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Research Article

A Proof of Łojasiewicz's Theorem

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We give a necessary and sufficient condition for a primitive of a distribution to have the value at a point in the sense of Łojasiewicz. A formula defining the indefinite integral of a distribution with a basepoint is introduced, and further structural results are discussed.

1. Introduction

Let $\mathscr{D} = \mathscr{D}(\mathbb{R})$ be the topological $\mathbb{C}$-vector space of complex valued compactly supported test functions on $\mathbb{R}$, and let $\mathscr{D}' = \mathscr{D}'(\mathbb{R})$ be the space of complex valued distributions on $\mathbb{R}$. In the following discussion, a distribution $f \in \mathscr{D}'$ is also denoted by $f(x)$, and the dual pairing between $f \in \mathscr{D}'$ and a test function $\phi \in \mathscr{D}$ is denoted by either $\langle f, \phi\rangle$ or $\langle f(x), \phi(x)\rangle$. On the other hand, the letter x_0 will always denote a point.

According to Łojasiewicz [1], a distribution $f \in \mathscr{D}'$ has the value $c \in \mathbb{C}$ at x_0 if

$$f(ax + x_0) \longrightarrow c \tag{1}$$

in $\mathscr{D}'$ as $a \to 0$. If such a value c exists at x_0, we will say that f is *evaluable* at x_0 and write $f(x_0) = c$. For f to be evaluable at x_0, it suffices for $\lim_{a\to 0} f(ax + x_0)$ to exist in $\mathscr{D}'$, as the limit can only be a constant. We can equivalently require that there exists $c \in \mathbb{C}$ such that $\lim_{a\to 0^+} f(ax + x_0) = c$, as this entails $\lim_{a\to 0^-} f(ax + x_0) = c$. Simply requiring the existence of $\lim_{a\to 0^+} f(ax+x_0)$ does not suffice, as the limit may in general be of the form $c_1 + c_2 H(x - x_0)$, where H is the Heaviside step function.

One interesting consequence of this definition is the following.

Theorem 1 (Łojasiewicz). *If a distribution f is evaluable at x_0, then any primitive F of f is also evaluable at x_0.*

This result is useful in various circumstances. For instance, if a distribution f is evaluable at a and b, then so is any primitive F of f, and we may define a definite integral of f as

$$\int_a^b f = F(b) - F(a). \tag{2}$$

These ideas are connected with an interesting construction of distributional integral in the work of Estrada and Vindas [2].

In view of the simplicity and naturality of Theorem 1, the known proof is somewhat indirect. The argument follows as a corollary of a more difficult result of Łojasiewicz, which is stated in Theorem 5. The first purpose of this paper is to give a short and direct proof. We then arrive at a formula of the indefinite integral of a distribution with a basepoint. In fact, we can reverse the usual direction of reasoning and use the arguments developed along these lines to give a different proof of Theorem 5.

Theorem 5 is an example of a structure theorem, which is interesting in its own right and has a generalization involving the notion of the quasiasymptotic behavior [3]. In the last section, we study how variations of the definition of the value at a point lead to some other nice analogous structural results.

2. A Proof of Theorem 1

In order to fix our notation, we briefly recall the following elementary notions [4]. Suppose we have a continuous family of distributions $\{f_u\}_{u\in I}$ depending on a parameter u in an interval I, meaning that $\langle f_u, \phi\rangle$ is continuous in u for each $\phi \in \mathscr{D}$. If $\langle f_u, \phi\rangle$ is differentiable at $u_0 \in I$ for each $\phi \in \mathscr{D}$, we say that $\{f_u\}_{u\in I}$ is differentiable with respect to u at u_0 and define $\partial_u f_u|_{u=u_0}$ by

$$\langle \partial_u f_u|_{u=u_0}, \phi\rangle = \partial_u \langle f_u, \phi\rangle|_{u=u_0}. \quad (3)$$

Evidently $\partial_u f_u|_{u=u_0}$ is a distribution as it is the limit of distributions given by the difference quotients. Similarly, for $a, b \in I$, we define $\int_a^b f_u du$ by

$$\left\langle \int_a^b f_u du, \phi \right\rangle = \int_a^b \langle f_u, \phi\rangle du, \quad (4)$$

which is again a distribution, being the limit of distributions given by the Riemann sums. By pairing with test functions, it follows from the fundamental theorem of calculus that if $\{f_u\}_{u\in I}$ and $\{F_u\}_{u\in I}$ are continuous families of distributions with $\partial_u F_u|_{u=u_0} = f_{u_0}$ for all $u_0 \in I$, then, for any $a, b \in I$,

$$\int_a^b f_u du = F_b - F_a. \quad (5)$$

Let us note that, for any distribution $f(x) \in \mathcal{D}'$, both $\{f(ax)\}_{a\in(-\infty,0)}$ and $\{f(ax)\}_{a\in(0,\infty)}$ are continuous families of distributions. If f is evaluable at $x_0 = 0$, namely, if $f(ax) \to c$ as $a \to 0$, then $\{f(ax)\}_{a\in\mathbb{R}}$ becomes a continuous family of distributions if we define $f(0) = c$. Our argument uses this simple observation.

Proof of Theorem 1. Let $f = \partial F$ in $\mathcal{D}'$ and suppose f is evaluable at $x_0 = 0$. As seen above, $\{f(ax)\}_{a\in\mathbb{R}}$ is a continuous family of distributions and so is the family $\{xf(ax)\}_{a\in\mathbb{R}}$. It is trivial to verify that the family $\{F(ax)\}_{a\in(0,\infty)}$ is differentiable with respect to $a \in (0,\infty)$ with $\partial_a F(ax)|_{a=a_0} = xf(a_0x)$. By (5), for $u, u_0 \in (0,\infty)$,

$$\int_u^{u_0} xf(ax)\,da = F(u_0x) - F(ux). \quad (6)$$

The left-hand side is well defined for $u \in \mathbb{R}$ and gives a continuous family as u ranges over the real line, and thus, taking the limit $u \to 0^+$ on both sides, we see that $F(ux) \to L(x)$ as $u \to 0^+$ for some $L(x) \in \mathcal{D}'$. Applying ∂_x gives $uf(ux) \to L'(x)$, but clearly $uf(ux) \to 0$. We conclude that $L(x)$ is a constant. □

It also follows that if F is a primitive of a distribution f such that $f(x_0) = c$, the family $\{F(ax + x_0)\}_{a\in\mathbb{R}}$ is differentiable with respect to a and we have $\partial_a F(ax + x_0)|_{a=a_0} = xf(a_0x + x_0)$. In particular, $\partial_a F(ax + x_0)|_{a=0} = cx$.

3. Distributions Integrable from a Basepoint

In the preceding proof, it is clear that the assumption that f is evaluable at x_0 was not entirely necessary. Let us say that $f \in \mathcal{D}'$ is *integrable from* x_0 if the following two conditions hold.

(i) For $u_0 > 0$, $\int_u^{u_0} xf(ax+x_0)da$ converges in $\mathcal{D}'$ as $u \to 0^+$.

(ii) $af(ax + x_0) \to 0$ in $\mathcal{D}'$ as $a \to 0^+$.

By the same argument, this definition gives a necessary and sufficient condition for a primitive F of f to be evaluable at x_0. Indeed, if we set $x_0 = 0$, then (i) is equivalent to the existence of $L(x) := \lim_{u\to 0^+} F(ux)$. In this case, since $uf(ux) \to L'(x)$ as $u \to 0^+$, (ii) is equivalent to $L(x)$ being a constant. We summarize this as follows.

Proposition 2. *Let F be a distribution and let $f = \partial F$. Then F is evaluable at x_0 if and only if f is integrable from x_0.*

We denote by $\mathcal{D}'_{x_0}$ the space of all distributions integrable from x_0. For $f \in \mathcal{D}'_{x_0}$, we define a distribution $\int_{x_0}^{x+x_0} f$ by the formula

$$\int_{x_0}^{x+x_0} f := \lim_{u\to 0^+} \int_u^1 xf(ax + x_0)\,da. \quad (7)$$

Let F be a primitive of f. For any $u > 0$,

$$\int_u^1 xf(ax + x_0)\,da = F(x + x_0) - F(ux + x_0), \quad (8)$$

and if f is integrable from x_0, taking the limit $u \to 0^+$, we have

$$\int_{x_0}^{x+x_0} f = F(x + x_0) - F(x_0), \quad (9)$$

as $F(x_0)$ exists by Proposition 2. Replacing x with $x - x_0$, we define the indefinite integral of $f \in \mathcal{D}'_{x_0}$ with basepoint x_0 by

$$\int_{x_0}^{x} f := \lim_{u\to 0^+} \int_u^1 (x - x_0) f(ax - ax_0 + x_0)\,da. \quad (10)$$

It follows that we have $\int_{x_0}^x f = F(x) - F(x_0)$ and $\partial_x \int_{x_0}^x f = f(x)$. We also note that $\int_{x_0}^x f$ is evaluable with value 0 at x_0.

It is easy to see that if f_n is a sequence in $\mathcal{D}'_{x_0}$, then $f_n \to f$ in $\mathcal{D}'$ for some $f \in \mathcal{D}'_{x_0}$ does not imply $\int_{x_0}^x f_n \to \int_{x_0}^x f$ in general. In order to remedy this, we introduce the following notions.

Suppose f_n is a sequence in $\mathcal{D}'_{x_0}$. We say f_n is *bounded at* x_0 if, for each $\phi \in \mathcal{D}$, $\langle f_n(ax + x_0), \phi(x)\rangle$ is bounded independently of n as well as of $a \in (0, 1]$. Let us say f_n converges *boundedly* to $f \in \mathcal{D}'_{x_0}$ if $f_n \to f$ in $\mathcal{D}'$ and $f_n - f$ is eventually bounded at x_0. Finally, we say f_n converges *uniformly* to $f \in \mathcal{D}'_{x_0}$ if, for each $\phi \in \mathcal{D}$, $\langle f_n(ax + x_0), \phi(x)\rangle$ converges to $\langle f(ax + x_0), \phi(x)\rangle$ uniformly in $a \in (0, 1]$. Clearly, uniform convergence implies bounded convergence.

Lemma 3. *If a sequence f_n in $\mathcal{D}'_{x_0}$ converges boundedly (resp., uniformly) to $f \in \mathcal{D}'_{x_0}$, then the sequence $\int_{x_0}^x f_n$ converges boundedly (resp., uniformly) to $\int_{x_0}^x f$.*

Proof. Suppose f_n in $\mathscr{D}'_{x_0}$ converges to 0 with f_n bounded at x_0, and let $F_n(x) = \int_{x_0}^x f_n$. We have

$$\begin{aligned}\langle F_n(ax+x_0), \phi(x)\rangle &= \lim_{u\to 0^+}\int_u^1 \langle axf_n(abx+x_0), \phi(x)\rangle\, db \\ &= \lim_{u\to 0^+}\int_{au}^a \langle f_n(bx+x_0), x\phi(x)\rangle db,\end{aligned} \tag{11}$$

which shows $\langle F_n(ax+x_0), \phi(x)\rangle$ is bounded independently of n and of $a \in (0,1]$, and by taking the limit in n under the integral sign, we see that F_n converges boundedly to 0. If f_n in fact converges uniformly to 0, the uniform convergence of F_n is also apparent from the same expression. □

Let us write $F_n \rightrightarrows F$ on Ω to mean that F_n and F are continuous functions on Ω such that F_n converges to F uniformly on Ω. Let us also denote by $\int_{x_0}^x : \mathscr{D}'_{x_0} \to \mathscr{D}'_{x_0}$ the map that sends $f(x)$ to $\int_{x_0}^x f$.

Lemma 4. *Let f_n, f be distributions in $\mathscr{D}'_{x_0}$. If f_n converges boundedly to f, then, for every bounded open neighborhood Ω of x_0, there exists an integer $k \geq 0$ such that $(\int_{x_0}^x)^k f_n \rightrightarrows (\int_{x_0}^x)^k f$ on Ω.*

Proof. Let I be a compact interval containing Ω. We can find $k \geq 0$ and a sequence of continuous functions F_n, F on I such that $f_n = \partial^k F_n$, $f = \partial^k F$ with $F_n \rightrightarrows F$ on I (see [5]). Thus, $(\int_{x_0}^x)^k f_n$ and F_n (resp., $(\int_{x_0}^x)^k f$ and F) differ by polynomials P_n (resp., P) of degree $< k$ on I. By Lemma 3, $f_n \to f$ boundedly implies $(\int_{x_0}^x)^k f_n \to (\int_{x_0}^x)^k f$ in $\mathscr{D}'$, and since $F_n \to F$ in $\mathscr{D}'(\Omega)$, we have $P_n \to P$ in $\mathscr{D}'(\Omega)$, which is the case only when $P_n \rightrightarrows P$ on Ω. Hence $(\int_{x_0}^x)^k f_n \rightrightarrows (\int_{x_0}^x)^k f$ on Ω. □

4. Structure Theorem of Łojasiewicz

These ideas lead to a proof of another result of Łojasiewicz that we have already mentioned (cf. [1, 5, 6]). The proof given below seems illustrative in the sense that the implication in one direction is obtained by applying ∂_x several times, and the converse is obtained by applying $\int_{x_0}^x$ several times.

Theorem 5 (Łojasiewicz). *Let $f \in \mathscr{D}'$. Then $f(ax+x_0) \to c$ as $a \to 0$ if and only if $f = \partial^k F$ for some $k \geq 0$, where F is a continuous function near x_0 such that $\lim_{x\to x_0}(F(x)/(x-x_0)^k) = c/k!$.*

Proof. Let $x_0 = 0$. If $f = \partial^k F$ and F is continuous near 0 with $\lim_{x\to 0} F(x)/x^k = c/k!$, then $F(ax)/a^k \to cx^k/k!$ in $\mathscr{D}'$ as $a \to 0$, and applying ∂_x^k we obtain $f(ax) \to c$ in $\mathscr{D}'$. Conversely, suppose $f(ax) \to c$ in $\mathscr{D}'$ as $a \to 0$. Letting $f_a(x) = f(ax)$, it is easily observed that $f_a(x)$ converges boundedly (in fact, uniformly) to c as $a \to 0$. By Lemma 4, there exist a neighborhood Ω of 0 and $k \geq 0$ such that $(\int_0^x)^k f_a \rightrightarrows (\int_0^x)^k c = cx^k/k!$ on Ω. As $(\int_0^x)^k f_a = F(ax)/a^k$ if $F := (\int_0^x)^k f$, we have $F(ax)/a^k \rightrightarrows cx^k/k!$ as $a \to 0$. For any fixed $x \neq 0$ in Ω, we have $F(ax)/(ax)^k \to c/k!$ as $a \to 0$; namely, $\lim_{a\to 0} F(a)/a^k = c/k!$. □

5. Further Structure Theorems

There are various notions of the value of a distribution at a point, some defined under stricter conditions with stronger properties while others applicable for more general distributions [7–11]. When a situation or an application demands some specific features from the evaluable distributions, one would like to know how the values that we obtain are associated with some structural qualities of the distributions. We now discuss some results of this type similar to Theorem 5.

The works of Shiraishi and Itano give a notion of evaluation at a point with stricter properties than that of Łojasiewicz [7–9]. Let us call a sequence (f_n) in $\mathscr{D}(\mathbb{R}^d)$ a *δ-sequence* if there is a sequence of positive real numbers $(a_n) \to 0$ such that, $\forall n \in \mathbb{N}$,

(i) $f_n(x) = 0$ for $|x| \geq a_n$,

(ii) $\int f_n = 1$,

(iii) $\int |f_n|$ is bounded independently of $n \in \mathbb{N}$.

We say that a distribution $T \in \mathscr{D}'(\mathbb{R}^d)$ has *δ-value $c \in \mathbb{C}$ at $x_0 \in \mathbb{R}^d$* if

$$\langle T, \tau_{x_0} f_n\rangle \longrightarrow c \tag{12}$$

as $n \to \infty$ for all δ-sequences (f_n), where $(\tau_{x_0} f_n)(x) = f_n(x - x_0)$. In fact, we can restrict this condition to real nonnegative δ-sequences (which are called δ-sequences in, e.g., [5, 7]) without affecting the definition. By the result in [7] (see also [12] for a proof based on ideas from nonstandard analysis), $T \in \mathscr{D}'(\mathbb{R}^d)$ has δ-value c at x_0 if and only if it can be represented as an L^∞-function near x_0 which is continuous at x_0 with value c. Thus, we have $T = c + \Psi$, with

$$\operatorname*{ess\,sup}_{|x-x_0|<a} |\Psi(x)| \longrightarrow 0 \tag{13}$$

as $a \to 0^+$. As this condition is quite strong, we can regard this as the most conservative notion of the value of a distribution at a point.

We can compare this with the previously discussed Łojasiewicz definition, as it is immediate that the Łojasiewicz value has the following sequential representation. A δ-sequence (f_n) of the form

$$f_n(x) = a_n^{-d} f\left(\frac{x}{a_n}\right), \tag{14}$$

where $f \in \mathscr{D}(\mathbb{R}^d)$ with $\int f = 1$ and $a_n > 0$ with $(a_n) \to 0$, is called a model sequence. One sees that a distribution T has Łojasiewicz's value c at x_0 if and only if $\langle T, \tau_{x_0} f_n\rangle \to c$ for all model sequences. A structural result given by Theorem 5 tells us that the condition imposed on T is much weaker.

In this section we find a continuous family of classes of distributions $\mathscr{D}_{p,x_0}$ for $1 \le p < \infty$ such that, for any $1 \le q \le p < \infty$,

$$\left\{\begin{array}{c}\text{Distributions}\\ \text{with } \delta\text{-value}\\ \text{at } x_0\end{array}\right\} \subseteq \mathscr{D}_{q,x_0} \subseteq \mathscr{D}_{p,x_0} \subseteq \left\{\begin{array}{c}\text{Distributions with}\\ \text{Łojasiewicz's value}\\ \text{at } x_0\end{array}\right\} \tag{15}$$

with analogous structural results involving L^p functions. These classes of distributions can be defined sequentially in a natural way.

Definition 6. Let $1 \le p < \infty$ be fixed. A sequence (f_n) in $\mathscr{D}(\mathbb{R}^d)$ is called a δ_p*-sequence* if there exists a sequence of positive real numbers $(a_n) \to 0$ such that, $\forall n \in \mathbb{N}$,

(i) $f_n(x) = 0$ for $|x| \ge a_n$,

(ii) $\int f_n = 1$,

(iii) $a_n^{d(p-1)} \int |f_n|^p$ is bounded independently of $n \in \mathbb{N}$.

A distribution $T \in \mathscr{D}'(\mathbb{R}^d)$ is said to have δ_p*-value* $c \in \mathbb{C}$ *at* $x_0 \in \mathbb{R}^d$ if

$$\langle T, \tau_{x_0} f_n \rangle \longrightarrow c \tag{16}$$

as $n \to \infty$ for all δ_p-sequences (f_n).

Remark 7. In the above definition, we will say that (a_n) is a contracting sequence of (f_n).

For any $1 \le q \le p \le \infty$, we have (e.g., [13]) that if $\Omega \subseteq \mathbb{R}^d$ is a nonempty open subset of finite measure $\mu(\Omega)$ and if $f \in L^p(\Omega)$, then $f \in L^q(\Omega)$ and

$$\|f\|_{L^q(\Omega)} \le \mu(\Omega)^{1/q-1/p} \|f\|_{L^p(\Omega)}. \tag{17}$$

Let $1 \le q \le p < \infty$, and suppose (f_n) is a δ_p-sequence, with a contracting sequence $(a_n) \to 0$. From (17) we obtain

$$a_n^{-d/q} \left(\int |f_n|^q \right)^{1/q} \le a_n^{-d/p} \left(\int |f_n|^p \right)^{1/p}, \tag{18}$$

and multiplying both sides by a_n^d gives

$$\left(a_n^{d(q-1)} \int |f_n|^q \right)^{1/q} \le \left(a_n^{d(p-1)} \int |f_n|^p \right)^{1/p}, \tag{19}$$

which shows that (f_n) is also a δ_q-sequence. Therefore, if a distribution T has δ_q-value c at x_0, then it has the same δ_p-value at x_0. This will also follow from Theorem 10 (iii), as we have, since $1 < p' \le q' \le \infty$,

$$a^{-d/p'} \|\Psi\|_{L^{p'}(B_{x_0}(a))} \le a^{-d/q'} \|\Psi\|_{L^{q'}(B_{x_0}(a))} \tag{20}$$

by (17). Hence, the condition of a distribution having δ_p-value at a point becomes less restrictive as p increases. As any model sequence is a δ_p-sequence for all $1 \le p < \infty$, if $T \in \mathscr{D}(\mathbb{R}^d)$ has δ_p-value c at $x_0 \in \mathbb{R}^d$ for some p, then it has the same value c at x_0 in the sense of Łojasiewicz.

For a nonempty open set $\Omega \subseteq \mathbb{R}^d$, we let $\mathscr{D}_{\mathbb{R}}(\Omega) \subseteq \mathscr{D}(\Omega)$ be the subspace of all real valued test functions and let $\mathscr{D}(\Omega)_+ \subseteq \mathscr{D}_{\mathbb{R}}(\Omega)$ (resp., $\mathscr{D}(\Omega)_- \subseteq \mathscr{D}_{\mathbb{R}}(\Omega)$) be the subset of all nonnegative (resp., nonpositive) test functions. Let

$$\mathscr{D}_p^1(\Omega) \subseteq \mathscr{D}(\Omega) \tag{21}$$

be the subset consisting of f such that $\int |f|^p = 1$, and let

$$\mathscr{D}_p^1(\Omega)_+ = \mathscr{D}_p^1(\Omega) \cap \mathscr{D}(\Omega)_+. \tag{22}$$

For $\Psi \in \mathscr{D}'(\Omega)$, we define

$$\begin{aligned} \|\Psi\|_{\mathscr{D}_p'(\Omega)} &= \sup_{f \in \mathscr{D}_p^1(\Omega)} |\langle \Psi, f \rangle|, \\ \|\Psi\|_{\mathscr{D}_p'(\Omega)_+} &= \sup_{f \in \mathscr{D}_p^1(\Omega)_+} |\langle \Psi, f \rangle|, \end{aligned} \tag{23}$$

taking values in $[0, \infty]$. We then have the following simple estimate.

Lemma 8. *We have*

$$\|\Psi\|_{\mathscr{D}_p'(\Omega)_+} \le \|\Psi\|_{\mathscr{D}_p'(\Omega)} \le 4\|\Psi\|_{\mathscr{D}_p'(\Omega)_+}. \tag{24}$$

Proof. The first inequality follows trivially since $\mathscr{D}_p^1(\Omega)_+ \subseteq \mathscr{D}_p^1(\Omega)$. In order to see the second inequality, suppose $f \in \mathscr{D}_{\mathbb{R}}(\Omega)$. We can write $f = f_+ + f_-$, where $f_+(x) = \max\{f(x), 0\}$ and $f_-(x) = \min\{f(x), 0\}$ for $x \in \Omega$. As f_+ and f_- are compactly supported continuous functions, we can find $f_1 \in \mathscr{D}(\Omega)_+$ (resp., $f_2 \in \mathscr{D}(\Omega)_-$) that is as close as we want to f_+ (resp., f_-) in the L^p-norm, such that $f = f_1 + f_2$. Hence, from

$$\begin{aligned} |\langle \Psi, f \rangle| &\le |\langle \Psi, f_1 \rangle| + |\langle \Psi, f_2 \rangle| \\ &\le \|\Psi\|_{\mathscr{D}_p'(\Omega)_+} \left(\|f_1\|_{L^p(\Omega)} + \|f_2\|_{L^p(\Omega)} \right), \end{aligned} \tag{25}$$

since $\|f_\pm\|_{L^p(\Omega)} \le \|f\|_{L^p(\Omega)}$, we have

$$|\langle \Psi, f \rangle| \le 2\|\Psi\|_{\mathscr{D}_p'(\Omega)_+} \|f\|_{L^p(\Omega)}. \tag{26}$$

By (26), if $f \in \mathscr{D}(\Omega)$, then since we have $\|\operatorname{Re}(f)\|_{L^p(\Omega)} \le \|f\|_{L^p(\Omega)}$ and $\|\operatorname{Im}(f)\|_{L^p(\Omega)} \le \|f\|_{L^p(\Omega)}$,

$$\begin{aligned} |\langle \Psi, f \rangle| &\le |\langle \Psi, \operatorname{Re}(f) \rangle| + |\langle \Psi, \operatorname{Im}(f) \rangle| \\ &\le 4\|\Psi\|_{\mathscr{D}_p'(\Omega)_+} \|f\|_{L^p(\Omega)}. \end{aligned} \tag{27}$$

□

Suppose $\|\Psi\|_{\mathscr{D}_p'(\Omega)} < \infty$ for some $1 \le p < \infty$. Since $\mathscr{D}(\Omega)$ is dense in $L^p(\Omega)$, Ψ extends to a continuous functional on $L^p(\Omega)$ and lies in the strong dual of $L^p(\Omega)$, which is isometric to $L^{p'}(\Omega)$ [13]. We thus have $\Psi \in L^{p'}(\Omega)$ and $\|\Psi\|_{\mathscr{D}_p'(\Omega)} = \|\Psi\|_{L^{p'}(\Omega)}$.

Lemma 9. *Let (a_n) be a sequence of positive real numbers such that $(a_n) \to 0$. Suppose we have two sequences (f_n) and (g_n) in $\mathscr{D}(\mathbb{R}^d)$ such that both $a_n^{d(p-1)} \int |f_n|^p$ and $a_n^{d(p-1)} \int |g_n|^p$ are bounded independently of $n \in \mathbb{N}$. Then $a_n^{d(p-1)} \int |f_n + g_n|^p$ is bounded independently of $n \in \mathbb{N}$.*

Proof. By multiplying by $a_n^{d(p-1)}$ on both sides of Minkowski's inequality for f_n and g_n, we obtain

$$a_n^{d(p-1)} \int |f_n + g_n|^p \leq \left(\left(a_n^{d(p-1)} \int |f_n|^p \right)^{1/p} + \left(a_n^{d(p-1)} \int |g_n|^p \right)^{1/p} \right)^p, \tag{28}$$

from which the lemma follows. □

We can now give a structure theorem on our notion of δ_p-value of a distribution. The only tricky part of the following argument seems to be that our definition is unaffected even if we only restrict ourselves to real nonnegative δ_p-sequences (Theorem 10 (ii)).

Theorem 10. *Let $T \in \mathscr{D}'(\mathbb{R}^d)$. Then, the following statements are equivalent.*

(i) *T has δ_p-value $c \in \mathbb{C}$ at $x_0 \in \mathbb{R}^d$.*

(ii) *$\langle T, \tau_{x_0} f_n \rangle \to c$ as $n \to \infty$ for all δ_p-sequences (f_n) such that $f_n \geq 0$.*

(iii) *$T = c + \Psi$, where Ψ can be represented as an $L^{p'}$-function in some open ball $B_{x_0}(a)$ of radius $a > 0$ around x_0, and*

$$a^{-d/p'} \|\Psi\|_{L^{p'}(B_{x_0}(a))} \longrightarrow 0 \tag{29}$$

as $a \to 0^+$, where $p' = p/(p-1) \in (1, \infty]$ is the Hölder conjugate of p.

Proof. As the implication (i)⇒(ii) is immediate, it only remains to show (ii)⇒(iii)⇒(i).

Let us assume (ii). It suffices to consider the special case $x_0 = 0$. Let T be a distribution such that $(\langle T, f_n \rangle) \to c$ for all nonnegative δ_p-sequences (f_n). For $\Psi = T - c$, since $\int f_n = 1$ and $\langle T - c, f_n \rangle = \langle T, f_n \rangle - c$, we have $(\langle T, f_n \rangle) \to c$ if and only if $(\langle \Psi, f_n \rangle) \to 0$. We now claim that

$$a^{-d/p'} \|\Psi\|_{\mathscr{D}'_p(B_0(a))_+} \longrightarrow 0 \tag{30}$$

as $a \to 0^+$. Otherwise, for some $\varepsilon_0 > 0$, we can find a sequence of positive real numbers $(a_n) \to 0$ and functions $g_n \in \mathscr{D}_p^1(B_0(a_n))_+$ such that $a_n^{-d/p'} |\langle \Psi, g_n \rangle| \geq \varepsilon_0$ for all $n \in \mathbb{N}$. We note

$$a_n^{d(p-1)} \int \left(a_n^{-d/p'} g_n \right)^p = a_n^{d(p-1)} a_n^{-d(p-1)} \int g_n^p = 1, \tag{31}$$

and, in particular, it is bounded independently of $n \in \mathbb{N}$. Applying inequality (19) to the functions $a_n^{-d/p'} g_n$ (with $q = 1$), we obtain

$$\int a_n^{-d/p'} g_n \leq 1 \tag{32}$$

for all $n \in \mathbb{N}$. Let (h_n) be any fixed nonnegative δ_p-sequence of which (a_n) is a contracting sequence, such as a nonnegative model sequence. We let

$$f_n = a_n^{-d/p'} g_n + b_n h_n, \tag{33}$$

where $b_n = 1 - \int a_n^{-d/p'} g_n \in [0, 1)$. Observe that $f_n \geq 0$ with $\int f_n = 1$, and applying Lemma 9 to the sequences $a_n^{-d/p'} g_n$ and $b_n h_n$, we see that (f_n) is in fact a nonnegative δ_p-sequence. Thus, we must have

$$(\langle \Psi, f_n \rangle) \longrightarrow 0. \tag{34}$$

But as $\langle \Psi, f_n \rangle = a_n^{-d/p'} \langle \Psi, g_n \rangle + b_n \langle \Psi, h_n \rangle$, the fact that $(b_n \langle \Psi, h_n \rangle) \to 0$ implies $(a_n^{-d/p'} \langle \Psi, g_n \rangle) \to 0$, a contradiction. Hence, (30) follows, which implies (iii) by Lemma 8 and the paragraph following it.

Lastly, we assume that (iii) holds for $x_0 = 0$. Let (f_n) be a δ_p-sequence with a contracting sequence $(a_n) \to 0$. By Hölder's inequality,

$$\begin{aligned} |\langle \Psi, f_n \rangle| &\leq \|\Psi\|_{L^{p'}(B(a_n))} \|f_n\|_{L^p(B(a_n))} \\ &= a_n^{-d/p'} \|\Psi\|_{L^{p'}(B(a_n))} \left(a_n^{d(p-1)} \int |f_n|^p \right)^{1/p} \longrightarrow 0 \end{aligned} \tag{35}$$

as $n \to \infty$, and (i) follows. □

It is often useful to relate a given notion of a value at a point, usually defined through the pairing of a distribution with test functions, to a statement revealing the internal structure of the distribution. One such result is Theorem 5, and the above theorem gives some others.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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