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Research Article

The *Congruence Class of the Solutions to a System of Matrix Equations

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We present the *congruence class of the least-square and the minimum norm least-square solutions to the system of complex matrix equation $AX = C, XB = D$ by generalized singular value decomposition and canonical correlation decomposition.

1. Introduction

Throughout we denote the complex $m \times n$ matrix space by $\mathbb{C}^{m\times n}$. The symbols I, A^*, and $\|A\|$ stand for the identity matrix with the appropriate size, the conjugate transpose, and the Frobenius norm of $A \in \mathbb{C}^{m\times n}$, respectively. Recall that matrices $X, Y \in \mathbb{C}^{n\times n}$ are in the same *congruence class if there is a nonsingular $P \in \mathbb{C}^{n\times n}$ such that $X = P^* Y P$ [1].

Investigating the classical system of matrix equations

$$AX = C, \qquad XB = D \tag{1}$$

has attracted many people's attention and many results have been obtained about system (1) with various constraints, such as Hermitian, positive definite, positive semidefinite, reflexive, and generalized reflexive solutions (see [2–10]). Studying the least-square solutions of the system of matrix equations (1) is also a very active research topic (see [11–16]). It is well known that Hermitian, positive definite and positive semidefinite matrices are the special case of *congruence. Therefore investigating the *congruence class of a solution of the matrix equation (1) is very meaningful.

In 2005, Horn et al. [1] studied the possible *congruence class of a square solution when linear matrix equation $AX = B$ is solvable. In 2009, Zheng et al. [17] describe *congruence class of least-square and minimum norm least-square solutions of the equation $AX = B$ when it is not solvable and discuss a *congruence class of the solutions of the system (1) when it is solvable. To our knowledge, so far there has been little investigation of *congruence class of the least-square and minimum norm least-square solutions to (1) when it is not solvable.

Motivated by the work mentioned above, we investigate the *congruence class of the least-square and the minimum norm least-square solutions to the system of complex matrix equation (1) by generalized singular value decomposition (GSVD) and canonical correlation decomposition (CCD).

2. The *Congruence Class of the Solutions to (1)

Lemma 1 (see [4]). *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$. Then the GSVD of A and B^* can be expressed as*

$$A = U\Sigma_A P, \qquad B^* = V\Sigma_B P, \tag{2}$$

where $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{p\times p}$ are unitary matrices, $P \in \mathbb{C}^{n\times n}$ is nonsingular matrix,

$$\Sigma_A \in \mathbb{C}^{m\times n}, \qquad \Sigma_B \in \mathbb{C}^{p\times n}, \qquad r = \operatorname{rank}\begin{pmatrix} A \\ B^* \end{pmatrix},$$

$$\Sigma_A = \begin{pmatrix} I_A & & & \cdot & \\ & S_A & & \cdot & 0 \\ & & O_A & \cdot & \end{pmatrix}, \quad \begin{matrix} t & s & r-s-t & & n-r \end{matrix}$$

$$\Sigma_B = \begin{pmatrix} O_B & & & \cdot & \\ & S_B & & \cdot & 0 \\ & & I_B & \cdot & \end{pmatrix}, \quad (3)$$
$$\qquad t \quad s \quad r-s-t \quad n-r$$

I_A and I_B are identity matrices, O_A and O_B are zero matrices, and

$$S_A = \operatorname{diag}(\alpha_1, \dots, \alpha_s), \qquad S_B = \operatorname{diag}(\beta_1, \dots, \beta_s) \quad (4)$$

with $1 > \alpha_1 \geq \cdots \geq \alpha_s > 0$, $0 < \beta_1 \leq \cdots \leq \beta_s < 1$, *and* $\alpha_i^2 + \beta_i^2 = 1$ $(i = 1, \dots, s)$.

For convenience, in the following theorem we denote

$$PXP^* = \begin{pmatrix} X_{11} & X_{12} & X_{13} & X_{14} \\ X_{21} & X_{22} & X_{23} & X_{24} \\ X_{31} & X_{32} & X_{33} & X_{34} \\ X_{41} & X_{42} & X_{43} & X_{44} \end{pmatrix}, \quad (5)$$
$$\qquad t \quad s \quad r-s-t \quad n-r$$

$$U^*CP^* = \begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{21} & C_{22} & C_{23} & C_{24} \\ C_{31} & C_{32} & C_{33} & C_{34} \end{pmatrix},$$
$$\qquad t \quad s \quad r-s-t \quad n-r$$

$$PDV = \begin{pmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \\ D_{41} & D_{42} & D_{43} \end{pmatrix}. \quad (6)$$
$$\qquad p-r+t \quad s \quad r-s-t$$

Theorem 2. *Let $A, C \in \mathbb{C}^{m\times n}$, $B, D \in \mathbb{C}^{n\times p}$, and the GSVD of A and B^* be expressed as (2), and then one has the following.*

(a) *The system of matrix equation (1) has a solution in $\mathbb{C}^{n\times n}$ if and only if*

$$C_{3i} = 0, \qquad D_{i1} = 0, \quad (i = 1, 2, 3, 4),$$
$$C_{12} = D_{12}S_B^{-1}, \qquad C_{13} = D_{13}, \quad (7)$$
$$S_A^{-1}C_{22} = D_{22}S_B^{-1}, \qquad S_A^{-1}C_{23} = D_{23}.$$

(b) *In that case, the general solutions of (1) are*

$$X = P^{-1}\begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ S_A^{-1}C_{21} & S_A^{-1}C_{22} & D_{23} & S_A^{-1}C_{24} \\ X_{31} & D_{32}S_B^{-1} & D_{33} & X_{34} \\ X_{41} & D_{42}S_B^{-1} & D_{43} & X_{44} \end{pmatrix}\left(P^{-1}\right)^*, \quad (8)$$

where X_{31}, X_{41}, X_{34}, and X_{44} are arbitrary.

(c) *For arbitrary X_{31}, X_{41}, X_{34}, and X_{44}, there exists a solution in $\mathbb{C}^{n\times n}$ of (1) which is *congruent to*

$$Y = \begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ S_A^{-1}C_{21} & S_A^{-1}C_{22} & D_{23} & S_A^{-1}C_{24} \\ X_{31} & D_{32}S_B^{-1} & D_{33} & X_{34} \\ X_{41} & D_{42}S_B^{-1} & D_{43} & X_{44} \end{pmatrix}. \quad (9)$$

(d) *There exists a minimum norm solution in $\mathbb{C}^{n\times n}$ of (1) which is *congruent to*

$$Y = \begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ S_A^{-1}C_{21} & S_A^{-1}C_{22} & D_{23} & S_A^{-1}C_{24} \\ 0 & D_{32}S_B^{-1} & D_{33} & 0 \\ 0 & D_{42}S_B^{-1} & D_{43} & 0 \end{pmatrix}. \quad (10)$$

Proof. Using the GSVD of A and B^* given by (2), we get

$$AX = C \Longleftrightarrow U\Sigma_A PX = C \Longleftrightarrow \Sigma_A PXP^* = U^*CP^*,$$
$$XB = D \Longleftrightarrow XP^*\Sigma_B^* V^* = D \Longleftrightarrow PXP^*\Sigma_B^* = PDV. \quad (11)$$

By (2) and (5), $\Sigma_A PXP^*$ and $PXP^*\Sigma_B^*$ have the following matrix decomposition:

$$\Sigma_A PXP^* = \begin{pmatrix} X_{11} & X_{12} & X_{13} & X_{14} \\ S_AX_{21} & S_AX_{22} & S_AX_{23} & S_AX_{24} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$
$$PXP^*\Sigma_B^* = \begin{pmatrix} 0 & X_{12}S_B & X_{13} \\ 0 & X_{22}S_B & X_{23} \\ 0 & X_{32}S_B & X_{33} \\ 0 & X_{42}S_B & X_{43} \end{pmatrix}, \quad (12)$$

and we have that system (1) is equivalent to

$$\begin{pmatrix} X_{11} & X_{12} & X_{13} & X_{14} \\ S_AX_{21} & S_AX_{22} & S_AX_{23} & S_AX_{24} \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{21} & C_{22} & C_{23} & C_{24} \\ C_{31} & C_{32} & C_{33} & C_{34} \end{pmatrix},$$
$$\begin{pmatrix} 0 & X_{12}S_B & X_{13} \\ 0 & X_{22}S_B & X_{23} \\ 0 & X_{32}S_B & X_{33} \\ 0 & X_{42}S_B & X_{43} \end{pmatrix} = \begin{pmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \\ D_{41} & D_{42} & D_{43} \end{pmatrix}; \quad (13)$$

obviously, the system of matrix equation (1) has a solution in $\mathbb{C}^{n\times n}$ if and only if

$$C_{3i} = 0, \qquad D_{i1} = 0, \qquad S_AX_{2i} = C_{2i},$$
$$X_{i2}S_B = D_{i2}, \qquad X_{1i} = C_{1i}, \qquad X_{i3} = D_{i3}, \quad (14)$$
$$(i = 1, 2, 3, 4).$$

Therefore, (1) has a solution in $\mathbb{C}^{n\times n}$ if and only if (7) holds, and a general form of the solutions can be expressed as (8); for arbitrary X_{31}, X_{41}, X_{34}, and X_{44}, there exists a solution in $\mathbb{C}^{n\times n}$ of (1) which is *congruent to (9), and the part (d) follows from the definition of Frobenius norm. □

Remark 3. In 2009, Zheng et al. [17] discuss a *congruence class of the solutions of the system (1) when it is solvable. Our result in Theorem 2 is different with the result mentioned above.

3. The *Congruence Class of the Least-Square Solutions to (1)

Lemma 4 (see [18]). *Let the CCD of matrix pair* $[A, C]$ *with* $A \in \mathbb{C}^{m\times n}$, $C \in \mathbb{C}^{m\times k}$, $\operatorname{rank} A = g$, *and* $\operatorname{rank} C = h$ *be given as*

$$A = U(\Sigma_A, 0)E_A^{-1}, \qquad C = U(\Sigma_C, 0)E_C^{-1}, \tag{15}$$

where U *is a unitary matrix and*

$$\Sigma_A = \begin{pmatrix} I_i & & \\ & \Lambda_j & \\ \overline{0} & - & \underline{0} \\ & \Delta_j & \\ & & I_t \end{pmatrix}, \qquad \Sigma_C = \begin{pmatrix} I_h \\ \overline{0} \end{pmatrix} \tag{16}$$

are nonsingular matrices with the same row partitioning, and $g = i + j + t$,

$$\Lambda_j = \operatorname{diag}(\lambda_{i+1}, \ldots, \lambda_{i+j}), \quad 1 > \lambda_{i+1} \ge \cdots \ge \lambda_{i+j} > 0,$$
$$\Delta_j = \operatorname{diag}(\Delta_{i+1}, \ldots, \Delta_{i+j}), \quad 0 > \Delta_{i+1} \ge \cdots \ge \Delta_{i+j} > 1,$$
$$\Lambda_j^2 + \Delta_j^2 = I_j,$$
$$U = \begin{pmatrix} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \\ i & j & h-i-j & m-h-j-t & j & t \end{pmatrix}. \tag{17}$$

Lemma 5 (see [18]). *Given* $E, F \in \mathbb{C}^{m\times n}$, *then there exists a unique matrix* $\overline{S} \in \mathbb{C}^{m\times n}$ *such that*

$$\|S - E\|^2 + \|S - F\|^2 = \min, \tag{18}$$

and $\overline{S}$ *can be expressed as*

$$\overline{S} = E + F. \tag{19}$$

Lemma 6 (see [10]). *Given* $E, F \in \mathbb{C}^{m\times n}$, $\Omega_1 = \operatorname{diag}(a_1, \ldots, a_m)$, $\Omega_2 = \operatorname{diag}(b_1, \ldots, b_n)$, $a_i > 0$ $(i = 1, \ldots, m)$, *and* $b_j > 0$ $(j = 1, \ldots, n)$, *then there exists a unique matrix* $\overline{S} \in \mathbb{C}^{m\times n}$ *such that*

$$\|\Omega_1 S - E\|^2 + \|S\Omega_2 - F\|^2 = \min, \tag{20}$$

and $\overline{S}$ *can be expressed as*

$$\overline{S} = \Phi * (\Omega_1 E + F\Omega_2), \tag{21}$$

where

$$\Phi = \left(\frac{1}{a_i^2 + b_j^2}\right) \in \mathbb{C}^{m\times n}. \tag{22}$$

Using Lemmas 5 and 6, we can easily obtain the following.

Lemma 7. *Given* $E, F, G \in \mathbb{C}^{m\times n}$, $\Omega_1 = \operatorname{diag}(a_1, \ldots, a_m)$, $\Omega_2 = \operatorname{diag}(b_1, \ldots, b_m)$, $I_n = \operatorname{diag}(i_1, \ldots, i_n)$, $a_i > 0$ $(i = 1, \ldots, m)$, $b_j > 0$ $(j = 1, \ldots, m)$, *and* $i_k = 1$ $(k = 1, \ldots, n)$, *then there exist unique matrices* S *and* W *such that*

$$\|\Omega_1 S + \Omega_2 W - E\|^2 + \|S - F\|^2 + \|W - G\|^2 = \min, \tag{23}$$

and S *and* W *can be expressed as*

$$S = F, \qquad W = \Phi * (\Omega_2(\Omega_1 F - E) + G), \tag{24}$$

where

$$\Phi = \left(\frac{1}{b_j^2 + i_k^2}\right) \in \mathbb{C}^{m\times n}. \tag{25}$$

Lemma 8. *Given* $E, F \in \mathbb{C}^{m\times n}$, $\Omega_1 = \operatorname{diag}(a_1, \ldots, a_m)$, $\Omega_2 = \operatorname{diag}(b_1, \ldots, b_n)$, $a_i > 0$ $(i = 1, \ldots, m)$, *and* $b_j > 0$ $(j = 1, \ldots, n)$, *then there exist unique matrices* S *and* W *such that*

$$\|\Omega_1 S + \Omega_2 W - E\|^2 = \min, \tag{26}$$

and S *and* W *can be expressed as*

$$S = 0, \qquad W = \Omega_2^{-1} E. \tag{27}$$

Let $A, C \in \mathbb{C}^{m\times n}$, $B, D \in \mathbb{C}^{n\times l}$, and $\operatorname{rank} A = p \ge \operatorname{rank} B = q$. According to Lemma 4, there exist a unitary matrix $U \in \mathbb{C}^{n\times n}$ and nonsingular matrices $R_A \in \mathbb{C}^{m\times m}$ and $R_B \in \mathbb{C}^{l\times l}$, such that the CCD of matrix pair $[A^*, B]$ is given as

$$A^* = U(\Sigma_A, 0)R_A^{-1}, \qquad B = U(\Sigma_B, 0)R_B^{-1}, \tag{28}$$

where $\Sigma_A \in \mathbb{C}^{n\times p}$, $\Sigma_B \in \mathbb{C}^{n\times q}$,

$$\Sigma_A = \begin{pmatrix} I_r & 0 & 0 \\ 0 & G_s & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & S_s & 0 \\ 0 & 0 & I_t \end{pmatrix}, \qquad \Sigma_B = \begin{pmatrix} I_r & 0 & 0 \\ 0 & I_s & 0 \\ 0 & 0 & I_{q-r-s} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{29}$$

where $p = r + s + t$,

$$G_s = \operatorname{diag}(g_{r+1}, \ldots, g_{r+s}), \quad 1 > g_{r+1} \ge \cdots \ge g_{r+s} > 0,$$
$$S_s = \operatorname{diag}(w_{r+1}, \ldots, w_{r+s}), \quad 0 > w_{r+1} \ge \cdots \ge w_{r+s} > 1,$$
$$G_s^2 + S_s^2 = I_s,$$
$$U = \begin{pmatrix} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \\ r & s & q-r-s & n-q-s-t & s & t \end{pmatrix}. \tag{30}$$

Without loss of generality, let $p = q$, and then we have the following results.

Theorem 9. *Let* $A, C \in \mathbb{C}^{m\times n}$, $B, D \in \mathbb{C}^{n\times l}$, *and the CCD of matrix pair* $[A^*, B]$ *be expressed as* (28), *and then one has the following.*

(a) *The least-square solutions to the system* (1) *are*

$$X = U \begin{pmatrix} C_{11}+D_{11} & C_{12}+D_{12} & C_{13}+D_{13} & C_{14} & C_{15} & C_{16} \\ D_{21} & D_{22} & D_{23} & 0 & 0 & 0 \\ D_{31} & D_{32} & D_{33} & X_{34} & X_{35} & X_{36} \\ D_{41} & D_{42} & D_{43} & X_{44} & X_{45} & X_{46} \\ Y_{51} & Y_{52} & Y_{53} & S^{-1}C_{24} & S^{-1}C_{25} & S^{-1}C_{26} \\ D_{31}+D_{61} & D_{32}+D_{62} & D_{33}+D_{63} & C_{34} & C_{35} & C_{36} \end{pmatrix} U^{*}, \tag{31}$$

where X_{34}, X_{35}, X_{36}, X_{44}, X_{45}, and X_{46} are arbitrary, $Y_{5i} = \Phi * (S(GD_{2i} - C_{2i}) + D_{5i})$, $i = 1,2,3$, $\Phi = (1/(w_{r+j}^2 + e_k^2)) \in \mathbb{C}^{s\times s}$, and $e_k = 1$, $j = 1,\ldots,s$, $k = 1,\ldots,s$.

(b) *For arbitrary X_{34}, X_{35}, X_{36}, X_{44}, X_{45}, and X_{46}, there exists a least-square solution in $\mathbb{C}^{n\times n}$ of (1) which is *congruent to*

$$Y = \begin{pmatrix} C_{11}+D_{11} & C_{12}+D_{12} & C_{13}+D_{13} & C_{14} & C_{15} & C_{16} \\ D_{21} & D_{22} & D_{23} & 0 & 0 & 0 \\ D_{31} & D_{32} & D_{33} & X_{34} & X_{35} & X_{36} \\ D_{41} & D_{42} & D_{43} & X_{44} & X_{45} & X_{46} \\ Y_{51} & Y_{52} & Y_{53} & S^{-1}C_{24} & S^{-1}C_{25} & S^{-1}C_{26} \\ D_{31}+D_{61} & D_{32}+D_{62} & D_{33}+D_{63} & C_{34} & C_{35} & C_{36} \end{pmatrix}, \tag{32}$$

where $Y_{5i} = \Phi * (S(GD_{2i} - C_{2i}) + D_{5i})$, $i = 1,2,3$, $\Phi = (1/(w_{r+j}^2 + e_k^2)) \in \mathbb{C}^{s\times s}$, and $e_k = 1$, $j = 1,\ldots,s$, $k = 1,\ldots,s$.

(c) *There exists a minimum norm least-square solution in $\mathbb{C}^{n\times n}$ of (1) which is *congruent to*

$$Y = \begin{pmatrix} C_{11}+D_{11} & C_{12}+D_{12} & C_{13}+D_{13} & C_{14} & C_{15} & C_{16} \\ D_{21} & D_{22} & D_{23} & 0 & 0 & 0 \\ D_{31} & D_{32} & D_{33} & 0 & 0 & 0 \\ D_{41} & D_{42} & D_{43} & 0 & 0 & 0 \\ Y_{51} & Y_{52} & Y_{53} & S^{-1}C_{24} & S^{-1}C_{25} & S^{-1}C_{26} \\ D_{31}+D_{61} & D_{32}+D_{62} & D_{33}+D_{63} & C_{34} & C_{35} & C_{36} \end{pmatrix}, \tag{33}$$

where $Y_{5i} = \Phi * (S(GD_{2i} - C_{2i}) + D_{5i})$, $i = 1,2,3$, $\Phi = (1/(w_{r+j}^2 + e_k^2)) \in \mathbb{C}^{s\times s}$, and $e_k = 1$, $j = 1,\ldots,s$, $k = 1,\ldots,s$.

Proof. It follows from (28) that

$$\begin{aligned} AX = C &\Longleftrightarrow \left(R_A^{-1}\right)^* \begin{pmatrix} \Sigma_A^* \\ 0 \end{pmatrix} U^* X = C \\ &\Longleftrightarrow \begin{pmatrix} \Sigma_A^* \\ 0 \end{pmatrix} U^* X = \left(R_A\right)^* C, \\ XB = D &\Longleftrightarrow XU\left(\Sigma_B, 0\right) R_B^{-1} = D \Longleftrightarrow XU\left(\Sigma_B, 0\right) = DR_B. \end{aligned} \tag{34}$$

Then,

$$\begin{aligned} &\|AX - C\|^2 + \|XB - D\|^2 \\ &= \left\| \begin{pmatrix} \Sigma_A^* \\ 0 \end{pmatrix} U^* X - \left(R_A\right)^* C \right\|^2 + \left\| XU(\Sigma_B, 0) - DR_B \right\|^2 \\ &= \left\| \begin{pmatrix} \Sigma_A^* \\ 0 \end{pmatrix} U^* XU - \left(R_A\right)^* CU \right\|^2 \\ &\quad + \left\| U^* XU(\Sigma_B, 0) - U^* DR_B \right\|^2. \end{aligned} \tag{35}$$

Assume that

$$U^* X U = \begin{pmatrix} X_{11} & X_{12} & X_{13} & X_{14} & X_{15} & X_{16} \\ X_{21} & X_{22} & X_{23} & X_{24} & X_{25} & X_{26} \\ X_{31} & X_{32} & X_{33} & X_{34} & X_{35} & X_{36} \\ X_{41} & X_{42} & X_{43} & X_{44} & X_{45} & X_{46} \\ X_{51} & X_{52} & X_{53} & X_{54} & X_{55} & X_{56} \\ X_{61} & X_{62} & X_{63} & X_{64} & X_{65} & X_{66} \end{pmatrix}, \quad (36)$$

$$(R_A)^* C U = \begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \end{pmatrix},$$

$$U^* D R_B = \begin{pmatrix} D_{11} & D_{12} & D_{13} & D_{14} \\ D_{21} & D_{22} & D_{23} & D_{24} \\ D_{31} & D_{32} & D_{33} & D_{34} \\ D_{41} & D_{42} & D_{43} & D_{44} \\ D_{51} & D_{52} & D_{53} & D_{54} \\ D_{61} & D_{62} & D_{63} & D_{64} \end{pmatrix}, \quad (37)$$

and then

$$\begin{aligned}
&\left\| \begin{pmatrix} \Sigma_A^* \\ 0 \end{pmatrix} U^* X U - (R_A)^* C U \right\|^2 + \left\| U^* X U (\Sigma_B, 0) - U^* D R_B \right\|^2 \\
&= \left\| \begin{pmatrix} X_{11} & X_{12} & \cdots & X_{16} \\ GX_{21} + SX_{51} & GX_{22} + SX_{52} & \cdots & GX_{26} + SX_{56} \\ X_{61} & X_{62} & \cdots & X_{66} \\ 0 & 0 & \cdots & 0 \end{pmatrix} - \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{16} \\ C_{21} & C_{22} & \cdots & C_{26} \\ C_{31} & C_{32} & \cdots & C_{36} \\ C_{41} & C_{42} & \cdots & C_{46} \end{pmatrix} \right\|^2 \\
&\quad + \left\| \begin{pmatrix} X_{11} & X_{12} & X_{13} & 0 \\ X_{21} & X_{22} & X_{23} & 0 \\ \vdots & \vdots & \vdots & \vdots \\ X_{61} & X_{62} & X_{63} & 0 \end{pmatrix} - \begin{pmatrix} D_{11} & D_{12} & D_{13} & D_{14} \\ D_{21} & D_{22} & D_{23} & D_{24} \\ \vdots & \vdots & \vdots & \vdots \\ D_{61} & D_{62} & D_{63} & D_{64} \end{pmatrix} \right\|^2 \\
&= \|X_{11} - C_{11}\|^2 + \|X_{11} - D_{11}\|^2 + \|X_{12} - C_{12}\|^2 \\
&\quad + \|X_{12} - D_{12}\|^2 + \|X_{13} - C_{13}\|^2 + \|X_{13} - D_{13}\|^2 \\
&\quad + \|X_{61} - C_{31}\|^2 + \|X_{61} - D_{61}\|^2 + \|X_{62} - C_{32}\|^2 \\
&\quad + \|X_{62} - D_{62}\|^2 + \|X_{63} - C_{33}\|^2 + \|X_{63} - D_{63}\|^2 \\
&\quad + \|GX_{21} + SX_{51} - C_{21}\|^2 + \|X_{21} - D_{21}\|^2 \\
&\quad + \|X_{51} - D_{51}\|^2 + \|GX_{22} + SX_{52} - C_{22}\|^2 \\
&\quad + \|X_{22} - D_{22}\|^2 + \|X_{52} - D_{52}\|^2 \\
&\quad + \|GX_{23} + SX_{53} - C_{23}\|^2 + \|X_{23} - D_{23}\|^2 \\
&\quad + \|X_{53} - D_{53}\|^2 + \|GX_{24} + SX_{54} - C_{24}\|^2 \\
&\quad + \|GX_{25} + SX_{55} - C_{25}\|^2 + \|GX_{26} + SX_{56} - C_{26}\|^2 \\
&\quad + \|X_{14} - C_{14}\|^2 + \|X_{15} - C_{15}\|^2 + \|X_{16} - C_{16}\|^2 \\
&\quad + \|X_{64} - C_{34}\|^2 + \|X_{65} - C_{35}\|^2 + \|X_{66} - C_{36}\|^2 \\
&\quad + \|X_{31} - D_{31}\|^2 + \|X_{32} - D_{32}\|^2 + \|X_{33} - D_{33}\|^2 \\
&\quad + \|X_{41} - D_{41}\|^2 + \|X_{42} - D_{42}\|^2 + \|X_{43} - D_{43}\|^2.
\end{aligned} \quad (38)$$

By Lemmas 5, 7, and 8, a general form of the least-square solutions can be expressed as (31); for arbitrary X_{34}, X_{35}, X_{36}, X_{44}, X_{45}, and X_{46}, there exists a least-square solution in $\mathbb{C}^{n\times n}$ of (1) which is *congruent to (32), and the part (c) follows from the definition of Frobenius norm. □

4. An Algorithm and Numerical Examples

Based on the main results of this paper, we in this section propose an algorithm for finding the least-square solutions to the system (1). All the tests are performed by MATLAB 6.5 which has a machine precision of around 10^{-16}.

Algorithm 1. (1) Input $A \in \mathbb{C}^{m\times n}$and $B \in \mathbb{C}^{n\times l}$, and compute $U \in \mathbb{C}^{n\times n}$, $R_A^{-1} \in \mathbb{C}^{m\times m}$, $R_B^{-1} \in \mathbb{C}^{l\times l}$, Σ_A, $\Sigma_B \in \mathbb{C}^{n\times p}$, and $G, S \in \mathbb{C}^{s\times s}$ by the CCD of matrix pair $[A^*, B]$.

(2) Input $C \in \mathbb{C}^{m\times n}$, $D \in \mathbb{C}^{n\times l}$, and compute C_{ij} ($i = 1,2,3,4$; $j = 1,2,3,4,5,6$) and D_{lk} ($l = 1,2,3,4,5,6$; $k = 1,2,3,4$) according to (37).

(3) Compute the least-square solutions of the system (1) by (31).

(4) Compute the *congruence class of the least-square and the minimum norm least-square solutions to the system (1) according to (32) and (33).

Example 1. Suppose

$$A = \begin{bmatrix} -1.625 & 0 & -0.6875i & 0.875i & 0.3438 & 0 \\ -2 & 0 & -0.5i & 0.875i & 0.25 & 0 \\ -0.75 & 0 & -0.125i & 0.25i & 0.0625 & 0 \\ 2.625 & 0 & 0.6875i & -0.875i & -0.3438 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 2 & -2 & 0 & -1 \\ -5i & 6i & 0 & 2i \\ 31i & -37i & i & -15i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 3 & 2 & 2 & 1 & 1 & 5 \\ 4 & i & i & -i & 3 & 1 \\ 5 & 6 & 7 & 4 & 3 & 1 \\ 2 & 1 & 1 & 4 & 3 & 4 \end{bmatrix}, \quad D = \begin{bmatrix} i & -i & -i & 1 \\ 2 & 3 & 1 & 4 \\ 4 & 1 & 2 & 3 \\ 5 & 7 & 6 & 9 \\ 9 & 7 & 5 & 6 \\ 1 & 2 & 3 & 2 \end{bmatrix}. \tag{39}$$

Applying Algorithm 1, we obtain the following:

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 \end{bmatrix},$$

$$\Sigma_A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0.25 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \Sigma_B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$R_A^{-1} = \begin{bmatrix} -1.625 & -2 & -0.75 & 2.625 \\ 1.375 & 1 & 0.25 & -1.375 \\ -0.875 & 0 & -0.25 & 0.875 \\ 0.375 & 0 & 0.25 & -0.375 \end{bmatrix},$$

$$R_B^{-1} = \begin{bmatrix} 2 & -2 & 0 & -1 \\ 31 & -37 & 1 & -15 \\ 5 & -6 & 0 & -2 \\ -44 & 52 & -1 & 21 \end{bmatrix},$$

$$G = [0.5], \qquad S = [0.25],$$

$$(R_A)^* CU = \begin{pmatrix} 5 & 3i & -3i & -9i & 4 & 5i \\ 14 & -1+6i & 1-6i & -19i & 11 & 1+10i \\ 32 & -2+26i & 2-23i & -22i & 25 & 2+25i \\ 53 & -1+57i & 1-50i & -34i & 38 & 1+46i \end{pmatrix},$$

$$U^* DR_B = \begin{pmatrix} 0.03-0.05i & 0.02-0.05i & 0.04-0.02i & 0.02-0.04i \\ -0.51i & -0.27i & -0.33i & -0.25i \\ 0.43i & 0.21i & 0.29i & 0.2i \\ 0.41i & 0.26i & 0.25i & 0.23i \\ 1.35 & 0.67 & 0.78 & 0.62 \\ -1.27i & -0.71i & -0.82i & -0.65i \end{pmatrix}. \tag{40}$$

The least-square solutions to the system (1) are

$$X = U \begin{pmatrix} 5.03-0.05i & 0.02+2.95i & 0.04-3.02i & -9i & 4 & 5i \\ -0.51i & -0.27i & -0.33i & 0 & 0 & 0 \\ 0.43i & 0.21i & 0.29i & X_{34} & X_{35} & X_{36} \\ 0.41i & 0.26i & 0.25i & X_{44} & X_{45} & X_{46} \\ 2.02-0.06i & 0.864-0.366i & 0.498+1.448i & 76i & 44 & 4+40i \\ -0.84i & -0.5i & -0.53i & -22i & 25 & 2+25i \end{pmatrix} U^*, \tag{41}$$

where X_{34}, X_{35}, X_{36}, X_{44}, X_{45}, and X_{46} are arbitrary.

For arbitrary X_{34}, X_{35}, X_{36}, X_{44}, X_{45}, and X_{46}, there exists a least-square solution in $\mathbb{C}^{6\times 6}$ of (1) which is *congruent to

$$Y = \begin{pmatrix} 5.03-0.05i & 0.02+2.95i & 0.04-3.02i & -9i & 4 & 5i \\ -0.51i & -0.27i & -0.33i & 0 & 0 & 0 \\ 0.43i & 0.21i & 0.29i & X_{34} & X_{35} & X_{36} \\ 0.41i & 0.26i & 0.25i & X_{44} & X_{45} & X_{46} \\ 2.02-0.06i & 0.864-0.366i & 0.498+1.448i & 76i & 44 & 4+40i \\ -0.84i & -0.5i & -0.53i & -22i & 25 & 2+25i \end{pmatrix}. \tag{42}$$

There exists a minimum norm least-square solution in $\mathbb{C}^{6\times 6}$ of (1) which is *congruent to

$$Y = \begin{pmatrix} 5.03-0.05i & 0.02+2.95i & 0.04-3.02i & -9i & 4 & 5i \\ -0.51i & -0.27i & -0.33i & 0 & 0 & 0 \\ 0.43i & 0.21i & 0.29i & 0 & 0 & 0 \\ 0.41i & 0.26i & 0.25i & 0 & 0 & 0 \\ 2.02-0.06i & 0.864-0.366i & 0.498+1.448i & 76i & 44 & 4+40i \\ -0.84i & -0.5i & -0.53i & -22i & 25 & 2+25i \end{pmatrix}. \tag{43}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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