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Research Article

New Difference Sequence Spaces Defined by Musielak-Orlicz Function

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We introduce new sequence spaces by using Musielak-Orlicz function and a generalized $B^{\mu}_{\wedge}$-difference operator on n-normed space. Some topological properties and inclusion relations are also examined.

1. Introduction and Preliminaries

The notion of the difference sequence space was introduced by Kızmaz [1]. It was further generalized by Et and Çolak [2] as follows: $Z(\Delta^{\mu}) = \{x = (x_k) \in \omega : (\Delta^{\mu} x_k) \in z\}$ for $z = \ell_{\infty}, c$, and c_0, where μ is a nonnegative integer and

$$\Delta^{\mu} x_k = \Delta^{\mu-1} x_k - \Delta^{\mu-1} x_{k+1}, \quad \Delta^0 x_k = x_k \quad \forall k \in \mathbb{N} \tag{1}$$

or equivalent to the following binomial representation:

$$\Delta^{\mu} x_k = \sum_{\nu=0}^{\mu} (-1)^{\nu} \binom{\mu}{\nu} x_{k+\nu}. \tag{2}$$

These sequence spaces were generalized by Et and Basarir [3] taking $z = \ell_{\infty}(p)$, $c(p)$, and $c_0(p)$.

Dutta [4] introduced the following difference sequence spaces using a new difference operator:

$$Z\left(\Delta_{(\eta)}\right) = \left\{x = (x_k) \in \omega : \Delta_{(\eta)} x \in z\right\} \quad \text{for } z = \ell_{\infty}, c, c_0, \tag{3}$$

where $\Delta_{(\eta)} x = (\Delta_{(\eta)} x_k) = (x_k - x_{k-\eta})$ for all $k, \eta \in \mathbb{N}$.

In [5], Dutta introduced the sequence spaces $\overline{c}(\|\cdot,\cdot\|, \Delta^{\mu}_{(\eta)}, p)$, $\overline{c_0}(\|\cdot,\cdot\|, \Delta^{\mu}_{(\eta)}, p)$, $\ell_{\infty}(\|\cdot,\cdot\|, \Delta^{\mu}_{(\eta)}, p)$, $m(\|\cdot,\cdot\|, \Delta^{\mu}_{(\eta)}, p)$, and $m_0(\|\cdot,\cdot\|, \Delta^{\mu}_{(\eta)}, p)$, where $\eta, \mu \in \mathbb{N}$ and $\Delta^{\mu}_{(\eta)} x_k = (\Delta^{\mu}_{(\eta)} x_k) = (\Delta^{\mu-1}_{(\eta)} x_k - \Delta^{\mu-1}_{(\eta)} x_{k-\eta})$ and $\Delta^{0}_{(\eta)} x_k = x_k$ for all $k, \eta \in \mathbb{N}$, which is equivalent to the following binomial representation:

$$\Delta^{\mu}_{(\eta)} x_k = \sum_{\nu=0}^{\mu} (-1)^{\nu} \binom{\mu}{\nu} x_{k-\eta\nu}. \tag{4}$$

The difference sequence spaces have been studied by authors [6–14] and references therein. Başar and Altay [15] introduced the generalized difference matrix $B = (b_{mk})$ for all $k, m \in \mathbb{N}$, which is a generalization of $\Delta_{(1)}$-difference operator by

$$b_{mk} = \begin{cases} r, & k = m \\ s, & k = m-1 \\ 0, & (k > m) \text{ or } (0 \le k < m-1). \end{cases} \tag{5}$$

Başarir and Kayikçi [16] defined the matrix $B^{\mu}(b^{\mu}_{mk})$ which reduced the difference matrix $\Delta^{\mu}_{(1)}$ in case $r = 1$, $s = -1$. The generalized B^{μ}-difference operator is equivalent to the following binomial representation:

$$B^{\mu} x = B^{\mu}(x_k) = \sum_{\nu=0}^{\mu} \binom{\mu}{\nu} r^{\mu-\nu} s^{\nu} x_{k-\nu}. \tag{6}$$

Let $\wedge = (\wedge_k)$ be a sequence of nonzero scalars. Then, for a sequence space E, the multiplier sequence space $E_\wedge$, associated with the multiplier sequence $\wedge$, is defined as

$$E_\wedge = \{x = (x_k) \in \omega : (\wedge_k x_k) \in E\}. \tag{7}$$

An Orlicz function M is a function, $M : [0, \infty) \to [0, \infty)$, which is continuous, nondecreasing, and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$, and $M(x) \to \infty$ as $x \to \infty$.

We say that an Orlicz function M satsfies the Δ_2-condition if there exists $K > 2$ and $x_0 \geq 0$ such that $M(2x) \leq KM(x)$ for all $x \geq x_0$. The Δ_2-condition is equivalent to $M(Lx) \leq KLM(x)$ for all $x > x_0 > 0$ and for $L, K > 1$.

Lindenstrauss and Tzafriri [17] used the idea of Orlicz function to define the following sequence space:

$$\ell_M = \left\{x \in \omega : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty\right\} \tag{8}$$

which is called an Orlicz sequence space. The space ℓ_M is a Banach space with the norm

$$\|x\| = \inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\right\}. \tag{9}$$

It is shown in [17] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to ℓ_p $(p \geq 1)$.

A sequence $\mathscr{M} = (M_k)$ of Orlicz function is called a Musielak-Orlicz function; see [18, 19]. A sequence $\mathscr{N} = (N_k)$ defined by

$$N_k(v) = \sup\{|v| u - M_k(u) : u \geq 0\}, \quad k = 1, 2, \ldots, \tag{10}$$

is called the complimentary function of a Musielak-Orlicz function $\mathscr{M}$. For a given Musielak-Orlicz function $\mathscr{M}$, the Musielak-Orlicz sequence space $t_{\mathscr{M}}$ and its subspace $h_{\mathscr{M}}$ are defined as follows:

$$\begin{aligned} t_{\mathscr{M}} &= \{x \in \omega : I_M(cx) < \infty \text{ for some } c > 0\}, \\ h_{\mathscr{M}} &= \{x \in \omega : I_M(cx) < \infty \ \forall c > 0\}, \end{aligned} \tag{11}$$

where $I_{\mathscr{M}}$ is a convex modular defined by

$$I_{\mathscr{M}}(x) = \sum_{k=1}^{\infty} M_k(x_k), \quad x = (x_k) \in t_M. \tag{12}$$

We consider $t_{\mathscr{M}}$ equipped with the Luxemburg norm

$$\|x\| = \inf\left\{k > 0 : I_M\left(\frac{x}{k}\right) \leq 1\right\} \tag{13}$$

or equipped with the Orlicz norm

$$\|x\|^0 = \inf\left\{\frac{1}{k}\left(1 + I_M(kx)\right) : k > 0\right\}. \tag{14}$$

By a lacunary sequence $\theta = (i_r)$, $r = 0, 1, 2, \ldots$, where $i_0 = 0$, we will mean an increasing sequence of nonnegative integers $h_r = (i_r - r_{r-1}) \to \infty$ $(r \to \infty)$. The intervals determined by θ are denoted by $I_r = (i_{r-1}, i_r]$ and the ratio i_r / i_{r-1} will be denoted by q_r. The space of lacunary strongly convergent sequences N_θ was defined by Freedman et al. [20] as follows:

$$N_\theta = \left\{x = (x_k) : \lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} |x_k - L| = 0, \text{for some } L\right\}. \tag{15}$$

The concept of 2-normed spaces was initially developed by Gähler [21] in the mid of 1960's, while that of n-normed spaces one can see in Misiak [22]. Since then, many others have studied this concept and obtained various results; see Gunawan [23, 24] and Gunawan and Mashadi [25]. For more details about sequence spaces see [26–33] and references therein. Let $n \in \mathbb{N}$ and X be linear space over the field $\mathbb{K}$, where $\mathbb{K}$ is the field of real or complex numbers of dimension d, where $d \geq n \geq 2$.

A real valued function $\|\cdot, \ldots, \cdot\|$ on X^n satisfying the following four conditions:

(1) $\|(x_1, x_2, \ldots, x_n)\| = 0$ if and only if $x_1, x_2, x_3, \ldots, x_n$ are linearly dependent in X;

(2) $\|(x_1, x_2, \ldots, x_n)\|$ is invariant under permutation;

(3) $\|(\alpha x_1, x_2, \ldots, x_n)\| = |\alpha| \|(x_1, x_2, \ldots, x_n)\|$ for any $\alpha \in \mathbb{K}$;

(4) $\|(x + x', x_2, \ldots, x_n)\| \leq \|(x, x_2, \ldots, x_n)\| + \|(x', x_2, \ldots, x_n)\|$

is called an n-norm on X and the pair $(X, \|\cdot, \ldots, \cdot\|)$ is called an n-normed space over the field $\mathbb{K}$. For example, we may take $X = \mathbb{R}^n$ being equipped with the Euclidean n-norm $\|(x_1, x_2, \ldots, x_n)\|_E =$ the volume of the n-dimensional parallelepiped spanned by the vectors $x_1, x_2, \ldots, x_n$ which may be given explicitly by the formula

$$\|(x_1, x_2, \ldots, x_n)\|_E = \left|\det\left(x_{ij}\right)\right|, \tag{16}$$

where $x_i = (x_1, x_2, x_3, \ldots, x_n) \in \mathbb{R}^n$ for each $i = 1, 2, 3, \ldots, n$ and $\|\cdot\|_E$ denotes the Euclidean norm. Let $(X, \|\cdot, \ldots, \cdot\|)$ be an n-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \ldots, a_n\}$ linearly independent set in X. Then the following function $\|(\cdot, \ldots, \cdot)\|_\infty$ on X^{n-1} defined by

$$\begin{aligned} &\|(x_1, x_2, \ldots, x_n)\|_\infty \\ &\quad = \max\{\|(x_1, x_2, \ldots, x_{n-1}, a_i)\| : i = 1, 2, \ldots, n\} \end{aligned} \tag{17}$$

defines an $(n-1)$ norm on X with respect to $\{(a_1, a_2, \ldots, a_n)\}$.

A sequence (x_k) in an n-normed space $(X, \|\cdot, \ldots, \cdot\|)$ is said to converge to some $L \in X$ if

$$\begin{gathered} \lim_{k \to \infty} \|(x_k - L, z_1, \ldots, z_{n-1})\| = 0, \\ \text{for every } z_1, \ldots, z_{n-1} \in X. \end{gathered} \tag{18}$$

A sequence (x_k) in a normed space $(X, \|\cdot, \ldots, \cdot\|)$ is said to be Cauchy if

$$\begin{gathered} \lim_{\substack{k \to \infty \\ p \to \infty}} \left\|\left(x_k - x_p, z_1, \ldots, z_{n-1}\right)\right\| = 0, \\ \text{for every } z_1, \ldots, z_{n-1} \in X. \end{gathered} \tag{19}$$

If every Cauchy sequence in X converges to some $L \in X$ then X is said to be complete with respect to the n-norm. Any complete n-normed space is said to be n-Banach space.

Let $(X, \|\cdot,\ldots,\cdot\|)$ be an n-normed space and let $s(\omega - x)$ denote the space of X-valued sequences. Let $p = (p_k)$ be any bounded sequence of positive real numbers and $\mathscr{M} = (M_k)$ a Musielak-Orlicz function. We define the following sequence spaces in this paper:

$$
\begin{aligned}
& w_0^\theta\left(\mathscr{M}, B_\wedge^\mu, p, \|\cdot,\ldots,\cdot\|\right) \\
&= \Bigg\{ x = (x_k) \in s(w - x) : \lim_{r\to\infty} \frac{1}{h_r} \\
&\quad \times \sum_{k\in I_r} M_k \left(\left\| \left(\frac{B_\wedge^\mu x_k}{\rho}, z_1, \ldots, z_{n-1} \right) \right\| \right)^{p_k} = 0,\ \rho > 0 \Bigg\}, \\
& w^\theta\left(\mathscr{M}, B_\wedge^\mu, p, \|\cdot,\ldots,\cdot\|\right) \\
&= \Bigg\{ x = (x_k) \in s(w - x) : \lim_{r\to\infty} \frac{1}{h_r} \\
&\quad \times \sum_{k\in I_r} M_k \left(\left\| \left(\frac{B_\wedge^\mu x_k - L}{\rho}, z_1, \ldots, z_{n-1} \right) \right\| \right)^{p_k} = 0, \\
&\quad \text{for some } L, \rho > 0 \Bigg\}, \\
& w_\infty^\theta\left(\mathscr{M}, B_\wedge^\mu, p, \|\cdot,\ldots,\cdot\|\right) \\
&= \Bigg\{ x = (x_k) \in s(w - x) : \lim_{r\to\infty} \frac{1}{h_r} \\
&\quad \times \sum_{k\in I_r} M_k \left(\left\| \left(\frac{B_\wedge^\mu x_k}{\rho}, z_1, \ldots, z_{n-1} \right) \right\| \right)^{p_k} < \infty,\ \rho > 0 \Bigg\};
\end{aligned} \tag{20}
$$

when $\mathscr{M}(x) = x$, we get

$$
\begin{aligned}
& w_0^\theta\left(B_\wedge^\mu, p, \|\cdot,\ldots,\cdot\|\right) \\
&= \Bigg\{ x = (x_k) \in s(w - x) : \lim_{r\to\infty} \frac{1}{h_r} \\
&\quad \times \sum_{k\in I_r} \left(\left\| \left(\frac{B_\wedge^\mu x_k}{\rho}, z_1, \ldots, z_{n-1} \right) \right\| \right)^{p_k} = 0,\ \rho > 0 \Bigg\}, \\
& w^\theta\left(B_\wedge^\mu, p, \|\cdot,\ldots,\cdot\|\right) \\
&= \Bigg\{ x = (x_k) \in s(w - x) : \lim_{r\to\infty} \frac{1}{h_r} \\
&\quad \times \sum_{k\in I_r} \left(\left\| \left(\frac{B_\wedge^\mu x_k - L}{\rho}, z_1, \ldots, z_{n-1} \right) \right\| \right)^{p_k} = 0, \\
&\quad \text{for some } L,\ \rho > 0 \Bigg\}, \\
& w_\infty^\theta\left(B_\wedge^\mu, p, \|\cdot,\ldots,\cdot\|\right) \\
&= \Bigg\{ x = (x_k) \in s(w - x) : \lim_{r\to\infty} \frac{1}{h_r} \\
&\quad \times \sum_{k\in I_r} \left(\left\| \left(\frac{B_\wedge^\mu x_k}{\rho}, z_1, \ldots, z_{n-1} \right) \right\| \right)^{p_k} < \infty,\ \rho > 0 \Bigg\};
\end{aligned} \tag{21}
$$

when $p_k = 1$, for all k, we get

$$
\begin{aligned}
& w_0^\theta\left(\mathscr{M}, B_\wedge^\mu, \|\cdot,\ldots,\cdot\|\right) \\
&= \Bigg\{ x = (x_k) \in w(s - x) : \lim_{r\to\infty} \frac{1}{h_r} \\
&\quad \times \sum_{k\in I_r} M_k \left(\left\| \left(\frac{B_\wedge^\mu x_k}{\rho}, z_1, \ldots, z_{n-1} \right) \right\| \right) = 0,\ \rho > 0 \Bigg\}, \\
& w^\theta\left(\mathscr{M}, B_\wedge^\mu, \|\cdot,\ldots,\cdot\|\right) \\
&= \Bigg\{ x = (x_k) \in w(s - x) : \lim_{r\to\infty} \frac{1}{h_r} \\
&\quad \times \sum_{k\in I_r} M_k \left(\left\| \left(\frac{B_\wedge^\mu x_k - L}{\rho}, z_1, \ldots, z_{n-1} \right) \right\| \right) = 0, \\
&\quad \text{for some } L,\ \rho > 0 \Bigg\}, \\
& w_\infty^\theta\left(\mathscr{M}, B_\wedge^\mu, \|\cdot,\ldots,\cdot\|\right) \\
&= \Bigg\{ x = (x_k) \in w(s - x) : \lim_{r\to\infty} \frac{1}{h_r} \\
&\quad \times \sum_{k\in I_r} M_k \left(\left\| \left(\frac{B_\wedge^\mu x_k}{\rho}, z_1, \ldots, z_{n-1} \right) \right\| \right) < \infty,\ \rho > 0 \Bigg\}.
\end{aligned} \tag{22}
$$

The following inequality will be used throughout the paper. If $0 \le p_k \le \sup p_k = H$, $k = \max(1, 2^{H-1})$, then

$$|a_k + b_k|^{p_k} \le K\left\{|a_k|^{p_k} + |b_k|^{p_k}\right\} \tag{23}$$

for all k and $a_k, b_k \in \mathbb{C}$. Also $|a|^{p_k} \le \max(1, |a|^H)$ for all $a \in \mathbb{C}$.

2. Main Results

Theorem 1. *Let $\mathscr{M} = (M_k)$ be a Musielak-Orlicz function and $p = (p_k)$ a bounded sequence of positive real numbers; the spaces $w_0^\theta(\mathscr{M}, B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|)$, $w^\theta(\mathscr{M}, B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|)$, and $w_\infty^\theta(\mathscr{M}, B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|)$ are linear over the field of complex numbers $\mathbb{C}$.*

Proof. Let $x = (x_k)$, $y = (y_k) \in w_0^\theta(\mathscr{M}, B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|)$, and $\alpha, \beta \in \mathbb{C}$. Then there exist positive real numbers ρ_1 and ρ_2 such that

$$\begin{gathered}\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r} M_k\left(\left\|\left(\frac{B_\wedge^\mu x_k}{\rho_1}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k} = 0,\\ \lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r} M_k\left(\left\|\left(\frac{B_\wedge^\mu y_k}{\rho_2}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k} = 0.\end{gathered} \tag{24}$$

Define $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since $\|\cdot, \ldots, \cdot\|$ is an n-norm on X and $M_k's$ are nondecreasing and convex functions so by using inequality (23) we have

$$\begin{aligned}&\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r} M_k\left(\left\|\left(\frac{B_\wedge^\mu(\alpha x_k + \beta y_k)}{\rho_3}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k}\\ &\quad\le \lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r} M_k\left(\left\|\left(\frac{B_\wedge^\mu \alpha x_k}{\rho_3}, z_1, \ldots, z_{n-1}\right)\right\| + \left\|\left(\frac{B_\wedge^\mu \beta y_k}{\rho_3}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k}\\ &\quad\le K\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\frac{1}{2^{p_k}} M_k\left(\left\|\left(\frac{B_\wedge^\mu x_k}{\rho_1}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k}\\ &\qquad + K\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\frac{1}{2^{p_k}} M_k\left(\left\|\left(\frac{B_\wedge^\mu y_k}{\rho_2}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k}\\ &\quad = 0.\end{aligned} \tag{25}$$

Thus, we have $\alpha x + \beta y \in w_0^\theta(\mathscr{M}, B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|)$. Hence $w_0^\theta(\mathscr{M}, B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|)$ is a linear space. Similarly, we can prove that $w^\theta(\mathscr{M}, B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|)$ and $w_\infty^\theta(\mathscr{M}, B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|)$ are linear spaces. This completes the proof of the theorem. □

Theorem 2. *Let $\mathscr{M} = (M_k)$ be a Musielak-Orlicz function and $p = (p_k)$ a bounded sequence of positive real numbers; the space $w_0^\theta(\mathscr{M}, B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|)$ is a topological linear space paranormed by*

$$g(x) = \inf\left\{\rho^{p_r/M} : \left(\frac{1}{h_r}\sum_{k\in I_r} M_k\left(\left\|\left(\frac{B_\wedge^\mu x_k}{\rho}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k}\right)^{1/M} \le 1\right\}, \tag{26}$$

where $M = \max(1, \sup_k p_k < \infty)$.

Proof. Clearly $g(x) \ge 0$ for $x = (x_k) \in w_0^\theta(\mathscr{M}, B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|)$. Since $M_k(0) = 0$, we get $g(0) = 0$. Again, if $g(x) = 0$, then

$$g(x) = \inf\left\{\rho^{p_r/M} : \left(\frac{1}{h_r}\sum_{k\in I_r} M_k\left(\left\|\left(\frac{B_\wedge^\mu x_k}{\rho}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k}\right)^{1/M} \le 1\right\} = 0. \tag{27}$$

This implies that, for a given $\varepsilon > 0$, there exist some ρ_ε $(0 < \rho_\varepsilon < \varepsilon)$ such that

$$\left(\frac{1}{h_r}\sum_{k\in I_r} M_k\left(\left\|\left(\frac{B_\wedge^\mu x_k}{\rho_\varepsilon}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k}\right)^{1/M} \le 1. \tag{28}$$

Thus

$$\begin{aligned}&\left(\frac{1}{h_r}\sum_{k\in I_r} M_k\left(\left\|\left(\frac{B_\wedge^\mu x_k}{\varepsilon}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k}\right)^{1/M}\\ &\quad\le \left(\frac{1}{h_r}\sum_{k\in I_r} M_k\left(\left\|\left(\frac{B_\wedge^\mu x_k}{\rho_\varepsilon}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k}\right)^{1/M}\\ &\quad\le 1\end{aligned} \tag{29}$$

for each r, and suppose that $x_k \ne 0$ for each $k \in \mathbb{N}$. This implies that $B_\wedge^\mu x_k \ne 0$ for each $k \in \mathbb{N}$. Let $\varepsilon \to 0$, then

$$\left(\left(\left\|\left(\frac{B_\wedge^\mu x_k}{\rho_\varepsilon}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k}\right)^{1/M} \longrightarrow \infty; \tag{30}$$

which is a contradiction. Therefore, $B^{\mu}_{\wedge}x_k = 0$ for each k and thus $x_k = 0$ for each $k \in \mathbb{N}$. Let $\rho_1 > 0$ and $\rho_2 > 0$ be such that

$$\begin{aligned}
&\left(\frac{1}{h_r}\sum_{k\in I_r} M_k\left(\left\|\left(\frac{B^{\mu}_{\wedge}x_k}{\rho_1}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k}\right)^{1/M} \leq 1,\\
&\left(\frac{1}{h_r}\sum_{k\in I_r} M_k\left(\left\|\left(\frac{B^{\mu}_{\wedge}y_k}{\rho_2}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k}\right)^{1/M} \leq 1 \quad \text{for each } r.
\end{aligned} \tag{31}$$

Let $\rho = \rho_1 + \rho_2$; then by using Minkowski's inequality, we have

$$\begin{aligned}
&\left(\frac{1}{h_r}\sum_{k\in I_r} M_k\left(\left\|\left(\frac{B^{\mu}_{\wedge}(x_k + y_k)}{\rho}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k}\right)^{1/M}\\
&\leq \left(\frac{1}{h_r}\sum_{k\in I_r} M_k\left(\left\|\left(\frac{B^{\mu}_{\wedge}x_k + B^{\mu}_{\wedge}y_k}{\rho_1 + \rho_2}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k}\right)^{1/M}\\
&\leq \left(\frac{1}{h_r}\sum_{k\in I_r} M_k \right.\\
&\quad \times \left(\left(\frac{\rho_1}{\rho_1 + \rho_2}\right)\left\|\left(\frac{B^{\mu}_{\wedge}x_k}{\rho_1}, z_1, \ldots, z_{n-1}\right)\right\|\right.\\
&\quad\quad \left.\left. + \left(\frac{\rho_2}{\rho_1 + \rho_2}\right)\left\|\left(\frac{B^{\mu}_{\wedge}y_k}{\rho_2}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k}\right)^{1/M}\\
&\leq \left(\frac{\rho_1}{\rho_1 + \rho_2}\right)\\
&\quad \times \left(\frac{1}{h_r}\sum_{k\in I_r} M_k\left(\left\|\left(\frac{B^{\mu}_{\wedge}x_k}{\rho_1}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k}\right)^{1/M}\\
&\quad + \left(\frac{\rho_2}{\rho_1 + \rho_2}\right)\\
&\quad \times \left(\frac{1}{h_r}\sum_{k\in I_r} M_k\left(\left\|\left(\frac{B^{\mu}_{\wedge}y_k}{\rho_2}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k}\right)^{1/M} \leq 1.
\end{aligned} \tag{32}$$

Since $\rho' s$ are nonnegative, we have

$$\begin{aligned}
&g(x+y)\\
&= \inf\left\{\rho^{p_r/M} : \left(\frac{1}{h_r}\sum_{k\in I_r} M_k \right.\right.\\
&\quad\quad \left.\times\left(\left\|\left(\frac{B^{\mu}_{\wedge}x_k}{\rho}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k}\right)^{1/M}\\
&\quad\quad \left. \leq 1 \right\}\\
&\leq \inf\left\{{\rho_1}^{p_r/M} : \left(\frac{1}{h_r}\sum_{k\in I_r} M_k \right.\right.\\
&\quad\quad \left.\times\left(\left\|\left(\frac{B^{\mu}_{\wedge}x_k}{\rho_1}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k}\right)^{1/M}\\
&\quad\quad \left. \leq 1 \right\}\\
&\quad + \inf\left\{{\rho_2}^{p_r/M} : \left(\frac{1}{h_r}\sum_{k\in I_r} M_k \right.\right.\\
&\quad\quad \left.\times\left(\left\|\left(\frac{B^{\mu}_{\wedge}y_k}{\rho_2}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k}\right)^{1/M}\\
&\quad\quad \left. \leq 1 \right\}.
\end{aligned} \tag{33}$$

Therefore, $g(x + y) \leq g(x) + g(y)$.

Finally, we prove that the scalar multiplication is continuous. Let ν be any complex number. By definition,

$$\begin{aligned}
&g(\nu x)\\
&= \inf\left\{\rho^{p_r/M} : \left(\frac{1}{h_r}\sum_{k\in I_r} M_k \right.\right.\\
&\quad\quad \left.\times\left(\left\|\left(\frac{\nu B^{\mu}_{\wedge}x_k}{\rho}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k}\right)^{1/M}\\
&\quad\quad \left. \leq 1 \right\}.
\end{aligned} \tag{34}$$

Then

$$g(\nu x) = \inf\left\{(|\nu| t)^{p_r/M} : \left(\frac{1}{h_r}\sum_{k\in I_r} M_k \times \left(\left\|\left(\frac{B^{\mu}_{\wedge}x_k}{t}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k}\right)^{1/M} \leq 1\right\}, \tag{35}$$

where $t = \rho/|\nu|$. Since $|\nu|^{p_r} \leq \max(1, |\nu| \sup p_k)$, we have

$$g(\nu x) = \max(1, |\nu| \sup p_k) \inf \times \left\{(t)^{p_r/M} : \left(\frac{1}{h_r}\sum_{k\in I_r} M_k \times \left(\left\|\left(\frac{B^{\mu}_{\wedge}x_k}{t}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k}\right)^{1/M} \leq 1\right\}. \tag{36}$$

So, the fact that scalar multiplication is continuous follows from the above inequality. This completes the proof of theorem. □

Theorem 3. *Let $\mathscr{M} = (M_k)$ be a Musielak-Orlicz function. If $\sup_k (M_k(x))^{p_k} < \infty$ for all fixed $x > 0$, then $w^{\theta}(\mathscr{M}, B^{\mu}_{\wedge}, p, \|\cdot, \ldots, \cdot\|) \subset w^{\theta}_{\infty}(\mathscr{M}, B^{\mu}_{\wedge}, p, \|\cdot, \ldots, \cdot\|)$.*

Proof. Let $x = (x_k) \in w^{\theta}(\mathscr{M}, B^{\mu}_{\wedge}, p, \|\cdot, \ldots, \cdot\|)$. Then there exists some positive number ρ_1 such that

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r} M_k\left(\left\|\left(\frac{B^{\mu}_{\wedge}x_k - L}{\rho_1}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k} = 0. \tag{37}$$

Define $\rho = 2\rho_1$. Since M_k is nondecreasing and convex and by using inequality (23), we have

$$\begin{aligned}
&\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r} M_k\left(\left\|\left(\frac{B^{\mu}_{\wedge}x_k}{\rho}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k} \\
&= \lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r} M_k\left(\left\|\left(\frac{B^{\mu}_{\wedge}x_k - L + L}{\rho}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k} \\
&\leq K\left\{\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r} M_k \frac{1}{2^{p_k}}\left(\left\|\left(\frac{B^{\mu}_{\wedge}x_k - L}{\rho_1}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k}\right. \\
&\quad \left. + \lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r} M_k \frac{1}{2^{p_k}}\left(\left\|\left(\frac{L}{\rho_1}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k}\right\} \\
&< K\left\{\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r} M_k\left(\left\|\left(\frac{B^{\mu}_{\wedge}x_k - L}{\rho_1}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k}\right. \\
&\quad \left. + \lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r} M_k\left(\left\|\left(\frac{L}{\rho_1}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k}\right\}.
\end{aligned} \tag{38}$$

Hence $x = (x_k) \in w^{\theta}_{\infty}(\mathscr{M}, B^{\mu}_{\wedge}, p, \|\cdot, \ldots, \cdot\|)$. This completes the proof of the theorem. □

Theorem 4. *Let $\mathscr{M} = (M_k)$ be a Musielak-Orlicz function and $0 < h = \inf p_k$. Then*

$$w^{\theta}_{\infty}(\mathscr{M}, B^{\mu}_{\wedge}, p, \|\cdot, \ldots, \cdot\|) \subset w^{\theta}_{0}(\mathscr{M}, B^{\mu}_{\wedge}, p, \|\cdot, \ldots, \cdot\|) \tag{39}$$

if and only if

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r} M_k(t)^{p_k} = \infty, \quad \textit{for some } t > 0. \tag{40}$$

Proof. Let $w^{\theta}_{\infty}(\mathscr{M}, B^{\mu}_{\wedge}, p, \|\cdot, \ldots, \cdot\|) \subset w^{\theta}_{0}(\mathscr{M}, B^{\mu}_{\wedge}, p, \|\cdot, \ldots, \cdot\|)$. Suppose (40) does not hold. Therefore there are a subinterval $I_{r(m)}$ of the set of intervals I_r and a number n_0, where $n_0 = \|(B^{\mu}_{\wedge}x_k/\rho, z_1, \ldots, z_{n-1})\|$ for all k, such that

$$\frac{1}{h_r(m)}\sum_{k\in I_{r(m)}} M(n_0)^{p_k} \leq N < \infty, \quad m = 1, 2, 3, \ldots. \tag{41}$$

Let us define $x = (x_k)$ as follows:

$$B^{\mu}_{\wedge}x_k = \begin{cases}\rho n_0, & k \in I_{r(m)} \\ 0, & k \notin I_{r(m)}.\end{cases} \tag{42}$$

Thus by (41), $x = (x_k) \in w^{\theta}_{\infty}(\mathscr{M}, B^{\mu}_{\wedge}, p, \|\cdot, \ldots, \cdot\|)$. But $x = (x_k) \notin w^{\theta}_{0}(\mathscr{M}, B^{\mu}_{\wedge}, p, \|\cdot, \ldots, \cdot\|)$. Hence (40) must hold.

Conversely, suppose that (40) holds and let $x = (x_k) \in w^{\theta}_{\infty}(\mathscr{M}, B^{\mu}_{\wedge}, p, \|\cdot, \ldots, \cdot\|)$. Then,

$$\frac{1}{h_r}\sum_{k\in I_r} M_k\left(\left\|\left(\frac{B^{\mu}_{\wedge}x_k}{\rho}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k} \leq N < \infty. \tag{43}$$

Suppose that $x = (x_k) \notin w^{\theta}_{0}(\mathscr{M}, B^{\mu}_{\wedge}, p, \|\cdot, \ldots, \cdot\|)$. Then for some number ε, $1 > \varepsilon > 0$, there is a number N_0 such that, for a subinterval $I_{r(m)}$ of the set of intervals I_r,

$$\left\|\left(\frac{B^{\mu}_{\wedge}x_k}{\rho}, z_1, \ldots, z_{n-1}\right)\right\| > \varepsilon \quad \text{for } N \geq N_0. \tag{44}$$

We have $M_k(\|(B^{\mu}_{\wedge}x_k/\rho, z_1, \ldots, z_{n-1})\|) \geq M(\varepsilon)^{p_k}$, which contradicts (40) by using (43). Hence we get

$$w^{\theta}_{\infty}(\mathcal{M}, B^{\mu}_{\wedge}, p, \|\cdot, \ldots, \cdot\|) \subset w^{\theta}_{0}(\mathcal{M}, B^{\mu}_{\wedge}, p, \|\cdot, \ldots, \cdot\|). \tag{45}$$

This completes the proof. □

Theorem 5. *Let $0 < h = \inf p_k \leq \sup p_k = H < \infty$. For any Musielak-Orlicz function $\mathcal{M} = (M_k)$ which satisfies Δ_2-condition, one has*

(i) $w^{\theta}_{0}(B^{\mu}_{\wedge}, p, \|\cdot, \ldots, \cdot\|) \subset w^{\theta}_{0}(\mathcal{M}, B^{\mu}_{\wedge}, p, \|\cdot, \ldots, \cdot\|)$

(ii) $w^{\theta}(B^{\mu}_{\wedge}, p, \|\cdot, \ldots, \cdot\|) \subset w^{\theta}(\mathcal{M}, B^{\mu}_{\wedge}, p, \|\cdot, \ldots, \cdot\|)$

(iii) $w^{\theta}_{\infty}(B^{\mu}_{\wedge}, p, \|\cdot, \ldots, \cdot\|) \subset w^{\theta}_{\infty}(\mathcal{M}, B^{\mu}_{\wedge}, p, \|\cdot, \ldots, \cdot\|)$.

Proof. (i) Let $x = (x_k) \in w^{\theta}_{0}(B^{\mu}_{\wedge}, p, \|\cdot, \ldots, \cdot\|)$. Then, we have

$$\frac{1}{h_r}\sum_{k\in I_r}\left(\left\|\left(\frac{B^{\mu}_{\wedge}x_k}{\rho}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k} \longrightarrow 0 \quad \text{as } r \longrightarrow \infty. \tag{46}$$

Let $\varepsilon > 0$, and choose δ with $0 < \delta < 1$ such that $M_k < \varepsilon$ for $0 \leq t \leq \delta$. We can write

$$\begin{aligned}
&\frac{1}{h_r}\sum_{k\in I_r}\left(\left\|\left(\frac{B^{\mu}_{\wedge}x_k}{\rho}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k} \\
&= \frac{1}{h_r}\sum_{\substack{k\in I_r \\ \|(B^{\mu}_{\wedge}x_k/\rho, z_1,\ldots,z_{n-1})\|\leq\delta}}\left(\left\|\left(\frac{B^{\mu}_{\wedge}x_k}{\rho}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k} \\
&\quad + \frac{1}{h_r}\sum_{\substack{k\in I_r \\ \|(B^{\mu}_{\wedge}x_k/\rho, z_1,\ldots,z_{n-1})\|>\delta}}\left(\left\|\left(\frac{B^{\mu}_{\wedge}x_k}{\rho}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k}.
\end{aligned} \tag{47}$$

For the first summation above, we can write

$$\begin{aligned}
&\frac{1}{h_r}\sum_{\substack{k\in I_r \\ \|(B^{\mu}_{\wedge}x_k/\rho, z_1,\ldots,z_{n-1})\|\leq\delta}}\left(\left\|\left(\frac{B^{\mu}_{\wedge}x_k}{\rho}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k} \\
&\quad < \max\left(\varepsilon, \varepsilon^{h}\right).
\end{aligned} \tag{48}$$

By using continuity of M_k, for the second summation we can write

$$\left\|\left(\frac{B^{\mu}_{\wedge}x_k}{\rho}, z_1, \ldots, z_{n-1}\right)\right\| < 1 + \frac{\left(\|(B^{\mu}_{\wedge}x_k/\rho, z_1, \ldots, z_{n-1})\|\right)}{\delta}. \tag{49}$$

Since each M_k is nondecreasing and convex and satisfies Δ_2-condition, it follows that

$$\begin{aligned}
&\frac{1}{h_r}\sum_{k\in I_r}M_k\left(\left\|\left(\frac{B^{\mu}_{\wedge}x_k}{\rho}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k} \\
&\leq \max\left(\varepsilon, \varepsilon^{h}\right) \\
&\quad + \max\left\{1, \left[\frac{2M_k\left(\|(B^{\mu}_{\wedge}x_k/\rho, z_1, \ldots, z_{n-1})\|\right)}{\delta}\right]^{h}\right\} \\
&\quad \times \frac{1}{h_r}\sum_{k\in I_r}\left(\left\|\left(\frac{B^{\mu}_{\wedge}x_k}{\rho}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k}.
\end{aligned} \tag{50}$$

Taking limit as $\varepsilon \to 0$ and $r \to \infty$, it follows that $x = (x_k) \in w^{\theta}_{0}(\mathcal{M}, B^{\mu}_{\wedge}, p, \|\cdot, \ldots, \cdot\|)$. Hence $w^{\theta}_{0}(B^{\mu}_{\wedge}, p, \|\cdot, \ldots, \cdot\|) \subset w^{\theta}_{0}(\mathcal{M}, B^{\mu}_{\wedge}, p, \|\cdot, \ldots, \cdot\|)$. Similarly, we can prove (ii) and (iii). This completes the proof of the theorem. □

Theorem 6. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function. Then the following statements are equivalent:*

(i) $w^{\theta}_{\infty}(B^{\mu}_{\wedge}, p, \|\cdot, \ldots, \cdot\|) \subset w^{\theta}_{\infty}(\mathcal{M}, B^{\mu}_{\wedge}, p, \|\cdot, \ldots, \cdot\|)$;

(ii) $w^{\theta}_{0}(B^{\mu}_{\wedge}, p, \|\cdot, \ldots, \cdot\|) \subset w^{\theta}_{\infty}(\mathcal{M}, B^{\mu}_{\wedge}, p, \|\cdot, \ldots, \cdot\|)$;

(iii) $\sup_r(1/h_r)\sum_{k\in I_r}M_k(t)^{p_k} < \infty$ *for all* $t > 0$, *where* $t = \|B^{\mu}_{\wedge}x_k/\rho, z_1, \ldots, z_{n-1}\|$.

Proof. (i) ⇒ (ii) Suppose (i) holds. In order to prove (ii) we have to show that

$$w^{\theta}(B^{\mu}_{\wedge}, p, \|\cdot, \ldots, \cdot\|) \subset w^{\theta}_{\infty}(\mathcal{M}, B^{\mu}_{\wedge}, p, \|\cdot, \ldots, \cdot\|). \tag{51}$$

Let $x = (x_k) \in w^{\theta}_{0}(B^{\mu}_{\wedge}, p, \|\cdot, \ldots, \cdot\|)$. Then for a given $\varepsilon > 0$ there exists $s > s_{\varepsilon}$ such that

$$\frac{1}{h_r}\sum_{k\in I_r}\left(\left\|\left(\frac{B^{\mu}_{\wedge}x_k}{\rho}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k} < \varepsilon. \tag{52}$$

Hence there exists $K > 0$ such that

$$\sup_r\frac{1}{h_r}\sum_{k\in I_r}\left(\left\|\left(\frac{B^{\mu}_{\wedge}x_k}{\rho}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k} < K. \tag{53}$$

This shows that $x = (x_k) \in w^{\theta}_{\infty}(\mathcal{M}, B^{\mu}_{\wedge}, p, \|\cdot, \ldots, \cdot\|)$.

(ii) ⇒ (iii) Suppose (ii) holds and (iii) fails to hold. Then for some $t > 0$,

$$\sup_r\frac{1}{h_r}\sum_{k\in I_r}M_k(\varepsilon)^{p_k} = \infty, \tag{54}$$

and, therefore, we can find a subinterval $I_{r(m)}$ of the set of intervals I_r such that

$$\frac{1}{h_r(m)}\sum_{k\in I_{r(m)}}M_k\left(\frac{1}{m}\right)^{p_k} \geq m, \quad m = 1, 2, 3, \ldots. \tag{55}$$

Let us define $x = (x_k)$ as follows:

$$B_\wedge^\mu x_k = \begin{cases} \dfrac{\rho}{m}, & k \in I_{r(m)} \\ 0, & k \notin I_{r(m)}. \end{cases} \quad (56)$$

Thus $x = (x_k) \in w_0^\theta(B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|)$. But by (55), $x = (x_k) \notin w_\infty^\theta(\mathscr{M}, B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|)$ which contradicts (ii). Hence (iii) must hold.

(iii) $\Rightarrow$ (i) Let (iii) hold. Suppose that $x = (x_k) \notin w_\infty^\theta(\mathscr{M}, B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|)$. Then for $x = (x_k) \in w_\infty^\theta(B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|)$.

$$\sup_r \frac{1}{h_r} \sum_{k \in I_r} M_k\left(\left\|\left(\frac{B_\wedge^\mu x_k}{\rho}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k} = \infty. \quad (57)$$

Let $t = \|B_\wedge^\mu x_k/\rho, z_1, \ldots, z_{n-1}\|$ for each k, and then by (57) $\sup_r (1/h_r) \sum_{k \in I_r} M_k(t)^{p_k} = \infty$, which contradicts (iii). Hence (i) must hold. This completes the proof of the theorem. □

Theorem 7. *Let $\mathscr{M} = (M_k)$ be a Musielak-Orlicz function. Then the following statements are equivalent:*

(i) $w_0^\theta(\mathscr{M}, B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|) \subset w_0^\theta(B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|)$;

(ii) $w_0^\theta(\mathscr{M}, B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|) \subset w_\infty^\theta(B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|)$;

(iii) $\inf_r (1/h_r) \sum_{k \in I_r} M_k(t)^{p_k} > 0$ *for all* $t > 0$.

Proof. (i) $\Rightarrow$ (ii) is obvious

(ii) $\Rightarrow$ (iii) Let (ii) hold and let (iii) fail to hold. Then

$$\inf_r \frac{1}{h_r} \sum_{k \in I_r} M_k(t)^{p_k} = 0 \quad \text{for some } t > 0, \quad (58)$$

and we can find a subinterval $I_{r(m)}$ of the set of intervals I_r such that

$$\frac{1}{h_r(m)} \sum_{k \in I_{r(m)}} M_k(m)^{p_k} < \frac{1}{m}, \quad m = 1, 2, 3, \ldots. \quad (59)$$

Let us define $x = (x_k)$ as follows:

$$B_\wedge^\mu x_k = \begin{cases} \rho m, & k \in I_{r(m)} \\ 0, & k \notin I_{r(m)}. \end{cases} \quad (60)$$

Thus by (iii), $x = (x_k) \in w_0^\theta(\mathscr{M}, B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|)$. But $x = (x_k) \notin w_\infty^\theta(B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|)$ which contradict (ii). Hence (iii) must hold.

(iii) $\Rightarrow$ (i) Let (iii) hold. Suppose that $x = (x_k) \in w_0^\theta(\mathscr{M}, B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|)$. Therefore,

$$\frac{1}{h_r} \sum_{k \in I_r} M_k\left(\left\|\left(\frac{B_\wedge^\mu x_k}{\rho}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k} \longrightarrow 0 \quad \text{as } r \longrightarrow \infty. \quad (61)$$

Again suppose $x = (x_k) \notin w_0^\theta(B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|)$ for some number $\varepsilon > 0$ and a subinterval $I_{r(m)}$ of the set of intervals I_r, we have

$$\left\|\left(\frac{B_\wedge^\mu x_k}{\rho}, z_1, \ldots, z_{n-1}\right)\right\| \geq \varepsilon \quad \forall k. \quad (62)$$

Then, from properties of the Orlicz function, we can write

$$M_k\left(\left\|\left(\frac{B_\wedge^\mu x_k}{\rho}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k} \geq M_k(\varepsilon)^{p_k}. \quad (63)$$

Consequently, by (61), we have $\lim_{r \to \infty} (1/h_r) \sum_{k \in I_r} M_k(\varepsilon)^{p_k} = 0$, which contradicts (iii). Hence (i) must hold. This completes the proof of the theorem. □

Theorem 8. *(i) If $0 < \inf p_k \leq p_k \leq 1$ for all k, then $w^\theta(\mathscr{M}, B_\wedge^\mu, \|\cdot, \ldots, \cdot\|) \subseteq w^\theta(\mathscr{M}, B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|)$.*

(ii) If $1 \leq p_k \leq \sup p_k = H < \infty$, then $w^\theta(\mathscr{M}, B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|) \subseteq w^\theta(\mathscr{M}, B_\wedge^\mu, \|\cdot, \ldots, \cdot\|)$.

Proof. (i) Let $x \in w^\theta(\mathscr{M}, B_\wedge^\mu, \|\cdot, \ldots, \cdot\|)$. Since $0 < \inf p_k \leq 1$, we get

$$\begin{aligned} &\frac{1}{h_r} \sum_{k \in I_r} M_k\left(\left\|\left(\frac{B_\wedge^\mu x_k - L}{\rho}, z_1, \ldots, z_{n-1}\right)\right\|\right) \\ &\quad \leq \frac{1}{h_r} \sum_{k \in I_r} M_k\left(\left\|\left(\frac{B_\wedge^\mu x_k - L}{\rho}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k}, \end{aligned} \quad (64)$$

and hence $x \in w^\theta(\mathscr{M}, B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|)$.

(ii) $1 \leq p_k \leq \sup p_k = H < \infty$ and $x = (x_k) \in w^\theta(\mathscr{M}, B_\wedge^\mu, p, \|\cdot, \ldots, \cdot\|)$. Then for each $0 < \varepsilon < 1$ there exists a positive integer s_0 such that

$$\begin{aligned} &\frac{1}{h_r} \sum_{k \in I_r} M_k\left(\left\|\left(\frac{B_\wedge^\mu x_k - L}{\rho}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k} \\ &\quad \leq \varepsilon < 1 \quad \forall r > s_0. \end{aligned} \quad (65)$$

This implies that

$$\begin{aligned} &\frac{1}{h_r} \sum_{k \in I_r} M_k\left(\left\|\left(\frac{B_\wedge^\mu x_k - L}{\rho}, z_1, \ldots, z_{n-1}\right)\right\|\right)^{p_k} \\ &\quad \leq \frac{1}{h_r} \sum_{k \in I_r} M_k\left(\left\|\left(\frac{B_\wedge^\mu x_k - L}{\rho}, z_1, \ldots, z_{n-1}\right)\right\|\right). \end{aligned} \quad (66)$$

Therefore $x = (x_k) \in w^\theta(\mathscr{M}, B_\wedge^\mu, \|\cdot, \ldots, \cdot\|)$. This completes the proof of the theorem. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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