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Research Article

An Existence Theorem for Fractional Hybrid Differential Inclusions of Hadamard Type with Dirichlet Boundary Conditions

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This paper studies the existence of solutions for a boundary value problem of nonlinear fractional hybrid differential inclusions by using a fixed point theorem due to Dhage (2006). The main result is illustrated with the aid of an example.

1. Introduction

The intensive development of fractional calculus in recent years clearly indicates the popularity of the subject. It has been mainly due to applications of the subject in various fields such as physics, mechanics, chemistry, and engineering [1–3]. In particular, the tools of fractional calculus have considerably improved the modelling techniques and several important models describing biological, ecological, and engineering phenomena are now based on fractional derivatives and integrals. Another factor attracting the attention of many scientists is the nonlocal nature of fractional-order operators which accounts for the hereditary properties of many materials and processes.

Much of the work on fractional differential equations involves either Riemann-Liouville derivative or Caputo derivative; for instance, see [4–33] and the references therein. However, there is another concept of fractional derivative in the literature which was introduced by Hadamard in 1892 [34]. This derivative is known as Hadamard fractional derivative and differs from aforementioned derivatives in the sense that the kernel of the integral in its definition contains logarithmic function of arbitrary exponent. Further details of Hadamard fractional derivatives and integrals can be found in [2].

In this paper, we study a Dirichlet boundary value problem of nonlinear fractional hybrid differential inclusions given by

$$D^{\alpha}\left(\frac{x(t)}{f(t,x(t))}\right) \in F(t,x(t)), \quad 1<t<e, \quad 1<\alpha\leq 2,$$
$$x(1)=x(e)=0, \tag{1}$$

where D^{α} is the Hadamard fractional derivative, $f \in C([1,e]\times\mathbb{R},\mathbb{R}\setminus\{0\})$, $F:[1,e]\times\mathbb{R}\to\mathscr{P}(\mathbb{R})$ is a multivalued map, and $\mathscr{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$.

The main objective of the present study is to establish an existence result for the problem (1) under Lipschitz and Carathéodory conditions by applying a fixed point theorem in Banach algebras due to Dhage [35]. Some recent details on hybrid fractional differential equations can be found in [36–40] and the references cited therein. We emphasize that our work is new in the present configuration and contributes to the present literature on Hadamard type fractional differential equations and inclusions [41–44].

The paper is organized as follows: in Section 2 we recall some preliminary facts that we need in the sequel and Section 3 contains our main result.

2. Preliminaries

2.1. Fractional Calculus

Definition 1 (see [2]). The Hadamard derivative of fractional order q for a function $g : [1,\infty) \to \mathbb{R}$ is defined as

$$D^q g(t) = \frac{1}{\Gamma(n-q)} \left(t\frac{d}{dt}\right)^n \int_1^t \left(\log\frac{t}{s}\right)^{n-q-1} \frac{g(s)}{s} ds, \quad n-1<q<n, \quad n=[q]+1, \tag{2}$$

where $[q]$ denotes the integer part of the real number q and $\log(\cdot) = \log_e(\cdot)$.

Definition 2 (see [2]). The Hadamard fractional integral of order q for a function g is defined as

$$I^q g(t) = \frac{1}{\Gamma(q)} \int_1^t \left(\log\frac{t}{s}\right)^{q-1} \frac{g(s)}{s} ds, \quad q>0, \tag{3}$$

provided the integral exists.

Lemma 3. *Let $y \in C([1,e],\mathbb{R})$. Then the integral solution of the problem*

$$D^\alpha \left(\frac{x(t)}{f(t,x(t))}\right) = y(t), \quad 1<t<e, \; 1<\alpha\le 2, \quad x(1)=x(e)=0, \tag{4}$$

is given by

$$x(t) = f(t,x(t)) \left(\frac{1}{\Gamma(\alpha)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} \frac{y(s)}{s} ds - (\log t)^{\alpha-1} \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log\frac{e}{s}\right)^{\alpha-1} \frac{y(s)}{s} ds\right), \quad t\in[1,e]. \tag{5}$$

Proof. As argued in [2], the solution of Hadamard differential equation in (4) can be written as

$$x(t) = f(t,x(t)) \left(\frac{1}{\Gamma(\alpha)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} \frac{y(s)}{s} ds + c_1(\log t)^{\alpha-1} + c_2(\log t)^{\alpha-2}\right), \tag{6}$$

where $c_1, c_2 \in \mathbb{R}$ are arbitrary constants. Using the given boundary conditions in (6), we find that

$$c_2 = 0, \qquad c_1 = -\frac{1}{\Gamma(\alpha)} \int_1^e \left(\log\frac{e}{s}\right)^{\alpha-1} \frac{y(s)}{s} ds. \tag{7}$$

Substituting the values of c_1, c_2 in (6), we obtain (5). □

Remark 4. It is interesting to note that solution (5) for $\alpha = 2$ corresponds to the one for a Dirichlet boundary value problem of Cauchy-Euler type hybrid differential equation:

$$t^2 \frac{d^2}{dt^2}\left(\frac{x(t)}{f(t,x(t))}\right) + t\frac{d}{dt}\left(\frac{x(t)}{f(t,x(t))}\right) = y(t). \tag{8}$$

2.2. Multivalued Analysis.

Let us recall some basic definitions on multivalued maps [45, 46].

For a normed space $(X, \|\cdot\|)$, let $\mathscr{P}_{cl}(X) = \{Y \in \mathscr{P}(X) : Y \text{ is closed}\}$, $\mathscr{P}_b(X) = \{Y \in \mathscr{P}(X) : Y \text{ is bounded}\}$, $\mathscr{P}_{cp}(X) = \{Y \in \mathscr{P}(X) : Y \text{ is compact}\}$, and $\mathscr{P}_{cp,cv}(X) = \{Y \in \mathscr{P}(X) : Y \text{ is compact and convex}\}$. A multivalued map $G : X \to \mathscr{P}(X)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. The map G is bounded on bounded sets if $G(\mathbb{B}) = \cup_{x\in\mathbb{B}} G(x)$ is bounded in X for all $\mathbb{B} \in \mathscr{P}_b(X)$ (i.e., $\sup_{x\in\mathbb{B}}\{\sup\{|y| : y \in G(x)\}\} < \infty$). G is called upper semicontinuous (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood $\mathscr{N}_0$ of x_0 such that $G(\mathscr{N}_0) \subseteq N$. G is said to be completely continuous if $G(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in \mathscr{P}_b(X)$. If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph; that is, $x_n \to x_*$, $y_n \to y_*$, and $y_n \in G(x_n)$ imply $y_* \in G(x_*)$. G has a fixed point if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by $\operatorname{Fix} G$. A multivalued map $G : [0;1] \to \mathscr{P}_{cl}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$, the function

$$t \longmapsto d(y, G(t)) = \inf\{|y-z| : z \in G(t)\} \tag{9}$$

is measurable.

Let $C([1,e],\mathbb{R})$ denote a Banach space of continuous functions from $[1,e]$ into $\mathbb{R}$ with the norm $\|x\| = \sup_{t\in[1,e]}|x(t)|$. Let $L^1([1,e],\mathbb{R})$ be the Banach space of measurable functions $x : [1,e] \to \mathbb{R}$ which are Lebesgue integrable and normed by $\|x\|_{L^1} = \int_1^e |x(t)| dt$.

Definition 5. A multivalued map $F : [1,e]\times\mathbb{R} \to \mathscr{P}(\mathbb{R})$ is said to be Carathéodory if

(i) $t \mapsto F(t,x)$ is measurable for each $x \in \mathbb{R}$;

(ii) $x \mapsto F(t,x)$ is upper semicontinuous for almost all $t \in [1,e]$.

Further, a Carathéodory function F is called L^1-Carathéodory if

(iii) there exists a function $g \in L^1([1,e],\mathbb{R}^+)$ such that

$$\|F(t,x)\| = \sup\{|v| : v \in F(t,x)\} \le g(t), \tag{10}$$

for all $x \in \mathbb{R}$ and for a.e. $t \in [1,e]$.

For each $y \in C([1,e],\mathbb{R})$, define the set of selections of F by

$$S_{F,y} := \left\{v \in L^1([1,e],\mathbb{R}) : v(t) \in F(t,y(t)) \text{ for a.e. } t \in [1,e]\right\}. \tag{11}$$

The following lemma is used in the sequel.

Lemma 6 (see [47]). *Let X be a Banach space. Let $F : [1,e] \times \mathbb{R} \to \mathcal{P}_{cp,cv}(X)$ be an L^1-Carathéodory multivalued map, and let Θ be a linear continuous mapping from $L^1([1,e],X)$ to $C([1,e],X)$. Then the operator*

$$\Theta \circ S_F : C([1,e],X) \longrightarrow \mathcal{P}_{cp,cv}(C([1,e],X)),\quad x \longmapsto (\Theta \circ S_F)(x) = \Theta(S_{F,x}) \tag{12}$$

is a closed graph operator in $C([1,e],X) \times C([1,e],X)$.

The following fixed point theorem due to Dhage [35] is fundamental in the proof of our main result.

Lemma 7. *Let X be a Banach algebra, let $A : X \to X$ be a single-valued, and let $B : X \to \mathcal{P}_{cp,cv}(X)$ be a multivalued operator satisfying the following:*

(a) *A is single-valued Lipschitz with a Lipschitz constant k,*

(b) *B is compact and upper semicontinuous,*

(c) *$2Mk < 1$, where $M = \|B(X)\|$.*

Then either

(i) *the operator inclusion $x \in AxBx$ has a solution, or*

(ii) *the set $\mathcal{E} = \{u \in X \mid \mu u \in AuBu, \mu > 1\}$ is unbounded.*

3. Main result

Definition 8. A function $x \in AC^1([1,e],\mathbb{R})$ is called a solution of the problem (1) if there exists a function $v \in L^1([1,e],\mathbb{R})$ with $v(t) \in F(t,x(t))$, a.e. on $[1,e]$ such that $D^\alpha(x(t)/f(t,x(t))) = v(t)$, a.e. on $[1,e]$ and $x(1) = x(e) = 0$.

Theorem 9. *Assume that*

(H_1) *the function $f : [1,e]\times\mathbb{R} \to \mathbb{R}\setminus\{0\}$ is continuous and there exists a bounded function ϕ, with bound $\|\phi\|$, such that $\phi(t) > 0$, a.e $t \in [1,e]$ and*

$$\left|f(t,x) - f(t,y)\right| \le \phi(t)\left|x(t) - y(t)\right|,\quad \text{a.e. } t \in [1,e],\ \forall x, y \in \mathbb{R}; \tag{13}$$

(H_2) *$F : [1,e] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is L^1-Carathéodory and has nonempty compact and convex values;*

(H_3) *there exists a positive real number R such that*

$$R > \frac{(2F_0/\Gamma(\alpha))\|g\|_{L^1}}{1 - (2\|\phi\|/\Gamma(\alpha))\|g\|_{L^1}}, \tag{14}$$

where $(2\|\phi\|/\Gamma(\alpha))\|g\|_{L^1} < 1/2$, $F_0 = \sup_{t\in[1,e]}|F(t,0)|$.

Then, the boundary value problem (1) has at least one solution on $[1,e]$.

Proof. Set $X = C([1,e],\mathbb{R})$. Transform the problem (1) into a fixed point problem. Consider the operator $\mathcal{N} : X \to \mathcal{P}(X)$ defined by

$$\begin{aligned}\mathcal{N}x(t) &= \Big\{h \in C([1,e],\mathbb{R}) : h(t) = f(t,x(t)) \\ &\quad \times \Big(\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{v(s)}{s}ds \\ &\quad - (\log t)^{\alpha-1}\frac{1}{\Gamma(\alpha)} \\ &\quad \times \int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}\frac{v(s)}{s}ds\Big), \\ &\qquad v \in S_{F,x}\Big\}.\end{aligned} \tag{15}$$

Now we define two operators $\mathcal{A} : X \to X$ by

$$\mathcal{A}x(t) = f(t,x(t)),\quad t \in [1,e], \tag{16}$$

and $\mathcal{B} : X \to \mathcal{P}(X)$ by

$$\begin{aligned}\mathcal{B}x(t) = \Big\{h \in C([1,e],\mathbb{R}) : h(t) &= \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{v(s)}{s}ds \\ &\quad - (\log t)^{\alpha-1}\frac{1}{\Gamma(\alpha)} \\ &\quad \times \int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}\frac{v(s)}{s}ds, \\ &\qquad v \in S_{F,x}\Big\}.\end{aligned} \tag{17}$$

Observe that $\mathcal{N}(x) = \mathcal{A}x\mathcal{B}x$. We will show that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Lemma 7. For the sake of convenience, we split the proof into several steps.

Step 1. $\mathcal{A}$ is a Lipschitz on X; that is, (a) of Lemma 7 holds.

Let $x, y \in X$. Then by (H_1), we have

$$\begin{aligned}\left|\mathcal{A}x(t) - \mathcal{A}y(t)\right| &= \left|f(t,x(t)) - f(t,y(t))\right| \\ &\le \phi(t)\left|x(t) - y(t)\right| \\ &\le \|\phi\|\,\|x - y\|\end{aligned} \tag{18}$$

for all $t \in [1,e]$. Taking the supremum over the interval $[1,e]$, we obtain

$$\|\mathcal{A}x - \mathcal{A}y\| \le \|\phi\|\,\|x - y\| \tag{19}$$

for all $x, y \in X$. So $\mathcal{A}$ is a Lipschitz on X with Lipschitz constant $\|\phi\|$.

Step 2. The multivalued operator $\mathcal{B}$ is compact and upper semicontinuous on X; that is, (b) of Lemma 7 holds.

First, we show that $\mathscr{B}$ has convex values. Let $u_1, u_2 \in \mathscr{B}x$. Then there are $v_1, v_2 \in S_{F,x}$ such that

$$u_i(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v_i(s)}{s} ds - (\log t)^{\alpha-1} \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{v_i(s)}{s} ds, \tag{20}$$

$i = 1, 2$, $t \in [1, e]$. For any $\theta \in [0, 1]$, we have

$$\begin{aligned} &\theta u_1(t) + (1-\theta) u_2(t) \\ &= \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{[\theta u_1(s) + (1-\theta) u_2(s)]}{s} ds \\ &\quad - (\log t)^{\alpha-1} \frac{1}{\Gamma(\alpha)} \\ &\quad \times \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{[\theta u_1(s) + (1-\theta) u_2(s)]}{s} ds \\ &= \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{\overline{v}(s)}{s} ds \\ &\quad - (\log t)^{\alpha-1} \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{\overline{v}(s)}{s} ds, \end{aligned} \tag{21}$$

where $\overline{v}(t) = \theta v_1(t) + (1-\theta) v_2(t) \in F(t, x(t))$ for all $t \in [1, e]$. Hence $\theta u_1(t) + (1-\theta) u_2(t) \in \mathscr{B}x$ and consequently $\mathscr{B}x$ is convex for each $x \in X$. As a result $\mathscr{B}$ defines a multivalued operator $\mathscr{B} : X \to \mathscr{P}_{cv}(X)$.

Next we show that $\mathscr{B}$ maps bounded sets into bounded sets in X. To see this, let Q be a bounded set in X. Then there exists a real number $r > 0$ such that $\|x\| \le r$, for all $x \in Q$.

Now for each $h \in \mathscr{B}x$, there exists a $v \in S_{F,x}$ such that

$$h(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds - (\log t)^{\alpha-1} \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds. \tag{22}$$

Then, for each $t \in [1, e]$, using (H_2) we have

$$\begin{aligned} |\mathscr{B}x(t)| &= \left| \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds - (\log t)^{\alpha-1} \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds \right| \\ &\le \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{g(s)}{s} ds \\ &\quad + (\log t)^{\alpha-1} \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{g(s)}{s} ds \\ &\le \frac{2}{\Gamma(\alpha)} \|g\|_{L^1}. \end{aligned} \tag{23}$$

This further implies that

$$\|h\| \le \frac{2}{\Gamma(\alpha)} \|g\|_{L^1}, \tag{24}$$

and so $\mathscr{B}(X)$ is uniformly bounded.

Next we show that $\mathscr{B}$ maps bounded sets into equicontinuous sets. Let Q be, as above, a bounded set and $h \in \mathscr{B}x$ for some $x \in Q$. Then there exists a $v \in S_{F,x}$ such that

$$h(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds - (\log t)^{\alpha-1} \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds, \quad t \in [1, e]. \tag{25}$$

Then, for any $\tau_1, \tau_2 \in [1, e]$, we have

$$\begin{aligned} &|h(\tau_2) - h(\tau_1)| \\ &\le \frac{\|g\|_{L^1}}{\Gamma(\alpha)} \left| \int_1^{\tau_1} \left(\log \frac{\tau_1}{s}\right)^{\alpha-1} \frac{1}{s} ds - \int_1^{\tau_2} \left(\log \frac{\tau_2}{s}\right)^{\alpha-1} \frac{1}{s} ds \right| \\ &\quad + \frac{\|g\|_{L^1} \left|(\log \tau_2)^{\alpha-1} - (\log \tau_1)^{\alpha-1}\right|}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{1}{s} ds \\ &\le \frac{\|g\|_{L^1}}{\Gamma(\alpha)} \left| \int_1^{\tau_1} \left[\left(\log \frac{\tau_1}{s}\right)^{\alpha-1} - \left(\log \frac{\tau_2}{s}\right)^{\alpha-1} \right] \frac{1}{s} ds \right| \\ &\quad + \frac{\|g\|_{L^1}}{\Gamma(\alpha)} \left| \int_{\tau_1}^{\tau_2} \left(\log \frac{\tau_2}{s}\right)^{\alpha-1} \frac{1}{s} ds \right| \\ &\quad + \frac{\|g\|_{L^1} \left|(\log \tau_2)^{\alpha-1} - (\log \tau_1)^{\alpha-1}\right|}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{1}{s} ds. \end{aligned} \tag{26}$$

Obviously the right hand side of the above inequality tends to zero independently of $x \in Q$ as $t_2 - t_1 \to 0$. Therefore, it follows by the Arzelá-Ascoli theorem that $\mathscr{B} : X \to \mathscr{P}(X)$ is completely continuous.

In our next step, we show that $\mathscr{B}$ *has a closed graph*. Let $x_n \to x_*$, $h_n \in \mathscr{B}(x_n)$ and $h_n \to h_*$. Then we need to show that $h_* \in \mathscr{B}$. Associated with $h_n \in \mathscr{B}(x_n)$, there exists $v_n \in S_{F,x_n}$ such that, for each $t \in [1, e]$,

$$h_n(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v_n(s)}{s} ds - (\log t)^{\alpha-1} \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{v_n(s)}{s} ds. \tag{27}$$

Thus it suffices to show that there exists $v_* \in S_{F,x_*}$ such that, for each $t \in [1, e]$,

$$h_*(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v_*(s)}{s} ds - (\log t)^{\alpha-1} \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{v_*(s)}{s} ds. \tag{28}$$

Let us consider the linear operator $\Theta : L^1([1,e],\mathbb{R}) \to C([1,e],\mathbb{R})$ given by

$$f \longmapsto \Theta(v)(t) = \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{v(s)}{s}ds - (\log t)^{\alpha-1}\frac{1}{\Gamma(\alpha)}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha-1}\frac{v(s)}{s}ds. \quad (29)$$

Observe that

$$\|h_n(t) - h_*(t)\| = \left\| \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{(v_n(s) - v_*(s))}{s}ds - (\log t)^{\alpha-1}\frac{1}{\Gamma(\alpha)} \times \int_1^e \left(\log\frac{e}{s}\right)^{\alpha-1}\frac{(v_n(s) - v_*(s))}{s}ds \right\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \quad (30)$$

Thus, it follows by Lemma 6 that $\Theta \circ S_F$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,x_n})$. Since $x_n \to x_*$, therefore, we have

$$h_*(t) = \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{v_*(s)}{s}ds - (\log t)^{\alpha-1}\frac{1}{\Gamma(\alpha)}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha-1}\frac{v_*(s)}{s}ds \quad (31)$$

for some $v_* \in S_{F,x_*}$.

As a result we have that the operator $\mathscr{B}$ is compact and upper semicontinuous operator on X.

Step 3. Now we show that $2Mk < 1$; that is, (c) of Lemma 7 holds.

This is obvious by (H_3) since we have $M = \|B(X)\| = \sup\{|\mathscr{B}x| : x \in X\} \le (2/\Gamma(\alpha))\|g\|_{L^1}$ and $k = \|\phi\|$.

Thus all the conditions of Lemma 7 are satisfied and a direct application of it yields that either conclusion (i) or conclusion (ii) holds. We show that conclusion (ii) is not possible.

Let $u \in \mathscr{E}$ be arbitrary. Then we have, for $\lambda > 1$, $\lambda u \in \mathscr{A}u(t)\mathscr{B}u(t)$. Then there exists $v \in S_{F,x}$ such that, for any $\lambda > 1$, one has

$$u(t) = \lambda^{-1}[f(t,u(t))] \times \left(\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{v(s)}{s}ds - (\log t)^{\alpha-1}\frac{1}{\Gamma(\alpha)}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha-1}\frac{v(s)}{s}ds\right), \quad (32)$$

for all $t \in [1,e]$. Then we have

$$\begin{aligned}|u(t)| &\le \lambda^{-1}|f(t,u(t))| \times \left(\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{|v(s)|}{s}ds + (\log t)^{\alpha-1}\frac{1}{\Gamma(\alpha)}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha-1}\frac{|v(s)|}{s}ds\right) \\ &\le [|f(t,u(t)) - f(t,0)| + |f(t,0)|] \times \left(\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{|v(s)|}{s}ds + (\log t)^{\alpha-1}\frac{1}{\Gamma(\alpha)}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha-1}\frac{|v(s)|}{s}ds\right) \\ &\le [\|\phi\|\,\|u\| + F_0]\frac{2}{\Gamma(\alpha)}\|g\|_{L^1},\end{aligned} \quad (33)$$

where we have put $F_0 = \sup_{t\in[1,e]}|f(t,0)|$. Then with $\|u\| = R$, we have

$$R \le \frac{(2F_0/\Gamma(\alpha))\|g\|_{L^1}}{1 - (2\|\phi\|/\Gamma(\alpha))\|g\|_{L^1}}. \quad (34)$$

Thus condition (ii) of Lemma 7 does not hold by (14). Therefore the operator equation $\mathscr{A}x\mathscr{B}x$ and consequently problem (1) have a solution on $[1,e]$. This completes the proof. □

Theorem 10. *Assume that (H_1) holds. In addition, one supposes that*

(H2) *there exists a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ and a function $p \in C([1,e],\mathbb{R}^+)$ such that*

$$\|F(t,x)\|_{\mathscr{P}} := \sup\{|y| : y \in F(t,x)\} \le p(t)\psi(|x|) \quad \text{for each } (t,x) \in [1,e]\times\mathbb{R}; \quad (35)$$

(H3) *there exists a constant $r > 0$ such that*

$$r > \frac{(2F_0/\Gamma(\alpha))\|p\|\psi(r)}{1 - (2\|\phi\|/\Gamma(\alpha))\|p\|\psi(r)}, \quad (36)$$

where

$$\frac{2\|\phi\|}{\Gamma(\alpha)}\|p\|\psi(r) < \frac{1}{2}, \quad (37)$$

and $F_0 = \sup_{t\in[1,e]}|F(t,0)|$.

Then the boundary value problem (1) *has at least one solution on* $[1,e]$.

Proof. The proof is similar to that of Theorem 9 and is omitted. □

Example 11. Consider the boundary value problem

$$D^{3/2}\left[\frac{x(t)}{(1/12)e^{1-t}\tan^{-1}x+2}\right]\in F(t,x(t)),\quad 1<t<e,$$
$$x(1)=x(e)=0, \tag{38}$$

where $F:[1,e]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ is a multivalued map given by

$$t\longrightarrow F(t,x)=\left[\frac{|x|^3}{10\left(|x|^3+3\right)},\frac{|\sin x|}{9(|\sin x|+1)}+\frac{8}{9}\right]. \tag{39}$$

By condition (H_1), $\phi(t)=e^{1-t}/12$ with $\|\phi\|=1/12$. For $\tilde{f}\in F$, we have

$$\left|\tilde{f}\right|\leq\max\left(\frac{|x|^3}{10\left(|x|^3+3\right)},\frac{|\sin x|}{9(|\sin x|+1)}+\frac{8}{9}\right)\leq 1,\quad x\in\mathbb{R},$$
$$\|F(t,x)\|=\sup\left\{|y|:y\in F(t,x)\right\}\leq 1=g(t),\quad x\in\mathbb{R}. \tag{40}$$

Clearly,

$$\frac{2\|\phi\|\,\|g\|_{L^1}}{\Gamma(\alpha)}=\frac{(e-1)}{3\sqrt{\pi}}\simeq 0.323146<\frac{1}{2} \tag{41}$$

and $R>24(e-1)/(1+3\sqrt{\pi}-e)$. Hence all the conditions of Theorem 9 are satisfied and, accordingly, the problem (38) has a solution on $[1,e]$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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