following iterative error curve in Figure 1.

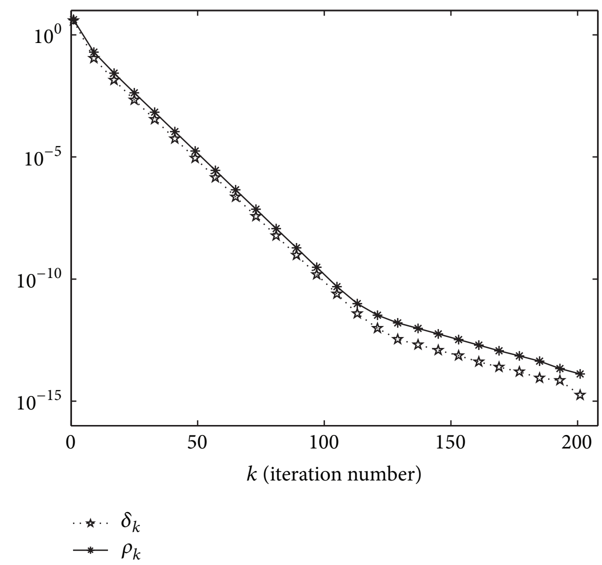

Figure 1: $\|Y_{k+1} - Y_k\|_F$ and $\|Z_{k+1} - Z_k\|_F$ for Example 12.

Example 13. Given matrices A, B, and C being the same as in Example 12, $D \in R^{4\times4}$ and $E \in R^{5\times5}$ are identity matrices, and F is given as follows:

$$F = \begin{pmatrix} 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \end{pmatrix}. \tag{53}$$

Following Algorithm 8, we get $Y^* = 0$,

$$Z^* = \begin{pmatrix} 1.3776 & 1.0824 & 0.6808 & 1.0713 & 1.0324 \\ 1.3952 & 1.2048 & 0.6016 & 0.9825 & 1.6110 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.9862 & 0.6338 & 0.5846 & 0.8894 & 0.1588 \end{pmatrix} \tag{54}$$

and a solution $\widetilde{G}$ to inequality (5) as follows:

$$\widetilde{G} = \begin{pmatrix} 0.2736 & 0.2066 & 0.1685 & 0.2373 & 0.1043 \\ 0.0275 & 0.0208 & 0.0169 & 0.0238 & 0.0398 \\ 0.8663 & 0.6541 & 0.5335 & 0.7511 & 0.4584 \\ 0.5129 & 0.3873 & 0.3158 & 0.4447 & 0.2916 \end{pmatrix}. \tag{55}$$

It follows from $AA^+CB^+B = C$ and $Y^* = [\widetilde{F} - D(\widetilde{G} - A^+A\widetilde{G}BB^+)E]_+ = 0$ that problem (1) is solvable. By substituting $\widetilde{G}$ into (3), we obtain a solution $\widetilde{X}$ to problem (1) as follows:

$$\widetilde{X} = A^+CB^+ + \widetilde{G} - A^+A\widetilde{G}BB^+$$
$$= \begin{pmatrix} 1.4776 & 1.1824 & 0.7808 & 1.1713 & 1.1324 \\ 1.4952 & 1.3048 & 0.7016 & 1.0825 & 1.7110 \\ 0.1000 & 0.1000 & 0.1000 & 0.1000 & 0.1000 \\ 1.0862 & 0.7338 & 0.6846 & 0.9894 & 0.2588 \end{pmatrix}. \tag{56}$$

Furthermore, we have the following iterative error curve in Figure 2.

6. Conclusion

In this paper, we propose Algorithm 8 to find solutions to the matrix equation $AXB = C$ subject to a matrix inequality constraint $DXE \geq F$. And the global convergence results are obtained. For least squares problem (27) in Algorithm 8, we use the modified conjugate gradient method to solve it. Numerical results also confirm the good theoretical properties of our approach.

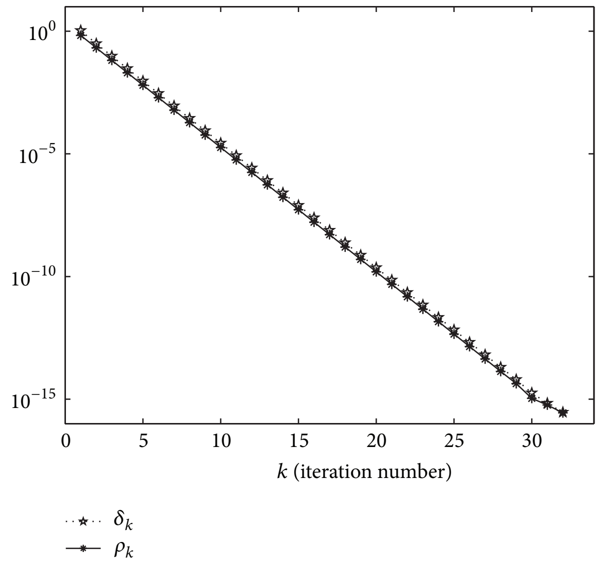

Figure 2: $\|Y_{k+1} - Y_k\|_F$ and $\|Z_{k+1} - Z_k\|_F$ for Example 13.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' Contribution

All authors contributed equally and significantly to the writing of this paper. All authors read and approved the final paper.

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