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Research Article

Estimates of Invariant Metrics on Pseudoconvex Domains of Finite Type in $\mathbb{C}^3$

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Let Ω be a smoothly bounded pseudoconvex domain in $\mathbb{C}^3$ and assume that $z_0 \in b\Omega$ is a point of finite 1-type in the sense of D'Angelo. Then, there are an admissible curve $\Gamma \subset \Omega \cup \{z_0\}$, connecting points $q_0 \in \Omega$ and $z_0 \in b\Omega$, and a quantity $M(z, X)$, along $z \in \Gamma$, which bounds from above and below the Bergman, Caratheodory, and Kobayashi metrics in a small constant and large constant sense.

1. Introduction

Let Ω be a smoothly bounded domain in $\mathbb{C}^n$ and let X be a holomorphic tangent vector at a point z in Ω, and let us denote the Bergman, Caratheodory, and Kobayashi metrics at z by $B_\Omega(z; X)$, $C_\Omega(z; X)$, and $K_\Omega(z; X)$, respectively. When Ω is a strongly pseudoconvex domain in $\mathbb{C}^n$, the optimal boundary behavior of the above metrics is well understood. For weakly pseudoconvex domains of finite type in $\mathbb{C}^n$, several authors found some results about these metrics. But in each case, the lower bounds are different from the upper bounds [1–5]. In [6], Catlin got optimal estimates in a small constant and large constant sense for pseudoconvex domains of finite type in $\mathbb{C}^2$. For pseudoconvex domains of finite type in $\mathbb{C}^n$, the first author and Herbort extended Catlin's result to the case that the Levi-form at z_0 has corank one [7, 8] or homogeneous finite diagonal type near $z_0 \in b\Omega$ [9, 10].

To estimate the above invariant metrics, we need a complete geometric analysis near $z_0 \in b\Omega$ of finite type, and then we construct a family of plurisubharmonic functions with maximal Hessian near $b\Omega$. However, this construction is really technical and known only for special types of domains mentioned above, but not for arbitrary pseudoconvex domains of finite type in $\mathbb{C}^n$, even for $n = 3$ case. Meanwhile, it is useful to understand the behavior of a holomorphic function near $z_0 \in b\Omega$ if we have precise estimates of the invariant metric along some curves.

In the sequel, we let Ω be a smoothly bounded pseudoconvex domain in $\mathbb{C}^3$ with smooth defining function r and let $z_0 \in b\Omega$. Let $\mathscr{M}(z_0) = (1, m, m_3)$ be Catlin's multitype [11]. Thus, $m = T_{\mathrm{BG}}(z_0)$ is the type in the sense of "Bloom-Graham." If $m_3 = \Delta_1(z_0)$, then Ω is an h-extensible domain [12] and Herbort [10] got an estimate in this case. Here, $\Delta_q(z_0)$ denotes finite q-type in the sense of D'Angelo. Thus, we assume that $m \le m_3 < \Delta_1(z_0)$. Regular finite 1-type at $z_0 \in b\Omega$ is the maximum order of vanishing of $r \circ \gamma$ for all one complex dimensional regular curves γ, $\gamma(0) = z_0$ and $\gamma'(0) \neq 0$. We denote the regular finite 1-type at z_0 by $T_\Omega^{\mathrm{reg}}(z_0)$. Note that $T_\Omega^{\mathrm{reg}}(z_0)$ is a positive integer and $T_\Omega^{\mathrm{reg}}(z_0) \le \Delta_1(z_0)$.

Assuming that $T_\Omega^{\mathrm{reg}}(z_0) = \eta < \infty$, there exist coordinate functions $z = (z_1, z_2, z_3)$ defined in a neighborhood V of z_0 such that $z_0 = 0$ and $|\partial r/\partial z_3| \ge c_0$ on V for a uniform constant $c_0 > 0$, and $|r(z_1, 0, 0)|$ vanishes to order η, and $(\partial r/\partial z_2)(0) = 0$ (Theorem 2.1 in [13]). With these coordinates at hand, set

$$L_k = \frac{\partial}{\partial z_k} - \left(\frac{\partial r}{\partial z_3}\right)^{-1} \frac{\partial r}{\partial z_k}\frac{\partial}{\partial z_3} := \frac{\partial}{\partial z_k} + e_k(z)\frac{\partial}{\partial z_3}, \quad k = 1, 2, \tag{1}$$

$$L_3 = \frac{\partial}{\partial z_3}.$$

Then, $e_k(0) = 0$, $k = 1, 2$, and $\{L_1, L_2, L_3\}$ form a basis of $\mathbb{C}T^{(1,0)}(V)$ provided V is sufficiently small. For any integer j, $k > 0$, set

$$\mathscr{L}_{j,k}\partial\bar{\partial}r(z) = \underbrace{L_2 \cdots L_2}_{(j-1)\text{times}} \underbrace{\bar{L}_2 \cdots \bar{L}_2}_{(k-1)\text{times}} \partial\bar{\partial}r(z)\left(L_2, \bar{L}_2\right)(z) \tag{2}$$

and define

$$C_l(z) = \max\left\{\left|\mathscr{L}_{j,k}\partial\bar{\partial}r(z)\right| : j + k = l\right\}. \tag{3}$$

Let $X = a_1L_1 + a_2L_2 + a_3L_3$ be a holomorphic tangent vector at $z \in \Omega$ and set

$$M(z; X) = |a_1| |r(z)|^{-1/\eta} + |a_2| \sum_{l=2}^{m} \left(\frac{C_l(z)}{|r(z)|}\right)^{1/l} + |a_3| |r(z)|^{-1}. \tag{4}$$

Let $\Gamma \subset \Omega \cup \{z_0\}$ be the admissible curve defined in (20). Our main result is as follows.

Theorem 1. *Let $\Omega \subset\subset \mathbb{C}^3$ be a smoothly bounded pseudoconvex domain and assume $z_0 \in b\Omega$ is a point of finite 1-type in the sense of D'Angelo; that is, $\Delta_1(z_0) < \infty$. Then, there exist a neighborhood V about z_0, an admissible curve $\Gamma \subset \Omega \cup \{z_0\}$ connecting $q_0 \in \Omega$ and z_0, and positive constants c and C such that, for all $X = a_1L_1 + a_2L_2 + a_3L_3$ at $z \in V \cap \Gamma \cap \Omega$,*

$$\begin{aligned} cM(z; X) &\le B_\Omega(z; X) \le CM(z; X) \\ cM(z; X) &\le C_\Omega(z; X) \le CM(z; X) \\ cM(z; X) &\le K_\Omega(z; X) \le CM(z; X). \end{aligned} \tag{5}$$

To prove Theorem 1, we use special coordinates constructed in Section 2 of [13]. Thus, there is a special direction d, $|d| = 1$, so that, for each $\delta > 0$, the two-dimensional slice $D_\delta := \{(z_2, z_3); r(d\delta^{1/\eta}, z_2, z_3) < 0\}$ becomes a pseudoconvex domain of finite type in $\mathbb{C}^2$, whose type is less than or equal to $m = T_{\mathrm{BG}}(z_0)$. We then apply the method which holds for the domains of finite type in $\mathbb{C}^2$ as in [6]. To avoid the difficulty to push out the domain in z_1-direction, we use a bumping theorem of Cho [14].

2. Special Coordinates

Let $\Omega \subset \mathbb{C}^3$ and $z_0 \in b\Omega$ be as in Section 1. We may assume that $z_0 = 0$. In this section, we consider special coordinates defined near $z_0 \in b\Omega$ and then construct "balls" which are of maximal size on which $r(z)$ changes by no more than some prescribed number $\delta > 0$. In the following, we let $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$ be multi-indices with respect to $z' = (z_1, z_2)$ variables. In Theorem 2.1 of [13], You constructed special coordinates which represent the local geometry of $b\Omega$ near z_0.

Theorem 2. *Let Ω be a smoothly bounded pseudoconvex domain in $\mathbb{C}^3$ with smooth defining function r and assume $T_\Omega^{reg}(0) = \eta < \infty$, $0 \in b\Omega$. Then, there is a holomorphic coordinate system $z = (z_1, z_2, z_3)$ about 0 such that*

$$\begin{aligned} &(1)\ r(z) = \operatorname{Re} z_3 + \sum_{\substack{|\alpha|+|\beta|=m \\ |\alpha|,|\beta|>0}}^{\eta} a_{\alpha,\beta} z'^{\alpha} \bar{z}'^{\beta} \\ &\qquad + \mathcal{O}\left(|z_3||z| + |z'|^{\eta+1}\right), \end{aligned} \tag{6}$$

$$(2)\ |r(t, 0, 0)| \lesssim |t|^\eta,$$

where $z' = (z_1, z_2)$ and where

$$a_{\alpha,\beta} \neq 0 \ \text{forsome}\ \alpha,\ \beta \ \text{with}\ \alpha_1 = \beta_1 = 0,\ \alpha_2 + \beta_2 = m. \tag{7}$$

Remark 3. (1) The second condition in (6) and the property (7) say that $r(z)$ vanishes to order η along z_1 axis and order m along z_2 axis. These properties are crucial for the construction of maximal polydiscs $Q_{c\delta}(z^\delta)$ contained in Ω.

(2) There are much more terms (mixed with z_1 and z_2 and their conjugates) in the summation part of (6) compared to the h-extensible domain cases.

According to Proposition 2.6 and Remark 2.7 of [13], there are pairs of integers (p_ν, q_ν), $\nu = 1, \dots, N$, such that the terms satisfying $\alpha_1 + \beta_1 = p_\nu$ and $\alpha_2 + \beta_2 = q_\nu$ with $\alpha_2 > 0$ and $\beta_2 > 0$ are dominant terms in the summation part of (6). Also, there is a small constant $a_0 > 0$ and a fixed direction d, $|d| = 1$, in z_1 direction, such that, for each fixed $\delta > 0$ and for all z_1 satisfying $|z_1 - d\delta^{1/\eta}| < a_0\delta^{1/\eta}$, those major terms in the summation part of (6) satisfy

$$\left|\frac{\partial^{q_\nu}}{\partial z_2^{\alpha_2}\partial\bar{z}_2^{\beta_2}} r(z_1, 0, 0)\right| \approx |z_1|^{p_\nu} \approx \delta^{p_\nu/\eta}, \tag{8}$$

where $\alpha_2 + \beta_2 = q_\nu$ and where $\alpha_2 > 0$ and $\beta_2 > 0$.

Now, let us fix z_1 with $|z_1 - d\delta^{1/\eta}| < a_0\delta^{1/\eta}$ and consider the two-dimensional slice $D_{z_1} := \{(z_2, z_3) : r(z_1, z_2, z_3) < 0\}$. For each $z = (z_1, 0, z_3)$ near $b\Omega$, set $\pi(z) = (z_1, 0, e_\delta) := \tilde{z}_1 \in b\Omega$, where $\pi(z)$ is the projection of z onto $b\Omega$ along z_3 direction. On D_{z_1}, following the argument in two-dimensional case as in the proof of Proposition 1.1 in [6], we construct special coordinates $\zeta = (\zeta_1, \zeta_2, \zeta_3) = (z_1, z_2, \zeta_3)$ about $\tilde{z}_1$ so that, in terms of new coordinates, there are no pure terms in z_2 variable in the expression of $r(z)$ in (6).

Proposition 4. *For each fixed $\tilde{z}_1 = (z_1, 0, e_\delta) \in V \cap b\Omega$, there exists a holomorphic coordinate system $z = \Phi_{\tilde{z}_1}(\zeta) = (z_1, z_2, \Phi_3(\zeta))$, $\zeta = (\zeta_1, \zeta_2, \zeta_3) = (z_1, z_2, \zeta_3)$, where $\Phi_3(\zeta)$ is defined by*

$$\begin{aligned} \Phi_3(\zeta) &= e_\delta + \left(\frac{\partial r}{\partial z_3}(\tilde{z}_1)\right)^{-1} \\ &\quad \times \left(\frac{\zeta_3}{2} - \sum_{l=2}^{m} c_l(\tilde{z}_1)\zeta_2^l - \frac{\partial r}{\partial z_2}(\tilde{z}_1)\zeta_2\right) \\ &:= e_\delta + d_0(\tilde{z}_1)\zeta_3 + \sum_{l=1}^{m} d_l(\tilde{z}_1)\zeta_2^l, \end{aligned} \tag{9}$$

and the function ρ, given by $\rho(z_1,\zeta'') := r \circ \Phi_{\tilde{z}_1}(z_1,\zeta'')$, $\zeta'' = (\zeta_2,\zeta_3)$, satisfies

$$\rho\left(z_1,\zeta''\right) = \operatorname{Re}\left(\Phi_3\left(\zeta\right)\right) + \sum_{\substack{j+k=2\\ j,k>0}}^{m} a_{j,k}\left(\tilde{z}_1\right)\zeta_2^j\bar{\zeta}_2^k + E\left(\zeta\right), \quad (10)$$

where

$$E\left(\zeta\right) = \mathcal{O}\left(\left|\Phi_3\left(\zeta\right)\right|\left|\zeta\right| + \sum_{\nu=1}^{N}\left|z_1\right|^{1+p_\nu}\left|\zeta_2\right|^{q_\nu} + \left|\zeta_2\right|^{m+1}\right). \quad (11)$$

In view of (6) and (8), the major terms in (10) are $a_{j,k}(\tilde{z}_1)\zeta_2^j\bar{\zeta}_2^k$ where $j+k=\alpha_2+\beta_2=q_\nu$ for some α_2 and β_2 with $\alpha_2>0$ and $\beta_2>0$. Also, from (8), it follows that

$$\left|a_{j,k}\left(\tilde{z}_1\right)\zeta_2^j\bar{\zeta}_2^k\right| \approx \left|z_1\right|^{p_\nu}\left|z_2\right|^{q_\nu}, \quad (12)$$

and these terms control the error terms $|z_1|^{1+p_\nu}|\zeta_2|^{q_\nu}$ in $E(\zeta)$. As in Section 1 in [6], set

$$A_l\left(\tilde{z}_1\right) = \max\left\{\left|a_{j,k}\left(\tilde{z}_1\right)\right|; j+k=l\right\}, \quad l=2,\ldots,m, \quad (13)$$

and for each sufficiently small $\delta>0$, we set

$$\tau\left(\tilde{z}_1,\delta\right) = \min\left\{\left(\frac{\delta}{A_l\left(\tilde{z}_1\right)}\right)^{1/l}; 2\le l\le m\right\}. \quad (14)$$

Thus, for all z_1 with $|z_1 - d\delta^{1/\eta}| < a_0\delta^{1/\eta}$, it follows from (8) and (14) that

$$\tau\left(\tilde{z}_1,\delta\right) \lesssim \left(\frac{\delta}{\left|z_1\right|^{p_\nu}}\right)^{1/q_\nu}, \quad \nu=1,\ldots,N, \quad (15)$$

and hence the summation part of (10) is dominated by $C\delta$.

For each $\tilde{z} = (z_1,0,z_3)$ near $b\Omega$, set $\tilde{\zeta} = \Phi_{\tilde{z}_1}^{-1}(\tilde{z}) = (z_1,0,\tilde{\zeta}_3)$, where $\Phi_{\tilde{z}_1}$ is the function defined in Proposition 4. For each small $e>0$, set

$$\begin{aligned} R_{e\delta}\left(\tilde{\zeta}\right) = \{\zeta : &\left|\zeta_1 - z_1\right| < e\delta^{1/\eta}, \left|\zeta_2\right| < e\tau\left(\tilde{z}_1,\delta\right), \\ &\left|\zeta_3 - \tilde{\zeta}_3\right| < e\delta\}, \end{aligned} \quad (16)$$

$$Q_{e\delta}\left(\tilde{z}\right) = \left\{z : z = \Phi_{\tilde{z}_1}\left(\zeta\right), \zeta \in R_{e\delta}\left(\tilde{\zeta}\right)\right\}.$$

For each $\sigma>0$, let $\Omega_\sigma = \{z; r(z)<\sigma\}$ and define

$$\begin{aligned} S\left(\sigma\right) &= \left\{z\in V : -\sigma < r\left(z\right) \le \sigma\right\} \\ S^-\left(\sigma\right) &= \left\{z\in V : -\sigma < r\left(z\right) \le 0\right\}, \end{aligned} \quad (17)$$

and set $\tilde{z}^\delta = (d\delta^{1/\eta},0,e_\delta) \in b\Omega$, where z_1 is replaced by $d\delta^{1/\eta}$ in $\tilde{z}_1 = (z_1,0,e_\delta)$. The following theorem is about the existence of plurisubharmonic function with maximal Hessian. In [6], for the domains in $\mathbb{C}^2$, Catlin constructed the functions with maximal Hessian on the strip $S(\delta)\cap V$. However, for regular finite type pseudoconvex domains in $\mathbb{C}^3$, we show that the functions have maximal Hessian on each ball $Q_{b\delta}(\tilde{z}^\delta)$ and this will suffice to prove the boundary behavior of the invariant metrics. The proof of the following theorem can be found in Theorem 3.2 in [9].

Theorem 5. *There is a small constant $b>0$ such that, for each small $\delta>0$, there is a plurisubharmonic function $g_\delta \in C_0^\infty(Q_{2b\delta}(\tilde{z}^\delta))$ with the following properties:*

(i) $|g_\delta(\zeta)| \le 1$, $z\in\Omega_\delta$,

(ii) *for all $L = b_1L_1 + b_2L_2 + b_3L_3$ at z, where $z\in Q_{b\delta}(\tilde{z}^\delta)\cap S(b\delta)$,*

$$\partial\bar{\partial}g_\delta\left(L,\bar{L}\right)\left(z\right) \gtrsim \delta^{-2/\eta}\left|b_1\right|^2 + \tau\left(\tilde{z}^\delta,\delta\right)^{-2}\left|b_2\right|^2 + \delta^{-2}\left|b_3\right|^2, \quad (18)$$

(iii) *if $\Phi(\zeta) = (\zeta_1,\zeta_2,\Phi_3(\zeta))$, where Φ_3 is defined in (10) for $\tilde{z}^\delta$, then*

$$\left|\tilde{D}^\alpha g_\delta\circ\Phi\left(\zeta\right)\right| \le C_\alpha\delta^{-\alpha_1/\eta}\tau\left(\tilde{z}^\delta,\delta\right)^{-\alpha_2}\delta^{-\alpha_3} \quad (19)$$

holds for all $\zeta\in R_{2b\delta}(\tilde{z}^\delta)$, where $\tilde{D}^\alpha = \tilde{D}_1^{\alpha_1}\tilde{D}_2^{\alpha_2}\tilde{D}_3^{\alpha_3}$.

Let $\Gamma\subset\Omega$ be a curve defined by

$$\Gamma := \left\{z^\delta : z^\delta = \left(d\delta^{1/\eta},0,e_\delta - \frac{b\delta}{2}\right), 0\le\delta\le\delta_0\right\}, \quad (20)$$

for sufficiently small $\delta_0>0$ and $b>0$. In the sequel, for each $z^\delta = (d\delta^{1/\eta},0,e_\delta - b\delta/2)\in\Gamma$, set $\zeta^\delta := \Phi_{\tilde{z}^\delta}^{-1}(z^\delta)$ and set $\tilde{\Omega} = \Phi_{\tilde{z}^\delta}^{-1}(\Omega)$. In view of Proposition 3.4 in [9], there is a uniform small constant $c>0$ such that $R_{c\delta}(\zeta^\delta) \subset\subset R_{b\delta}(\tilde{z}^\delta)\cap\tilde{\Omega}$, and hence

$$Q_{c\delta}\left(z^\delta\right) = \left\{z : z = \Phi_{\tilde{z}^\delta}\left(\zeta\right), \zeta\in R_{c\delta}\left(\zeta^\delta\right)\right\} \subset\subset Q_{b\delta}\left(\tilde{z}^\delta\right)\cap\Omega, \quad (21)$$

provided $c>0$ and $\delta_0>0$ are sufficiently small. In particular, we have $\Gamma\subset\Omega\cup\{z_0\}$. Note that $\tau(z^\delta,\delta)\approx\tau(\tilde{z}^\delta,\delta)$, and for $z\in Q_{c\delta}(z^\delta)\subset\Omega$, we note that $|r(z)|\approx\delta$. Thus, as in Proposition 1.3 and Corollary 1.4 in [6], we obtain that

$$\tau\left(z^\delta,\delta\right)^{-1} \approx \sum_{k=2}^{m}\left(\frac{C_k\left(z\right)}{\left|r\left(z\right)\right|}\right)^{1/k}, \quad z\in Q_{c\delta}\left(z^\delta\right), \quad (22)$$

where $C_k(z)$ is defined in (3). In the sequel, we set $\tau_1 = \delta^{1/\eta}$, $\tau_2 = \tau(\tilde{z}^\delta,\delta)$, and $\tau_3=\delta$. If we use the plurisubharmonic weight functions constructed in Theorem 5 and follow the method to prove Theorem 6.1 in [6], we get the following estimates of the Bergman kernel along Γ.

Theorem 6. *Let $z_0\in b\Omega$ be a point of regular finite 1-type and $T_\Omega^{reg}(z_0) = \eta$. Then, $K_\Omega(z^\delta,z^\delta)$, the Bergman kernel function of Ω at $z^\delta\in\Gamma$, $\delta>0$, satisfies*

$$K_\Omega\left(z^\delta,z^\delta\right) \approx \delta^{-2}\tau_1^{-2}\tau_2^{-2}. \quad (23)$$

3. Metric Estimates

In this section, we estimate the behavior of the invariant metric along Γ. In [15], Hahn got the following inequalities:

$$C_\Omega\left(z;X\right) \le B_\Omega\left(z;X\right), \qquad C_\Omega\left(z;X\right) \le K_\Omega\left(z;X\right). \quad (24)$$

Therefore, the estimates for the lower bounds of $C_\Omega(z;X)$ will suffice for the lower bounds of $B_\Omega(z;X)$ and $K_\Omega(z;X)$. First, we recall the following bumping theorem [14].

Theorem 7 (Theorem 2.3 in [14]). *Let z_0 be a point of finite 1-type in the boundary of a pseudoconvex domain $\Omega \subset \mathbb{C}^n$, defined by $\Omega = \{z : r(z) < 0\}$. Then, there exist $V \ni z_0$ and a smooth 1-parameter family of pseudoconvex domains Ω_t, $0 \le t < t_0$, each defined by $\Omega_t = \{z; r(z,t) < 0\}$, where $r(z,t)$ has the following properties:*

(1) *$r(z,t)$ is smooth in z for z near $b\Omega$ and in t for $0 \le t < t_0$;*

(2) *$r(z,t) = r(z)$, for $z \notin V$;*

(3) *$(\partial r/\partial t)(z,t) \le 0$;*

(4) *$r(z,0) = r(z)$;*

(5) *for z in V, $\partial r/\partial t < 0$.*

Proof of Theorem 1. In the sequel, let us fix $\delta > 0$ and, for each $z^\delta \in \Gamma$, set $\pi(z^\delta) = \tilde{z}^\delta = (d\delta^{1/\eta}, 0, e_\delta) \in b\Omega$ and consider the special coordinates $\zeta = (z_1, z_2, \zeta_3)$ and $\Phi_{\tilde{z}^\delta}(\zeta) = (z_1, z_2, \Phi_3(\zeta)) = z$, where Φ_3 is defined in Proposition 4. From (9), we see that $\zeta^\delta = (d\delta^{1/\eta}, 0, -b\delta/2d_0(\tilde{z}^\delta)) := (\tilde{\zeta}_1, \tilde{\zeta}_2, \tilde{\zeta}_3)$. We will estimate the metrics at ζ^δ. For all small $\delta > 0$ and for each $\zeta'' = (\zeta_2, \zeta_3)$, define

$$J_\delta(\zeta'') = \left(\delta^2 + |\zeta_3|^2 + \sum_{k=2}^{m} \left(A_k(\tilde{z}^\delta)\right)^2 |\zeta_2|^{2k}\right)^{1/2}, \quad (25)$$

where $A_k(\tilde{z}^\delta)$ is defined in (13) with $\tilde{z}_1$ replaced by $\tilde{z}^\delta$. Let $c > 0$ be the fixed constant determined in (21). Note that $\Phi_{\tilde{z}^\delta}(d\delta^{1/\eta}, 0, 0) = \tilde{z}^\delta$. Set

$$\begin{aligned} \widetilde{\Omega}_{a,\delta} = \{\zeta; &|\zeta_1 - d\delta^{1/\eta}| < c\delta^{1/\eta}, |\zeta_2| < a, |\zeta_3| < a, \\ &\rho(\zeta_1, \zeta_2, \zeta_3) < 0\}, \end{aligned} \quad (26)$$

and, for each $\epsilon > 0$, define

$$\begin{aligned} \widetilde{\Omega}^\epsilon_{a,\delta} = \{\zeta; &|\zeta_1 - d\delta^{1/\eta}| < c\delta^{1/\eta}, |\zeta_2| < a, |\zeta_3| < a, \\ &\rho(d\delta^{1/\eta}, \zeta'') < \epsilon J_\delta(\zeta'')\}, \end{aligned} \quad (27)$$

and for all small $e > 0$ set $B_e = R_{e\delta}(\zeta^\delta)$. By (21), we see that $\zeta^\delta \in B_e \subset \widetilde{\Omega}$ for all $e \le c$. Note that the domains $\widetilde{\Omega}^\epsilon_{a,\delta}$ are pushed out only in ζ_2 and ζ_3 directions but not in ζ_1 direction. To avoid the difficulty to push out $\widetilde{\Omega}$ in ζ_1 direction, we use a bumping family of Theorem 7. Consider a bumping family of pseudoconvex domains $\{\Omega_t\}_{0 \le t \le t_0}$ with front V and set $D = \Omega_{t_0}$. For each $r > 0$, let $U_r(z)$ be a ball of radius $r > 0$ with center at z and set $\widetilde{U}_r(\zeta) = \Phi^{-1}_{\tilde{z}^\delta}(U_r(\Phi_{\tilde{z}^\delta}(\zeta)))$. Then, there is $r_0 > 0$ such that

$$Q_{c\delta}(z^\delta) \subset \Omega^\epsilon_{a,\delta} = \Phi_{\tilde{z}^\delta}(\widetilde{\Omega}^\epsilon_{a,\delta}) \subset U_{r_0/4}(0) \subset U_{r_0}(0) \subset\subset D, \quad (28)$$

for all sufficiently small $a > 0$, $\epsilon > 0$, and $\delta > 0$.

In view of the proof in Section 3 of [13], we have $\widetilde{\Omega}_{a,\delta} \subset \widetilde{\Omega}^{\epsilon/2}_{a,\delta} \subset \widetilde{\Omega}^\epsilon_{a,\delta}$ and there is a uniformly (independent of $\delta > 0$) bounded function $\tilde{f} = \tilde{f}(\zeta_2, \zeta_3)$ which is holomorphic on $\widetilde{\Omega}^\epsilon_{a,\delta}$ and satisfies

$$\left|Y'' \tilde{f}(\zeta^\delta)\right| \gtrsim |b_2| \tau_2^{-1} + |b_3| \tau_3^{-1}, \quad (29)$$

where $Y'' = b_2(\partial/\partial\zeta_2) + b_3(\partial/\partial\zeta_3)$. Here, we may assume that $\tilde{f}(0, -b\delta/d_0(\tilde{z}^\delta)) = 0$. In the sequel, we let Y be a vector field given by $Y = b_1(\partial/\partial\zeta_1) + b_2(\partial/\partial\zeta_2) + b_3(\partial/\partial\zeta_3)$. If $|b_1|\tau_1^{-1} \ge |b_2|\tau_2^{-1} + |b_3|\tau_3^{-1}$, then set $v_\delta = \tau_1^{-1}(\zeta_1 - d\delta^{1/\eta})$. Otherwise, set $v_\delta = \tilde{f}(\zeta_2, \zeta_3)$. From (29), we note that

$$\left|Y v_\delta(\zeta^\delta)\right| \gtrsim \sum_{i=1}^{3} |b_i| \tau_i^{-1}. \quad (30)$$

Let $\psi \in C_0^\infty(U)$, where U is the unit polydisc in $\mathbb{C}^3$, such that $\psi(z) = 1$ if $|z_i| \le 1/2$, $i = 1, 2, 3$, and set

$$\psi_d(\zeta) = \psi\left(\frac{\zeta_1 - \tilde{\zeta}_1}{d\tau_1}, \frac{\zeta_2}{d\tau_2}, \frac{\zeta_3 - \tilde{\zeta}_3}{d\tau_3}\right), \quad (31)$$

and set $\beta_\delta = v_\delta \psi_d$. Then, $\beta_\delta(\zeta^\delta) = 0$. Since $\tilde{f}$ is bounded independent of δ (and hence independent of ζ^δ), there exists a constant $C > 0$, independent of δ, such that $|\beta_\delta| \le C$. We want to correct β_δ so that the corrected function f_δ becomes a uniformly bounded holomorphic function on $\widetilde{\Omega}$ satisfying the estimate (30) with β_δ replaced by f_δ. With bumped domain $D = \Omega_{t_0}$ at hand, set $\widetilde{D} = \Phi^{-1}_{\tilde{z}^\delta}(D)$. On $\widetilde{D}$, instead of $\widetilde{\Omega}$, we will employ weighted estimates of $\bar\partial$ that is essentially a replication of the proof of Theorem 6.1 in [6].

Let g_δ be the weight function defined in Theorem 5 and set $\tilde{g}_\delta = \Phi^*_{\tilde{z}^\delta} g_\delta$. By replacing $\tilde{g}_\delta$ by $\tilde{g}_\delta + |\zeta|^2 := \phi$, we can obviously assume that ϕ is strictly plurisubharmonic on $\widetilde{D}$ and $\phi(\zeta^\delta) = 0$. In view of Theorem 5, we also have

$$\begin{aligned} \partial\bar\partial\phi(Y, \overline{Y})(\zeta) \gtrsim \tau_1^{-2}|b_1|^2 + \tau_2^{-2}|b_2|^2 + \tau_3^{-2}|b_3|^2, \\ \zeta \in R_{c\delta}(\zeta^\delta). \end{aligned} \quad (32)$$

From property (iii) in Theorem 5, there is a small constant a, $0 < a \le c$ (independent of τ_i, $i = 1, 2, 3$), so that

$$\phi(\zeta) \ge 2\,\mathrm{Re}\, h(\zeta) + a\sum_{i=1}^{3} \tau_i^{-2} \left|\zeta_i - \tilde{\zeta}_i\right|^2, \quad \zeta \in R_{c\delta}(\zeta^\delta), \quad (33)$$

where

$$\begin{aligned} h(\zeta) = &\sum_{i=1}^{3} \frac{\partial\phi}{\partial\zeta_i}(\zeta^\delta)(\zeta_i - \tilde{\zeta}_i) \\ &+ \frac{1}{2}\sum_{i,j=1}^{3} \frac{\partial^2\phi}{\partial\zeta_i\partial\zeta_j}(\zeta^\delta)(\zeta_i - \tilde{\zeta}_i)(\zeta_j - \tilde{\zeta}_j). \end{aligned} \quad (34)$$

If we set $\tilde{a} = a^3/3$, it follows, from (33), that

$$\mathrm{Re}\, h(\zeta) \le -\tilde{a}, \quad \zeta \in \{\zeta; \phi(z) \le \tilde{a}\} \cap \mathrm{supp}\, \bar\partial\psi_d. \quad (35)$$

In the sequel, we set $B_e = R_{e\delta}(\zeta^\delta)$ for each small $e > 0$. For each $s \geq 0$, set

$$\alpha_s = \bar{\partial}\left(\beta_\delta e^{sh}\right) = \nu_\delta e^{sh}\bar{\partial}\psi_d(\zeta) := \sum_{i=1}^{3}\alpha_{s,i}d\bar{\zeta}_i. \quad (36)$$

Then, α_s is a $\bar{\partial}$-closed smooth $(0,1)$-form with $\operatorname{supp}\alpha_s \subset R_{c\delta}(\zeta^\delta) = B_c$. Let χ be a smooth convex increasing function that satisfies $\chi(t) = 0$ for $t \leq \tilde{a}/2$ and $\chi''(t) > 0$ for $t > \tilde{a}/2$. Now, define

$$\lambda_s(\zeta) = \phi(\zeta) + s^2\chi(\phi(\zeta)). \quad (37)$$

According to the weighted estimates of $\bar{\partial}$-equation on $\widetilde{D}$ (instead of $\widetilde{\Omega}$) and by using estimate (32) for each $s \geq 0$, there is u_s which satisfies $\bar{\partial}u_s = \alpha_s$, and

$$\|u_s\|_{\lambda_s} \lesssim \int_{\widetilde{D}-B_c}|\alpha_s|^2 e^{-\lambda_s} + \int_{B_c}\sum_{i=1}^{3}\tau_i^2|\alpha_{s,i}|^2 e^{-\lambda_s}dV. \quad (38)$$

Since $|\alpha_{s,i}| \lesssim e^{s\operatorname{Re}h}\tau_i^{-1}$ and $\operatorname{supp}\alpha_s \subset B_c$, it follows from (38) that

$$\begin{aligned}\int_{\widetilde{D}}|u_s|^2 e^{-\lambda_s}dV &\lesssim \int_{B_c}\sum_{i=1}^{3}\tau_i^2|\alpha_{s,i}|^2 e^{-\lambda_s}dV \\ &\lesssim \int_{\operatorname{supp}\bar{\partial}\psi_d} e^{2s\operatorname{Re}h-\phi-s^2\chi(\phi)}dV.\end{aligned} \quad (39)$$

We consider the integrand of the last integral. If $\phi(z) \geq \tilde{a}$, then $\chi(\phi(z)) \geq \chi(\tilde{a}) > 0$, so the s^2-term in the exponent predominates. On the other hand, if $z \in \operatorname{supp}\bar{\partial}\psi_d$ and $\phi(z) \leq \tilde{a}$, then (35) shows that the integrand tends to zero. Thus, for any $\epsilon_0 > 0$, there exist $s_0 > 0$ and a function u_{s_0} so that $\bar{\partial}u_{s_0} = \alpha_{s_0}$ and

$$\int_{\widetilde{D}}|u_{s_0}|^2 e^{-\lambda_{s_0}}dV \lesssim \int_{\operatorname{supp}\bar{\partial}\psi_d}\epsilon_0 dV \lesssim \epsilon_0\prod_{i=1}^{3}\tau_i^2. \quad (40)$$

Since $\phi(\zeta^\delta) = 0$, it follows, from the property (iii) of Theorem 5, that there is $e > 0$, independent of ζ^δ, such that $\psi_d(z) = 1$ and $\phi(z) < \tilde{a}/2$ for all $z \in B_e$. Note that λ_s is independent of s for $z \in B_e$, and u_{s_0} is holomorphic in B_e. By mean value theorem, we have

$$\left|\frac{\partial u_{s_0}}{\partial\zeta_k}(\zeta^\delta)\right|^2 \lesssim \tau_k^{-2}\prod_{i=1}^{3}\tau_i^{-2}\int_{B_e}|u_{s_0}|^2 e^{-\lambda_{s_0}}dV \lesssim \epsilon_0\tau_k^{-2}, \quad k = 1,2,3, \quad (41)$$

and hence it follows that

$$\left|Yu_{s_0}(\zeta^\delta)\right| \lesssim \sqrt{\epsilon_0}\max\left(|b_k|\tau_k^{-1}\right). \quad (42)$$

Now, set $f_\delta = \beta_\delta e^{s_0 h} - u_{s_0}$. Then, f_δ is holomorphic on $\widetilde{D} = \Phi_{\tilde{z}^\delta}^{-1}(D)$. Since $\beta_\delta(\zeta^\delta) = h(\zeta^\delta) = 0$, it follows, from (30) and (42), that f_δ satisfies

$$\left|Yf_\delta(\zeta^\delta)\right| \gtrsim \sum_{i=1}^{3}|b_i|\tau_i^{-1}, \quad (43)$$

provided ϵ_0 is sufficiently small.

We want to show that $\sup_{\widetilde{\Omega}}|f_\delta| \leq C$, where $C > 0$ is independent of δ. Recall that $s_0 > 0$ is fixed. Thus, from the property (iii) of Theorem 5, there is a uniform constant $C_0 > 0$ such that $|\beta_\delta e^{s_0 h}| \leq C_0$. Let $r_0 > 0$ be the constant satisfying (28) and assume that $\zeta \in \widetilde{U}_{r_0/2}(0) = \Phi_{\tilde{z}^\delta}^{-1}(U_{r_0/2}(0))$. Since f_δ is holomorphic on $\widetilde{D}$, it follows, by (40) and mean value theorem, that there exists a constant $C_1 > 0$, independent of $\delta > 0$, such that

$$|f_\delta(\zeta)|^2 \lesssim r_0^{-6}\int_{\widetilde{U}_{r_0/2}(\zeta)}|f_\delta|^2 dV \leq C_1. \quad (44)$$

We need to show the boundedness of f_δ outside $\widetilde{U}_{r_0/2}(0)$. Let χ_1 and χ_2 be smooth cutoff functions with

$$\begin{aligned}&\text{(i)}\ \chi_1(z) = 1 \quad \text{if } |z| \geq \frac{r_0}{2}, \qquad \chi_2(z) = 1 \quad \text{if } z \in \operatorname{supp}\chi_1 \\ &\text{(ii)}\ \chi_2(z) = 0 \quad \text{if } |z| \leq \frac{r_0}{4},\end{aligned} \quad (45)$$

and set $\tilde{\chi}_i = \Phi_{\tilde{z}^\delta}^*(\chi_i)$, $i = 1,2$. By Kohn's theorem on global regularity for the $\bar{\partial}$-equation, the following estimate for the solution of $\bar{\partial}u = \alpha$,

$$\left\|\tilde{\chi}_1 u_{s_0}\right\|_4^2 \lesssim \left\|\tilde{\chi}_2\alpha_{s_0}\right\|_4^2 + \left\|u_{s_0}\right\|^2, \quad (46)$$

holds on D provided $s_0 > 0$ is sufficiently large. Note that $\tilde{\chi}_2\alpha_{s_0} = 0$ because $\operatorname{supp}\alpha_{s_0} \subset R_{c\delta}(\zeta^\delta) \subset \widetilde{U}_{r_0/4}(0)$ for all sufficiently small $\delta > 0$. Thus, we conclude from (40), (46), and the Sobolev lemma that

$$\sup_{\widetilde{D}}\left|\tilde{\chi}_1 u_{s_0}\right| \lesssim \left\|\tilde{\chi}_1 u_{s_0}\right\|_4^2 \lesssim \left\|u_{s_0}\right\|^2 \leq C_2, \quad (47)$$

where C_2 is independent of δ.

Combining (44) and (47) and by the fact that $|\beta_\delta e^{s_0 h}| \leq C_0$, we conclude that

$$\sup_{\widetilde{D}}\left|f^\delta\right| \leq C, \quad (48)$$

where C is independent of ζ^δ and δ. Therefore, it follows from (43) and (48) that

$$C_{\widetilde{\Omega}}\left(\zeta^\delta; Y\right) \geq C_{\widetilde{D}}\left(\zeta^\delta; Y\right) \gtrsim \sum_{i=1}^{3}|b_i|\tau_i^{-1}. \quad (49)$$

On the other hand, the polydisc $B_c = R_{c\delta}(\zeta^\delta)$ about ζ^δ lies in $\widetilde{\Omega}$. So one obtains that

$$C_{\widetilde{\Omega}}\left(\zeta^\delta; Y\right) \leq C_{B_c}\left(\zeta^\delta; Y\right) = \max\left\{|b_k|\left(c\tau_k\right)^{-1} : k = 1,2,3\right\}. \quad (50)$$

Thus, one concludes from (49) and (50) that

$$C_{\widetilde{\Omega}}\left(\zeta^\delta; Y\right) \approx \sum_{i=1}^{3}|b_i|\tau_i^{-1}. \quad (51)$$

Set $L_k' = (d\Phi_{\tilde{z}^\delta}^{-1})L_k$, $k = 1,2,3$, where L_k's are defined in (1) in terms of z-coordinates defined in Theorem 1.

At $\zeta^\delta = (d\delta^{1/\eta}, 0, -b\delta/d_0(\tilde{z}^\delta))$, from the holomorphic coordinate change of $\Phi_{\tilde{z}^\delta}$ in Proposition 4, we see that

$$L_1' = \frac{\partial}{\partial \zeta_1} + e_1\left(z^\delta\right) d_0\left(\tilde{z}^\delta\right) \frac{\partial}{\partial \zeta_3} := \frac{\partial}{\partial \zeta_1} + \tilde{e}_1\left(z^\delta\right) \frac{\partial}{\partial \zeta_3},$$
$$L_2' = \frac{\partial}{\partial \zeta_2} + \left[d_1\left(\tilde{z}^\delta\right) + e_2\left(z^\delta\right)\right] \frac{\partial}{\partial \zeta_3} := \frac{\partial}{\partial \zeta_2} + \tilde{e}_1\left(z^\delta\right) \frac{\partial}{\partial \zeta_3}, \quad (52)$$

and that

$$L_3' = d_0\left(\tilde{z}^\delta\right) \frac{\partial}{\partial \zeta_3},$$

where $d_0(\tilde{z}^\delta) = (1/2)((\partial r/\partial z_3)(\tilde{z}^\delta))^{-1}$ and $d_1(\tilde{z}^\delta) = -((\partial r/\partial z_3)(\tilde{z}^\delta))^{-1}(\partial r/\partial z_2)(\tilde{z}^\delta)$ and where $e_i = -(\partial r/\partial z_3)^{-1}(\partial r/\partial z_i)$, $i = 1, 2$. Since $(\partial r/\partial z_i)(0) = 0$, $i = 1, 2$, and $|\partial r/\partial z_3| \approx 1$, independent of δ, it follows that $|\tilde{e}_i| \lesssim \delta$, $i = 1, 2$. Thus, if the vector $Y = \sum_{i=1}^3 b_i(\partial/\partial \zeta_i)$ is written as $Y = \sum_{i=1}^3 a_i L_i'$, then it follows that

$$\max\left(\left|b_i\right| \tau_i^{-1}\right) \approx \sum_{i=1}^3 \left|a_i\right| \tau_i^{-1}. \quad (53)$$

Let us write $X = \sum_{i=1}^3 a_i L_i$, and $Y = (\Phi_{\tilde{z}^\delta}^{-1})_* X = \sum_{i=1}^3 a_i L_i' = \sum_{i=1}^3 b_i(\partial/\partial \zeta_i)$. From (51), (53), and the invariance property of the metric, it follows that

$$C_\Omega\left(z^\delta; X\right) = C_{\tilde{\Omega}}\left(\zeta^\delta; Y\right) \approx \sum_{i=1}^3 \left|a_i\right| \tau_i^{-1}. \quad (54)$$

To obtain an upper bound for the Bergman metric, we note that $R_{c\delta}(\zeta^\delta) \subset \tilde{\Omega}$. Thus, by elementary estimates, for any $f \in A^2(\tilde{\Omega}) := L^2(\tilde{\Omega}) \cap A(\tilde{\Omega})$, we obtain that

$$\left|\frac{\partial f}{\partial \zeta_k}\left(\zeta^\delta\right)\right|^2 \lesssim \tau_k^{-2} \prod_{j=1}^3 \tau_j^{-2} \|f\|_{L^2(\tilde{\Omega})}^2, \quad (55)$$

for $k = 1, 2, 3$. Therefore, it follows that

$$b_{\tilde{\Omega}}\left(\zeta^\delta; Y\right) \lesssim \left(\sum_{k=1}^3 \left|b_k\right| \tau_k^{-1}\right) \prod_{j=1}^3 \tau_j^{-1}, \quad (56)$$

where

$$b_{\tilde{\Omega}}\left(\zeta^\delta; Y\right) = \sup\left\{\left|Yf\left(\zeta^\delta\right)\right| : f \in A^2\left(\tilde{\Omega}\right), f(z) = 0, \|f\|_{L^2(\tilde{\Omega})} \le 1\right\}. \quad (57)$$

Combining (23) and (56), one concludes that

$$B_{\tilde{\Omega}}\left(\zeta^\delta; Y\right) = \frac{b_{\tilde{\Omega}}\left(\zeta^\delta; Y\right)}{K_{\tilde{\Omega}}\left(\zeta^\delta, \zeta^\delta\right)^{1/2}} \lesssim \sum_{k=1}^3 \left|b_k\right| \tau_k^{-1}. \quad (58)$$

To estimate the upper bound of the Kobayashi metric, set

$$R = \min\left\{c\tau_k \left|b_k\right|^{-1} : k = 1, 2, 3\right\}. \quad (59)$$

Then,

$$f(t) = \left(b_1 t, b_2 t, -\frac{b\delta}{2} + b_3 t\right) \quad (60)$$

defines a map $f : D_R \subset \mathbb{C} \to B_c = R_{c\delta}(\zeta^\delta) \subset \tilde{\Omega}$ with $f_*((\partial/\partial t)|_0) = Y = \sum_{k=1}^3 b_k(\partial/\partial \zeta_k)$. Hence,

$$K_{\tilde{\Omega}}\left(\zeta^\delta; Y\right) \le K_{B_c}\left(\zeta^\delta; Y\right) \le R^{-1} \le \max\left\{\left|b_k\right| \left(c\tau_k\right)^{-1} : k = 1, 2, 3\right\} \lesssim \sum_{k=1}^3 \left|b_k\right| \tau_k^{-1}. \quad (61)$$

Combining (51), (58), and (61), we obtain that

$$C_{\tilde{\Omega}}\left(\zeta^\delta; Y\right) \approx B_{\tilde{\Omega}}\left(\zeta^\delta; Y\right) \approx K_{\tilde{\Omega}}\left(\zeta^\delta; Y\right) \approx \sum_{i=1}^3 \left|b_i\right| \tau_i^{-1}, \quad (62)$$

and hence the invariance property implies that

$$C_\Omega\left(z^\delta; X\right) \approx B_\Omega\left(z^\delta; X\right) \approx K_\Omega\left(z^\delta; X\right) \approx \sum_{i=1}^3 \left|a_i\right| \tau_i^{-1}. \quad (63)$$

If we combine (3), (4), (22), and (63), a proof of Theorem 1 is completed. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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