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Research Article

Asymptotic Limit to Shocks for Scalar Balance Laws Using Relative Entropy

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We consider a scalar balance law with a strict convex flux. In this paper, we study inviscid limit to shocks for scalar balance laws up to a shift function, which is based on the relative entropy.

1. Introduction

We consider the following balance law in one-dimensional space $\mathbb{R}$:

$$\begin{gathered}\partial_t U + \partial_x A(U) = g(U) + \epsilon \partial^2_{xx} U, \\ U(0,x) = U_0(x), \\ t > 0,\ x \in \mathbb{R},\end{gathered} \tag{1}$$

where the flux $A''(v) := a'(v) \geq c$ for some constant $c > 0$ and $U_0 \in L^\infty(\mathbb{R})$. The existence of global unique weak solutions of (1) has been studied by Kruzkov. In this paper, we are interested in getting the optimal rate of convergence linked to a layer.

Let us consider the shock solutions of the scalar conservation laws with the given source term (1) with the initial data

$$S_0(x) = \begin{cases} C_L & \text{if } x < 0, \\ C_R & \text{if } x \geq 0, \end{cases} \tag{2}$$

with two constants $C_L > C_R$, where the source term g is defined as follows:

$$g \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R}), \qquad g(C_L) = g(C_R) = 0. \tag{3}$$

Then, the Rankine-Hugoniot condition ensures that the function

$$S_0(x - \sigma t) \quad \text{with } \sigma := \frac{A(C_L) - A(C_R)}{C_L - C_R} \tag{4}$$

is a solution to (1) with $\epsilon = 0$. Notice that the condition $C_L > C_R$ implies that they verify the entropy conditions; that is,

$$\partial_t \eta(U) + \partial_x G(U) - \eta'(U) g(U) \leq 0, \quad t > 0,\ x \in \mathbb{R}, \tag{5}$$

for any convex functions η, and $G' = \eta' A'$. An easy dimensional analysis shows that, because of those layers, we may have in general

$$\|U(t) - S(\cdot - \sigma t)\|^2_{L^2} \geq C\varepsilon, \tag{6}$$

for some $\epsilon > 0$ which means that the L^2 stability for two solutions U, S does not hold. We are interested in deriving the extremal L^2 stability up to a shift function. The main result is as follows.

Theorem 1. *Let $C_L > C_R$, $T > 0$ be any number, and let $U_0 \in L^\infty(\mathbb{R})$ be such that*

$$(U_0 - S_0) \in L^2(\mathbb{R}), \qquad \left(\frac{d}{dx} U_0\right)_+ \in L^2(\mathbb{R}). \tag{7}$$

Suppose that U is a solution of (1). Then there exists a Lipschitz curve $X \in L^\infty(0,T)$, $C := C(\|\eta''\|_{L^\infty}, \|g'\|_{L^\infty}, T)$, and $\epsilon_0 > 0$ such that $X(0) = 0$ and for any $0 < \epsilon < \epsilon_0$,

$$\begin{gathered}\|U(t) - S(t)\|^2_{L^2(\mathbb{R})} \leq C\left(\|U_0 - S_0\|^2_{L^2(\mathbb{R})} + \epsilon \log \frac{1}{\epsilon}\right), \\ t \in (0,T),\end{gathered} \tag{8}$$

where $S(t,x) := S_0(x - X(t))$*, and* S_0 *is defined by* (2)*. Moreover, this curve satisfies*

$$\begin{gathered}\left|\dot{X}(t)\right| \leq C,\\ |X(t) - \sigma t|^2 \leq Ct^{1/4}\left(\|U_0 - S_0\|^2_{L^2(\mathbb{R})} + \epsilon \log \frac{1}{\epsilon}\right).\end{gathered} \tag{9}$$

This is L^2 stability result to a shock for balance laws up to a shift function. The main point is how to construct a shift function $X(t)$ such that the time derivative of the relative entropy is smaller than convergence rate. Our method is based on the method developed in Leger and Vasseur [1, 2] together with using the relative entropy idea and the result cannot be true without shift (see [1]).

The relative entropy method introduced by Dafermos [3, 4] and Diperna [5] provides an efficient tool to study the stability and asymptotic limits among thermomechanical theories, which is related to the second law of thermodynamics. They showed, in particular, that if $\overline{U}$ is a Lipschitzian solution of a suitable conservation law on a lapse of time $[0,T]$, then for any bounded weak entropic solution U it holds

$$\int_{\mathbb{R}} \left|U(t) - \overline{U}(t)\right|^2 dx \leq C \int_{\mathbb{R}} \left|U(0) - \overline{U}(0)\right|^2 dx, \tag{10}$$

for a constant C depending on $\overline{U}$ and T. Since Dafermos [3] and Diperna [5]'s works, there has been much recent progress as applications of the relative entropy method. Chen et al. [6] have applied the relative entropy method to obtain the stability estimates to shocks for gas dynamics which derive the time asymptotic stability of Riemann solutions with large oscillation for the 3×3 system of Euler equations. For incompressible limits, see Bardos et al. [7, 8], Lions and Masmoudi [9], and Saint Raymond et al. [10–13] who have studied incompressible limit problems. There are also many recent results of the weak-uniqueness for the compressible Navier-Stokes equations together with using relative entropy by Germain [14] and Feireisl and Novotný [15]. For the relaxation there is an application for compressible models by Lattanzio and Tzavaras [16, 17] and we can also see Berthelin et al. [18, 19] as some applications of hydrodynamical limit problems. However, in all those cases, the method works as long as the limit solution has a good regularity such that the solution is Lipschitz. This is due to the fact that strong stability as (10) is not true when $\overline{U}$ has a discontinuity. It has been proven in [1, 2], however, that some shocks are strongly stable up to a shift. Choi and Vasseur [20] have recently used this stability property to study sharp estimates for the inviscid limit of viscous scalar conservation laws to a shock. With the same idea, Kwon and Vasseur [21] develop sharp estimates of hydrodynamical limits to shocks for BGK models. For this paper, we derive the optimal rate of convergence to shocks for scalar balance laws up to a shift function $X(t)$. Thus, it generalizes Choi and Vasseur's work [20]. The outline of this paper is as follows. In Section 2 we introduce relative entropy and some properties used in Leger [1]. In Section 3 we will derive some estimates of the hyperbolic and parabolic part of relative entropy. In Section 4, we will give the proof of Theorem 1 together with combining the estimates in Section 3. Finally, in the Appendix section, we will add the appendix to give the proof of Proposition 7.

2. Relative Entropy and Some Properties

In this section we introduce a special drift function $X(t)$, $t \in (0,T)$, defined in Leger [1] and relative entropy. To begin with we need some notations and properties provided in Leger [1]. Fix any strictly convex function $\eta \in C^2$; we first define the normalized relative entropy flux $g(\cdot,\cdot)$ by

$$f(x,y) := \frac{F(x,y)}{\eta(x \mid y)}, \tag{11}$$

where the associated relative entropy functional $\eta(\cdot \mid \cdot)$ is given by

$$\eta(x \mid y) := \eta(x) - \eta(y) - \eta'(y)(x-y) \tag{12}$$

and the flux of the relative entropy $F(\cdot,\cdot)$ is defined by

$$F(x,y) := G(x) - G(y) - \eta'(y)(A(x) - A(y)). \tag{13}$$

Note that for any fixed y and any weak entropic solution U of (1), we have

$$\partial_t \eta(U \mid y) + \partial_x F(U,y) = \left(\eta'(U) - \eta'(y)\right)\left(\epsilon\partial^2_{xx}U + g(U)\right). \tag{14}$$

Hence, f can be seen as a typical velocity associated to the relative entropy $\eta(\cdot, y)$.

Using the strict convexity of the function η, Leger showed in [1] the following lemma.

Lemma 2. *Let* $x, y \in [-L, L]$ *for any* $L > 0$*. There exists a constant* $\Lambda > 0$*, such that one has*

(i) $1/\Lambda \leq \eta''(x) \leq \Lambda$,

(ii) $(1/2\Lambda)(x-y)^2 \leq \eta(x \mid y) \leq (1/2)\Lambda(x-y)^2$,

(iii) $|F(x,y)| \leq \Lambda(x-y)^2$,

(iv) $0 \leq (\partial_x f)(x,y) \leq \Lambda$,

(v) $1/\Lambda \leq (\partial_y f)(x,y)$.

In the spirit of Leger [1], we consider the solution of the following differential equation in order to define the shift function X:

$$\begin{gathered}\dot{X}(t) = f\left(U(t, X(t)), \frac{C_L + C_R}{2}\right),\\ X(0) = 0.\end{gathered} \tag{15}$$

Note that the existence and uniqueness of X come from the Cauchy-Lipschitz theorem.

First, X is Lipschitz, since we have from Lemma 2

$$\left|\dot{X}(t)\right| \leq \frac{\left|F(U(t,X(t)), (C_L + C_R)/2)\right|}{\eta(U(t,X(t)) \mid (C_L + C_R)/2)} \leq 2\Lambda^2, \tag{16}$$

where we used the fact $\|U(t)\|_{L^\infty} \leq L$ for $t > 0$ in the following.

Lemma 3. *Let U be a solution of* (1). *Then, for every $t \in (0,T)$, one has*

$$\|U(t,\cdot)\|_{L^\infty(\mathbb{R})} \leq \|u_0\|_{L^\infty(\mathbb{R})} + \|g\|_{L^\infty(\mathbb{R})} t. \tag{17}$$

Proof. From the scalar balance law in (1), we get

$$\partial_t U + \partial_x A(U) - \epsilon \partial_{xx}^2 U \leq \|g\|_{L^\infty(\mathbb{R})}. \tag{18}$$

Since $\|u_0\|_{L^\infty(\mathbb{R})} + t\|g\|_{L^\infty(\mathbb{R})}$ satisfies (18) and $|u_0(x)| \leq \|u_0\|_{L^\infty(\mathbb{R})}$ for all $x \in \mathbb{R}$, the comparison principle for parabolic equations provides

$$U(t,x) \leq \|u_0\|_{L^\infty(\mathbb{R})} + t\|g\|_{L^\infty(\mathbb{R})} \tag{19}$$

for all $(t,x) \in (0,T) \times \mathbb{R}$. In the same method, we also get

$$U(t,x) \geq -\left(\|u_0\|_{L^\infty(\mathbb{R})} + t\|g\|_{L^\infty(\mathbb{R})}\right) \tag{20}$$

for all $(t,x) \in (0,T) \times \mathbb{R}$. □

The idea of the proof is to study the evolution of the relative entropy of the solution with respect to the shock, outside of a small region centered at $X(t)$ (this small region corresponds to the layer localization):

$$\int_{-\infty}^{X(t)-\delta\varepsilon} \eta(U(t,x) \mid C_L)\,dx + \int_{X(t)+\delta\varepsilon}^{\infty} \eta(U(t,x) \mid C_R)\,dx. \tag{21}$$

Indeed, for $\eta(x \mid y) = (x-y)^2$, the following holds:

$$\begin{aligned}\int_{\{|x-X(t)| \leq C\eta\}} \eta(U(t,x) \mid S(t,x))\,dx \\ \leq C\left|\{|x-X(t)| \leq C\eta\}\right| \leq C\eta,\end{aligned} \tag{22}$$

for any $\eta > 0$, where the constant C depends on T. From now on we will take a reasonable $\delta > 0$ and it will be mentioned in (40) later.

For the rigorous proof, we define the evolution of the integration in (21) as follows:

$$\mathscr{E}_\epsilon^\delta(t) := \int_{-\infty}^{\infty} \left[\phi_\delta\left(\frac{|x-X(t)|}{\varepsilon}\right)\right]^2 \eta(U(t,x) \mid S(t,x))\,dx \tag{23}$$

for any fixed $\delta > 0$ and $X \in C^1([0,T])$, where an increasing function ϕ_δ is defined by

$$\phi_\delta(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ 1 & \text{if } x \geq \delta. \end{cases} \tag{24}$$

From now on we delete δ in ϕ_δ. Thus, the derivative of $\mathscr{E}_\epsilon^\delta(t)$ implies the following lemma.

Lemma 4. *The function $\mathscr{E}_\epsilon^\delta(t)$, defined in* (23), *satisfies the following on* $(0,T)$:

$$\begin{aligned}
&\frac{d}{dt}\mathscr{E}_\epsilon^\delta(t) \\
&\quad = \int_{X(t)-\delta\varepsilon}^{X(t)} \frac{2}{\varepsilon}\phi'\left(\frac{-x+X(t)}{\varepsilon}\right)\phi\left(\frac{-x+X(t)}{\varepsilon}\right) \\
&\qquad \times\left[\dot{X}(t)\eta(U \mid C_L) - F(U,C_L)\right]dx \\
&\quad + \int_{-\infty}^{X(t)}\left[\phi\left(\frac{-x+X(t)}{\varepsilon}\right)\right]^2 \epsilon\partial_{xx}^2 U(t,x) \\
&\qquad \times\left(\eta'(U(t,x)) - \eta'(C_L)\right)dx \\
&\quad + \int_{-\infty}^{X(t)}\left[\phi\left(\frac{-x+X(t)}{\varepsilon}\right)\right]^2 g(U(t,x)) \\
&\qquad \times\left(\eta'(U(t,x)) - \eta'(C_L)\right)dx \\
&\quad - \int_{X(t)}^{X(t)+\delta\varepsilon} \frac{2}{\varepsilon}\phi\left(\frac{x-X(t)}{\varepsilon}\right)\phi'\left(\frac{x-X(t)}{\varepsilon}\right) \\
&\qquad \times\left[\dot{X}(t)\eta(U \mid C_R) - F(U,C_R)\right]dx \\
&\quad + \int_{x(t)}^{\infty}\left[\phi\left(\frac{x-X(t)}{\varepsilon}\right)\right]^2 \epsilon\partial_{xx}^2 U(t,x) \\
&\qquad \times\left(\eta'(U(t,x)) - \eta'(C_R)\right)dx \\
&\quad + \int_{X(t)}^{\infty}\left[\phi\left(\frac{x-X(t)}{\varepsilon}\right)\right]^2 g(U(t,x)) \\
&\qquad \times\left(\eta'(U(t,x)) - \eta'(C_R)\right)dx \\
&\quad := L_1 + L_2 + L_3 + R_1 + R_2 + R_3.
\end{aligned} \tag{25}$$

The proof is provided in Choi and Vasseur [20]. We next need a regularity to control hyperbolic part the lemma is as follows.

Lemma 5. *For any $t \in (0,T)$, there exists $C := C(\|g'\|_{L^\infty(\mathbb{R})}, T) > 0$ such that one gets*

$$\|(\partial_x U)_+\|_{L^2(\mathbb{R})} \leq C\left\|\left(\frac{d}{dx}U_0\right)_+\right\|_{L^2(\mathbb{R})}. \tag{26}$$

Proof. Taking derivative to (1) for variable x, multiplying $(\partial_x U)_+$, and integrating for variable x imply

$$\begin{aligned}
&\int_{\mathbb{R}} (\partial_x U)_+ \left(\partial_t \partial_x U + A''(U)\left|\partial_x U\right|^2 + A'(U)\partial_{xx}^2 U\right. \\
&\qquad \left. - \partial_{xxx}^3 U\right) dx - \int_{\mathbb{R}} g'(U)(\partial_x U)_+^2 dx \\
&\quad = \frac{1}{2}\partial_t \int_{\mathbb{R}} (\partial_x U)_+^2 dx \\
&\qquad + \int_{\mathbb{R}} \left[A''(U)(\partial_x U)_+^3 + A'(U)\partial_x\left(\frac{(\partial_x U)_+^2}{2}\right)\right. \\
&\qquad \left. + \epsilon \mid \partial_x (\partial_x U)_+^2 \right] dx \\
&\qquad - \int_{\mathbb{R}} g'(U)(\partial_x U)_+^2 dx = 0.
\end{aligned} \tag{27}$$

We apply the integration by parts to obtain the following regularity (26):

$$\frac{1}{2}\partial_t \int_{\mathbb{R}} (\partial_x U)_+^2 dx \le \left\|g'\right\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} (\partial_x U)_+^2 dx. \tag{28}$$

Thus, integrating (28) for time variable and using Gronwall's inequality provide the result (26). □

3. Estimates on the Hyperbolic and Parabolic Terms

In this section, we prove that the hyperbolic part $L_1 + R_1$ in equality (25) is strictly negative and the parabolic part $L_2 + R_2$ has a small rate of convergence. Applying Lemmas 2 and 5, we are able to show the main proposition for this section.

Proposition 6. *Let L_1 and R_1 be as in Lemma 4. Then, there exists a constant $\theta > 0$ such that, for any ε, δ, and satisfying*

$$\varepsilon\delta \le \theta, \tag{29}$$

we have

$$\begin{aligned}
&L_1 + R_1 \\
&\le -\frac{\theta}{\varepsilon}\int_{X(t)-\delta\varepsilon}^{X(t)+\delta\varepsilon} \phi\left(\frac{|x - X(t)|}{\varepsilon}\right)\phi'\left(\frac{|x - X(t)|}{\varepsilon}\right)\eta(U \mid S)\, dx.
\end{aligned} \tag{30}$$

Proof. Let us start with proving that L_1 is strictly negative. The proof of R_1 is similar. With the definition of $X(t)$, we write L_1 as

$$\begin{aligned}
L_1 = &\int_{X(t)-\delta\varepsilon}^{X(t)} \frac{2}{\varepsilon}\phi\left(\frac{-x + X(t)}{\varepsilon}\right)\phi'\left(\frac{-x + X(t)}{\varepsilon}\right) \\
&\cdot \eta(U \mid C_L) \cdot G(t, x)\, dx,
\end{aligned} \tag{31}$$

where $G(t, x) := [f(U(t, X(t)), (C_L + C_R)/2) - f(U(t, x), C_L)]$. Using Lemma 5 we find

$$\begin{aligned}
U(t, X(t)) - U(t, x) &= \int_x^{X(t)} (\partial_x U)(t, y)\, dy \\
&\le \int_x^{X(t)} (\partial_x U)_+(t, y)\, dy \\
&\le \sqrt{|X(t) - x|}\left\|(\partial_x U)_+\right\|_{L^2(\mathbb{R})} \\
&\le C\sqrt{|X(t) - x|}\left\|\left(\frac{d}{dx}U_0\right)_+\right\|_{L^2(\mathbb{R})}.
\end{aligned} \tag{32}$$

We next observe that $G(t, x)$, $t \in (0, T)$, $x \in \mathbb{R}$, is strictly negative. To do this, we rewrite the function G as

$$\begin{aligned}
G(t, x) = &f\left(U(t, X(t)), \frac{C_L + C_R}{2}\right) - f\left(U(t, x), \frac{C_L + C_R}{2}\right) \\
&+ f\left(U(t, x), \frac{C_L + C_R}{2}\right) - f(U(t, x), C_L).
\end{aligned} \tag{33}$$

For $x \in [X(t) - \delta\epsilon, X(t)]$, Lemma 2 and the inequality (32) imply that

$$\begin{aligned}
G(t, x) \le &f\left(U(t, x) + C\left\|\left(\frac{d}{dx}U_0\right)_+\right\|_{L^2(\mathbb{R})}\sqrt{\epsilon\delta}, \frac{C_L + C_R}{2}\right) \\
&- f\left(U(t, x), \frac{C_L + C_R}{2}\right) + f\left(U(t, x), \frac{C_L + C_R}{2}\right) \\
&- f(U(t, x), C_L) \\
\le &\Lambda C\left\|\left(\frac{d}{dx}U_0\right)_+\right\|_{L^2(\mathbb{R})}\sqrt{\epsilon\delta} - \frac{C_L - C_R}{2\Lambda} \\
\le &-\frac{\theta}{2} < 0,
\end{aligned} \tag{34}$$

for $\sqrt{\epsilon\delta} > 0$ small enough. Since $\phi(\cdot), \phi'(\cdot)$, and $\eta(\cdot \mid \cdot) \ge 0$, we get

$$\begin{aligned}
L_1 \le -\theta &\int_{X(t)-\delta\varepsilon}^{X(t)} \frac{1}{\varepsilon}\phi\left(\frac{-x + X(t)}{\varepsilon}\right) \\
&\times \phi'\left(\frac{-x + X(t)}{\varepsilon}\right)\eta(U \mid C_L)\, dx.
\end{aligned} \tag{35}$$

Similarly, we also obtain that

$$\begin{aligned}
R_1 \le -\theta &\int_{X(t)}^{X(t)+\delta\varepsilon} \frac{1}{\varepsilon}\phi\left(\frac{x - X(t)}{\varepsilon}\right) \\
&\times \phi'\left(\frac{x - X(t)}{\varepsilon}\right)\eta(U \mid C_R)\, dx.
\end{aligned} \tag{36}$$

Consequently, combining the two last inequalities gives the desired result. □

We are now going to introduce the parabolic term, L_2+R_2, and the proof is provided in [20] (see Appendix).

Proposition 7. *Let L_2, R_2 be given in Lemma 4. Then, there exists a constant $C > 0$ such that the following inequality holds:*

$$L_2 + R_2 \le \frac{C}{\varepsilon} \int_{X(t)-\delta\varepsilon}^{X(t)+\delta\varepsilon} \left[\phi'\left(\frac{|x - X(t)|}{\varepsilon}\right)\right]^2 dx. \tag{37}$$

4. Proof of Theorem 1

From Lemma 4, Proposition 6, and Proposition 7, we get

$$\begin{aligned} \frac{d}{dt}\mathscr{E}_\epsilon^\delta(t) \le{}& \frac{C}{\varepsilon} \int_{X(t)-\delta\varepsilon}^{X(t)+\delta\varepsilon} \left[\phi'\left(\frac{|x - X(t)|}{\varepsilon}\right)\right]^2 \chi_{\{\phi' - C^* \phi > 0\}} dx \\ &+ L_3 + R_3. \end{aligned} \tag{38}$$

Applying the change of variables $z = (x - X(t))/\varepsilon$ and changing θ by $\inf(\theta, C^*)$ if necessary, we find

$$\begin{aligned} \frac{d}{dt}\mathscr{E}_\epsilon^\delta(t) \le{}& \frac{C}{\varepsilon} \int_{X(t)-\delta\varepsilon}^{X(t)+\delta\varepsilon} \left[\left(\phi'\right)^2 \chi_{\{\phi' - \theta\phi > 0\}}\right] \left(\frac{|x - X(t)|}{\varepsilon}\right) dx \\ &+ L_3 + R_3 \\ \le{}& C \int_0^\delta \left(\phi'\right)^2 (z)\, \chi_{\{\phi' - \theta\phi > 0\}}(z)\, dz + L_3 + R_3. \end{aligned} \tag{39}$$

To get good estimate, we take a specific ϕ_δ. For any $\delta \ge 1/\theta$, we now fix the function ϕ_δ in the following explicit way:

$$\phi_\delta(x) = \begin{cases} \theta e^{1-\theta\delta} x, & \text{for } x \in \left[0, \dfrac{1}{\theta}\right), \\ e^{\theta(x-\delta)}, & \text{for } x \in \left[\dfrac{1}{\theta}, \delta\right]. \end{cases} \tag{40}$$

We use the computation:

$$\int_0^\delta \left(\phi_\delta'(x)\right)^2 \chi_{\{\phi_\delta' > \theta\phi_\delta\}}\, dx = C_\theta \cdot e^{-2\theta\delta}. \tag{41}$$

For the proof of (I), we integrate the estimate of Proposition 7 between 0 and $t \in (0, T)$ such that, for any ε, δ with $1/\theta \le \delta$ and $\varepsilon\delta \le \theta$, where θ is the constant from Proposition 7, it follows that

$$\frac{d}{dt}\mathscr{E}_\epsilon^\delta(t) \le C e^{-\theta\delta} + L_3 + R_3, \tag{42}$$

which implies that

$$\begin{aligned} &\int_{\{|x - X(t)| \ge \delta\varepsilon\}} \eta(U(t,x) \mid S(t,x))\, dx \\ &\quad\le \mathscr{E}_\epsilon^\delta(0) + \int_0^t \frac{d}{dt}\mathscr{E}_\epsilon^\delta(t)\, ds \\ &\quad\le \int_{\mathbb{R}} \eta(U_0 \mid S_0)\, dx + \int_0^t L_3 + R_3 ds + CTe^{-\theta\delta}. \end{aligned} \tag{43}$$

By taking $\varepsilon_0 := \theta^2$, we have, for any $\varepsilon \le \beta \le \varepsilon_0$,

$$\begin{aligned} &\int_{\{|x - X(t)| \ge \beta/\theta\}} \eta(U(t,x) \mid S(t,x))\, dx \\ &\quad\le \int_{\mathbb{R}} \eta(U_0 \mid S_0)\, dx + \int_0^t L_3 + R_3 ds + CTe^{-\beta/\varepsilon}. \end{aligned} \tag{44}$$

Let us observe that

$$\begin{aligned} \int_{\mathbb{R}} \eta(U \mid S)\, dx ={}& \int_{\{|x - X(t)| \ge C\beta\}} \eta(U \mid S)\, dx \\ &+ \int_{\{|x - X(t)| < C\beta\}} \eta(U \mid S)\, dx, \end{aligned} \tag{45}$$

$$\begin{aligned} \int_0^t L_3 + R_3 ds \le{}& \int_0^t \int_{\mathbb{R}} \left|g(U(t,x)) - g(S(t,x))\right| \\ &\times \left|\eta'(U(t,x)) - \eta'(S(t,x))\right| dx\, ds \\ \le{}& C \int_0^t \int_{\mathbb{R}} (U(t,x) - S(t,x))^2 dx\, ds \\ \le{}& C \int_0^t \int_{\mathbb{R}} \eta(U(t,x) \mid S(t,x))\, dx\, ds, \end{aligned} \tag{46}$$

where we have here used the mean value theorem and the definition of source term (3). Consequently, using inequlaities (22), (44), and (45) and taking $\beta = \varepsilon \log(1/\varepsilon)$, we get, for any $t \in (0, T)$,

$$\begin{aligned} \int_{\mathbb{R}} \eta(U \mid S)\, dx \le{}& \int_{\mathbb{R}} \eta(U_0 \mid S_0)\, dx + \int_0^t \int_{\mathbb{R}} \eta(U \mid S)\, dx\, ds \\ &+ C\varepsilon \log\left(\frac{1}{\varepsilon}\right), \end{aligned} \tag{47}$$

for any $\varepsilon \le \varepsilon_0$, which proves (6) by taking $\eta(v) = v^2$ and applying Gronwall's inequlity.

To end with the proof, it remains to prove (9). Let us define the function ψ by

$$\psi(x) := \begin{cases} 0 & \text{if } |x| > 2, \\ 1 & \text{if } |x| \le 1 \\ 2 - |x| & \text{if } 1 < |x| \le 2. \end{cases} \tag{48}$$

Let $s \in (0,t)$ and let $R > 0$. Multiplying $\Psi_R(s,x) := \psi((x - X(s))/R)$ to (1) and integrating in x, we get

$$\begin{aligned}
0 = & -\frac{d}{ds}\int \Psi_R \cdot U dx + \int \partial_x(\Psi_R) A(U)\, dx \\
& + \int \partial_t(\Psi_R) U dx + \int \Psi_R \left(\epsilon \partial_{xx}^2 U + g(U)\right) dx \\
= & \underbrace{-\frac{d}{ds}\int \psi\left(\frac{x - X(s)}{R}\right) \cdot U(s,x)\, dx}_{(I)} \\
& + \underbrace{\frac{1}{R}\int \psi'\left(\frac{x - X(s)}{R}\right) \cdot \left(A(U(s,x)) - \dot{X}(s) U(s,x)\right) dx}_{(II)} \\
& \underbrace{-\frac{\epsilon}{R}\int \psi'\left(\frac{x - X(s)}{R}\right) \partial_x U(s,x)\, dx}_{(III)} \\
& + \underbrace{\int \psi\left(\frac{x - X(s)}{R}\right) g(U) dx}_{(IV)}.
\end{aligned} \tag{49}$$

By using the above observation, we have

$$\begin{aligned}
\left(\sigma - \dot{X}(s)\right) &= \frac{1}{C_L - C_R}\left(A(C_L) - A(C_R) - (C_L - C_R)\dot{X}(s)\right) \\
&= \frac{1}{C_L - C_R}\left(\mathrm{A}(C_L) - A(C_R) - (C_L - C_R)\dot{X}(s)\right. \\
&\qquad \left. - (II) + (I) + (III) - (IV)\right).
\end{aligned} \tag{50}$$

Then we integrate the above equation in time on $[0,t]$ to get

$$\begin{aligned}
& |\sigma t - X(t)| \\
& \leq C\left(t \cdot \max_{s\in(0,t)} \underbrace{\left|A(C_L) - A(C_R) - (C_L - C_R)\dot{X}(s) - (II)\right|}_{(II')}\right. \\
& \qquad \left. + \left|\int_0^t (I)\, ds\right| + t \cdot \max_{s\in(0,t)} |(III)| + t \cdot \max_{s\in(0,t)} |(IV)|\right).
\end{aligned} \tag{51}$$

From the result of Choi and Vasseur [20], we already know the following results:

$$\left(II'\right)^2 \leq \frac{C}{R} \cdot \int_{\mathbb{R}} \eta(U(s) \mid S(s))\, dx, \qquad |(III)| \leq \frac{C\epsilon}{R}, \tag{52}$$

$$\begin{aligned}
& \left|\int_0^t (I)\, ds\right|^2 \\
& \qquad \leq CR\left(\int_{\mathbb{R}} \eta(U(t) \mid S(t))\, dx + \int_{\mathbb{R}} \eta(U_0 \mid S_0)\, dx\right).
\end{aligned} \tag{53}$$

We now estimate (IV). This directly follows from the definition of source term (3) and Holder's inequality:

$$\begin{aligned}
|(IV)| \leq & \int_{-2R+X(s)}^{X(s)} \left|g(U) - g(C_L)\right| dx \\
& + \int_{X(s)}^{2R+X(s)} \left|g(U) - g(C_R)\right| dx \\
\leq & C\sqrt{R}\|U - S\|_{L^2(\mathbb{R})} \\
\leq & C\sqrt{R}\left\|U_0 - S_0\right\|_{L^2(\mathbb{R})}.
\end{aligned} \tag{54}$$

Finally, by using (54), we combine (52) and (53) together with (51) to get, for any $t \in (0,T)$,

$$\begin{aligned}
|\sigma t - X(t)|^2 \leq & C\left(\frac{t^2}{R^2} + R + t^2 R\right) \\
& \cdot \left(\int_{\mathbb{R}} \left|U_0 - S_0\right|^2 dx + \epsilon \log\frac{1}{\epsilon}\right).
\end{aligned} \tag{55}$$

Consequently, taking $R = t^{1/2}$ provides the estimate (9).

Appendix

In this section we are going to give the proof of Proposition 7. First, we estimate the term L_2. Integrating by parts, we obtain

$$\begin{aligned}
L_2 = & \int_{-\infty}^{X(t)} 2\phi\left(\frac{-x + X(t)}{\varepsilon}\right)\phi'\left(\frac{-x + X(t)}{\varepsilon}\right) \\
& \times \partial_x U\left(\eta'(U) - \eta'(C_L)\right) dx \\
& - 2\varepsilon\int_{-\infty}^{X(t)} \left[\phi\left(\frac{-x + X(t)}{\varepsilon}\right)\right]^2 \eta''(U)\left|\partial_x U\right|^2 dx.
\end{aligned} \tag{A.1}$$

Then, using Hölder's inequality and Lemma 2, we get

$$\begin{aligned}
L_2 \leq & \frac{2\varepsilon}{\Lambda}\int_{-\infty}^{X(t)} \left[\phi\left(\frac{-x + X(t)}{\varepsilon}\right)\right]^2 \left|\partial_x U\right|^2 dx \\
& + \frac{\Lambda}{8\varepsilon}\int_{\infty}^{X(t)} \left[2\phi'\left(\frac{-x + X(t)}{\varepsilon}\right)\left(\eta'(U) - \eta'(C_L)\right)\right]^2 dx \\
& - \frac{2\varepsilon}{\Lambda}\int_{-\infty}^{X(t)} \left[\phi\left(\frac{-x + X(t)}{\varepsilon}\right)\right]^2 \left|\partial_x U\right|^2 dx \\
\leq & \frac{C}{\varepsilon}\int_{X(t)-\delta\varepsilon}^{X(t)} \left[\phi'\left(\frac{-x + X(t)}{\varepsilon}\right)\right]^2 \left|U - C_L\right|^2\, dx.
\end{aligned} \tag{A.2}$$

In the same way, we obtain the following estimate for $(R)_2$:

$$R_2 \leq \frac{C}{\varepsilon}\int_{X(t)}^{X(t)+\delta\varepsilon} \left[\phi'\left(\frac{x - X(t)}{\varepsilon}\right)\right]^2 \left|U - C_R\right|^2\, dx. \tag{A.3}$$

Combining the two last inequalities, we find

$$\begin{aligned}
& L_2 + R_2 \\
& \leq \frac{C}{\varepsilon}\int_{X(t)-\delta\varepsilon}^{X(t)+\delta\varepsilon} \left[\phi'\left(\frac{|x - X(t)|}{\varepsilon}\right)\right]^2 |U(t,x) - S(t,x)|^2\, dx,
\end{aligned} \tag{A.4}$$

which provides the proof of Proposition 7 thanks to the boundedness of U and S.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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