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Research Article

Exterior Dirichlet Problem for Translating Solutions of Gauss Curvature Flow in Minkowski Space

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We prove the existence of solutions to a class of Monge-Ampère equations on exterior domains in $\mathbb{R}^n$ $(n \geq 2)$ and the solutions are close to a cone. This problem comes from the study of the flow by powers of Gauss curvature in Minkowski space.

1. Introduction and Main Results

The Euclidean space $\mathbb{R}^{n+1}$ endowed with the Lorentz metric $ds^2 = \sum_{i=1}^n dx_i^2 - dx_{n+1}^2$ is called Minkowski space. We denote it by $\mathbb{R}^{n,1}$. A space-like hypersurface in $\mathbb{R}^{n,1}$ is a Riemanian n-manifold, having an everywhere lightlike normal field ν which we assume to be future directed and thus satisfy the condition $\langle \nu, \nu \rangle = -1$. Locally, such surfaces can be expressed as graphs of functions $x_{n+1} = u(x_1, \ldots, x_n) : \mathbb{R}^n \to \mathbb{R}$ satisfying the space-like condition $|Du(x)| < 1$ for all $x \in \mathbb{R}^n$.

If a family of space-like hypersurfaces $X_t = X(\cdot, t) : \mathbb{R}^n \to \mathbb{R}^{n,1}$ satisfies the evolution equation

$$\frac{\partial X}{\partial t} = -K^{\alpha} \nu \tag{1}$$

on some time interval, we say that the surfaces $M_t := X_t(M)$ are evolved by K^{α}-flow, where $K(\cdot, t)$ is the Gauss curvature of M_t and $\alpha \neq 0$ is a constant. When the initial surface is a graph over a domain $\Omega \subset \mathbb{R}^n$, (1) is equivalent, up to a diffeomorphism in $\mathbb{R}^n$, to

$$\frac{\partial V}{\partial t} = \sqrt{1 - |DV|^2} \left[\frac{\det(D^2 V)}{\left(1 - |DV|^2\right)^{(n+2)/2}} \right]^{\alpha} \tag{2}$$

with $|DV(\cdot, t)| < 1$, where V is a function defined in $\Omega \times [0, T)$.

The flow (2) was studied in [1] for the special case $\alpha = 1$. In fact, the authors in [1] used the flow (2) to prove existence and stability of smooth entire strictly convex space-like hypersurfaces of prescribed Gauss curvature and give a new proof of Theorem 3.5 in [2].

A function $u = u(x)$ is called a translating solution to the K^{α}-flow if the function $V(x, t) = u(x) + t$ solves (2). Equivalently, $u(x)$ is an initial hypersurface satisfying

$$\det\left(D^2 u\right) = \left(1 - |Du|^2\right)^{((n+2-(1/\alpha))/2)}. \tag{3}$$

The space-like condition reads as

$$|Du(x)| < 1. \tag{4}$$

The space-like hypersurfaces evolved by mean curvature flow in Minkowski space were studied in [3–6]. The translating solutions were introduced in [3, 4] and studied in [7, 8].

In this paper, we consider strictly convex space-like hypersurfaces of translating solutions to K^{α}-flow as graphs over $\mathbb{R}^n \setminus D$, where $D \subset \mathbb{R}^n$ is an open domain whose boundary ∂D is a smooth submanifold of $\mathbb{R}^n$. We want to look for a function $u \in C^{\infty}(\mathbb{R}^n \setminus D)$, which solves the problem (3)-(4) with the boundary condition

$$u = \phi \quad \text{on } \partial D, \tag{5}$$

where $\phi \in C^{\infty}(\partial D)$ is a given function.

There are similar problems for the equation of translating solution of Gauss curvature flow in Eucliden space [9], the equation of prescribed Gauss curvature in Eucliden space [10], and the equation of prescribed Gauss curvature in

Minkowski space [11], respectively. It was shown that there are convex solutions to the Dirichlet problems for the three equations on exterior domains, and the solution is close to the rotationally symmetric one at infinity for the first equation and close to a cone for the second and third equation under the assumption that there exists a strictly convex subsolution which is close to a cone up to the third order (see (7) and (8)).

In this paper, we will show that the same results as in [10, 11] hold for the problem (3)–(5). We would like to point out that (3) is essentially different from the equations in [9–11]. For example, the equation of prescribed Gauss curvature in Minkowski space, $\det(D^2u) = (1-|Du|^2)^{(n+2)/2}$, has an explicit solution $u = \sqrt{1+|x|^2}$, from which one can easily construct subsolution or supersolution for given Dirichlet problems. However, it is unknown if there is such a solution to (3). In particular, it has no solution in the form of $u = (1+|x|^2)^{\gamma}$.

Definition 1. A function $\underline{u} \in C^{\infty}(\mathbb{R}^n \setminus D)$ is called a subsolution of (3)–(5), if $\underline{u}$ is strictly convex and satisfies

$$\det\left(D^2\underline{u}\right) \ge \left(1-\left|D\underline{u}\right|^2\right)^{(n+2-\beta)/2}, \quad \text{in } \mathbb{R}^n \setminus \overline{D},$$
$$\left|D\underline{u}\right| < 1, \quad \text{in } \mathbb{R}^n \setminus \overline{D}, \tag{6}$$
$$\underline{u} = \phi, \quad \text{on } \partial D.$$

Here and below, we set $\beta = 1/\alpha$.

The main result of this paper is the following theorem.

Theorem 2. *Let $D \subset \mathbb{R}^n$ $(n \ge 2)$ be an open set whose boundary ∂D is a smooth submanifold of $\mathbb{R}^n$ and $\phi \in C^{\infty}(\partial D)$. Suppose that $\beta < (3/2) - 2n$ and $\underline{u} \in C^{\infty}(\mathbb{R}^n \setminus D)$ is a subsolution of* (3)–(5) *which is close to a cone, that is,*

$$\sup_{\mathbb{R}^n\setminus D} \left|\underline{u} - |x|\right| < \infty \tag{7}$$

and satisfies the following decay conditions at infinity:

$$\left|D\left(\underline{u}-|x|\right)\right| = O\left(\frac{1}{|x|}\right),$$
$$\left|D^2\left(\underline{u}-|x|\right)\right| + \left|D^3\underline{u}\right| = O\left(\frac{1}{|x|^2}\right). \tag{8}$$

Then there exists a smooth, strictly convex hypersurface of the exterior Dirichlet problem (3)–(5) *and the solution u is close to a cone in the sense that*

$$\sup_{\mathbb{R}^n\setminus D} \left|u - |x|\right| < \infty. \tag{9}$$

Although the above theorem has an obvious disadvantage that it assumes the existence of a locally strictly convex subsolution, this assumption is reasonable and necessary in some case for the Dirichlet problems on nonconvex domains; see [12] for the details. However, in the special case when $D = B_{\rho_0}(0)$ is a ball and the boundary values are zero, we can construct an explicit subsolution.

Theorem 3. *Let $D = B_{\rho_0}(0)$ with $\rho_0 > 0$ and $\phi \equiv 0$. If $\beta \le 0$, then there is a strictly convex subsolution $\underline{u}$ of* (3)–(5) *such that* (7) *and* (8) *are satisfied.*

We consider the local problem

$$\det\left(D^2u^R\right) = \left(1-\left|Du^R\right|^2\right)^{(n+2-\beta)/2}, \quad \text{in } B_R \setminus \overline{D},$$
$$\sup_{B_R\setminus\overline{D}} \left|Du^R\right| < 1, \tag{10}$$
$$u^R = \underline{u}, \quad \text{on } \partial D \cup \partial B_R,$$

where $R > 4R_0$ and $D \subset B_{R_0}$ for some constant $R_0 > 1$. It is well known from the standard continuity method as in [13] that the Dirichlet problem (10) has a locally strict convex solution in $C^{\infty}(\overline{B}_R \setminus D)$. Our main task is to show that the C^2-norms of u^R are uniformly bounded in R. Once this is established, by the standard Krylov/Shafanov theory, Schauder regularity theory, and a diagonal sequence argument, we can obtain a smooth locally strictly convex solution u to (3)–(5) on exterior domain $\mathbb{R}^n \setminus D$.

The paper is organized as follows. In Section 2, we prove the C^0 and C^1 a priori estimates for u^R. The C^2-estimates are given in Section 3. Finally, we prove Theorem 3 in the last section.

2. C^0 and C^1 A Priori Estimates

From now on, we assume D and $\underline{u}$ as in Theorem 2 and u^R as in (10); lower indices denote partial derivatives in $\mathbb{R}^n$, for example, $u_i = \partial u/\partial x_i$. The inverse of the Hessian of u is denoted by $(u^{ij}) = (u_{ij})^{-1}$. We use the Einstein summation convention. The letter c denotes a constant independent of R which may change its value from line to line throughout the text.

Without loss of generality we can assume that $0 \in D$. It is easy to check that $\overline{u} = \sqrt{1+|x|^2} + L$ is a supersolution to (3) for $\alpha < 0$, where the constant $L > \max_{\partial D}(\phi - \sqrt{1+|x|^2})$.

Owing to the maximum principle, we can obtain the following lemma as Lemma 2.2 in [10].

Lemma 4. *The functions u^R converge locally uniformly to a continuous function u as $R \to \infty$. Moreover, $\underline{u} \le u \le \overline{u}$ in $B_R \setminus D$.*

Proof. From the maximum principle we obtain that

$$\underline{u} \le u^R \le \overline{u} \quad \text{in } B_R \setminus D \tag{11}$$

for any $R > 4R_0$ and

$$u^{R_1} = \underline{u} \le u^{R_2} \quad \text{on } \partial B_{R_1} \cup \partial D \tag{12}$$

for $4R_0 < R_1 < R_2$. Again by the maximum principle, we have

$$u^{R_1} \le u^{R_2} \quad \text{in } B_{R_1} \setminus D. \tag{13}$$

We conclude that u^R are monotone in R and converge locally uniformly to a continuous function u according to Dini's theorem. □

To simplify the notation, we will omit the index R and from now on assume that u is a solution of (10) with R fixed sufficiently large. The estimate for the first derivatives is stated in the following lemma.

Lemma 5. *For $R/2 \leq |x| \leq R$, there is a constant c independent of R such that*

$$\left|\nabla_\nu (u - \underline{u})(x)\right| \leq \frac{c}{R}, \tag{14}$$

$$\left|\nabla_\tau u(x)\right| \leq \frac{c}{\sqrt{R}}, \tag{15}$$

$$\sqrt{1 - \frac{c}{R}} \leq |Du(x)| < 1, \tag{16}$$

where $\nu = x/|x|$ and τ are unit vectors parallel and orthogonal to x, respectively.

Proof. From the convexity of u and Lemma 4, we can prove (14) and (15) by using the similar proof techniques of (2.2) and (2.3) in [10]. Then, we need only to prove (16). Since u is strictly convex, for $R/2 \leq |x| \leq R$, $|Du|$ attains its maximum at ∂B_R. In view of (8), we may take

$$\left|D\underline{u}\right|^2 = O\left(1 - \frac{c}{R}\right). \tag{17}$$

Hence for $x \in \partial B_R$, by (14) and (17) we have

$$\begin{aligned} |Du(x)|^2 &= \left|\nabla_\tau u(x)\right|^2 + \left|\nabla_\nu u(x)\right|^2 \\ &= \left|\nabla_\tau \underline{u}(x)\right|^2 + \left|\nabla_\nu \underline{u}(x) + \nabla_\nu (u - \underline{u})(x)\right|^2 \\ &\geq \left|D\underline{u}(x)\right|^2 - 2\left|\nabla_\nu (u - \underline{u})(x)\right| \geq 1 - \frac{c}{R}. \end{aligned} \tag{18}$$

On the other hand, by the proof of Theorem 4.1 in [12],

$$\max_{\partial B_R} |Du| \leq \max_{\partial B_R} \left|D\underline{u}\right| < 1. \tag{19}$$

The lemma is completed. □

3. C^2 A Priori Estimates

In this section, we prove the C^2 a priori estimates for solutions of (10) under the assumption of Theorem 2. As in [12], one obtains that the second derivatives on ∂D are bounded uniformly in R. Furthermore, by considering the function

$$w = \frac{a}{2}|Du|^2 + \log u_{\xi\xi} \tag{20}$$

for some constant $a > 0$ and assuming its maximum over $(x, \xi) \in \overline{B_R \setminus D} \times S^{n-1}$ is attained at an interior, one can prove that

$$\max_{B_R \setminus \overline{D}} |Du|^2 \leq c + \max_{\partial B_R \cup \partial D} \left|D^2 u\right|. \tag{21}$$

Therefore, it suffices to bound $|D^2 u|$ on the outer boundary ∂B_R.

Next, we will give estimates for the tangential second derivatives, the mixed second derivatives, and the normal second derivatives on the outer boundary ∂B_R, respectively.

Theorem 6 (tangential second derivatives at the outer boundary). *Let $x_0 \in \partial B_R$ and τ_1, τ_2 be tangential directions at x_0. Then we have at x_0,*

$$\left|u_{\tau_1\tau_2} - |x|_{\tau_1\tau_2}\right| \leq \frac{c}{R^2}. \tag{22}$$

Proof. We may assume that $x_0 = R \cdot e_n \equiv R \cdot (0, \ldots, 0, 1)$. Then ∂B_R is represented locally as graph of ω, where

$$\omega(\hat{x}) = \sqrt{R^2 - |\hat{x}|^2}, \quad \hat{x} = (x_1, x_2, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}. \tag{23}$$

Note that the Dirichlet boundary condition implies

$$(u - \underline{u})(\hat{x}, \omega(\hat{x})) = 0. \tag{24}$$

We differentiate twice with respect to $\hat{x}_i$, $\hat{x}_j$, $1 \leq i, j \leq n-1$ to obtain that, at x_0,

$$(u - \underline{u})_{ij} + (u - \underline{u})_n \omega_{ij} + 2(u - \underline{u})_{nj}\omega_i = 0. \tag{25}$$

According to the decay conditions at infinity (8), we have $|\underline{u}_{ij} - |x|_{ij}| = O(1/R^2)$. Observing that

$$w_i(x_0) = 0, \qquad \omega_{ij}(x_0) = -\frac{\delta_{ij}}{R}. \tag{26}$$

Then, by Lemma 5 we have

$$\left|u_{ij} - |x|_{ij}\right| = \left|-(u - \underline{u})_n \omega_{ij} + (\underline{u} - |x|)_{ij}\right| \leq \frac{c}{R^2}. \tag{27}$$

□

Theorem 7 (mixed second derivatives at the outer boundary). *For $x_0 \in \partial B_R$, let τ, ν be unit vectors in tangential and normal directions, respectively. Then*

$$\left|u_{\tau\nu}(x_0)\right| \leq \frac{c}{\sqrt{R}}. \tag{28}$$

The proof is going to be put in three lemmas and will be finished below Lemma 10. Similar to Theorem 6, we may assume that $x_0 = R \cdot e_n$ and represent ∂B_R locally as graph of ω with $\omega(\hat{x}) = \sqrt{R^2 - |\hat{x}|^2}$. We take the logarithm of (3),

$$\log \det u_{ij} - \frac{n + 2 - \beta}{2} \log\left(1 - |Du|^2\right) = 0, \tag{29}$$

and differentiate with respect to x_k,

$$u^{ij} u_{ijk} + \frac{n + 2 - \beta}{1 - |Du|^2} u_i u_{ik} = 0, \tag{30}$$

where $(u^{ij}) = (u_{ij})^{-1}$. We introduce the linear differential operator L by

$$Lw := u^{ij} w_{ij} + \frac{n+2-\beta}{1-|Du|^2} u_i w_i \tag{31}$$

and define the linear operator for $t < n$:

$$T := \frac{\partial}{\partial x_t} + \sum_{\gamma=1}^{n-1} \omega_{\text{tr}}(0)\, x_\gamma \frac{\partial}{\partial x_n} \equiv \frac{\partial}{\partial x_t} - \frac{x_t}{R}\frac{\partial}{\partial x_n}. \tag{32}$$

In the following we restrict attention to the domain $\Omega_\delta := B_\delta(x_0) \cap B_R$ with $x_0 \in \partial B_R$ and $\delta \le R/2$. Notice that $\Omega_\delta \subset B_R \setminus \overline{D}$.

Lemma 8. *The function $u - \underline{u}$ satisfies the following estimates:*

$$|T(u-\underline{u})| \le \frac{c}{\sqrt{R}} \quad \text{in } \Omega_\delta, \tag{33}$$

$$|T(u-\underline{u})| \le \frac{c}{R^2}|x-x_0|^2 \quad \text{on } \partial B_R, \tag{34}$$

$$|LT(u-\underline{u})| \le cR + \frac{c}{R^2}\operatorname{tr}\left(u^{ij}\right) \quad \text{in } \Omega_\delta, \tag{35}$$

where $\operatorname{tr}(u^{ij}) \equiv \Sigma_{i=1}^n u^{ii}$.

Proof. For (33), by the assumption (8) and C^1 estimates of Lemma 5, we get

$$\begin{aligned}
|T(u-\underline{u})| &\le |(u-\underline{u})_t| + |(u-\underline{u})_n| \\
&\le 2|D(u-\underline{u})| \\
&\le 2|(u-\underline{u})_\nu| + 2|(u-\underline{u})_\tau| \\
&\le 2|(u-|x|)_\tau| + 2|(\underline{u}-|x|)_\tau| + \frac{c}{R} \\
&\le 2|u_\tau| + \frac{c}{R} \le \frac{c}{\sqrt{R}},
\end{aligned} \tag{36}$$

where $\nu = x/|x|$ and τ is unit vector orthogonal to x.

For the second inequality (34) we use that $(u-\underline{u})_t + (u-\underline{u})_n\omega_t = 0$ and note that $\omega_t(0) = 0$, $|\omega_i| \le c$, $|\omega_{ij}| \le c/R$, $|\omega_{ijk}| \le c/R^2$. Then for $x \in \partial B_R$,

$$\begin{aligned}
T(u-\underline{u}) &= -\left(\omega_t - \sum_{\gamma=1}^{n-1}\omega_{t\gamma}(0)\, x_\gamma\right)(u-\underline{u})_n \\
&= -\sum_{\gamma,s=1}^{n-1} x_\gamma \omega_{t\gamma s}(\theta\hat{x})\, x_s (u-\underline{u})_n
\end{aligned} \tag{37}$$

with $0 < \theta < 1$, which implies (34).

To prove (35), by Lemma 5, we may take $1/(1-|Du|^2) = O(R)$. In view of (8), (30), and $u^{ij}u_{jk} = \delta_{ik}$, we obtain

$$\begin{aligned}
&|LT(u-\underline{u})| \\
&\quad = \left|L\left((u-\underline{u})_t - \frac{x_t}{R}(u-\underline{u})_n\right)\right| \\
&\quad = \left|u^{ij}\left[(u-\underline{u})_{tij} - \left(\frac{x_t}{R}(u-\underline{u})_n\right)_{ij}\right]\right. \\
&\qquad \left. + \frac{n+2-\beta}{1-|Du|^2}u_i\left[(u-\underline{u})_{ti} - \left(\frac{x_t}{R}(u-\underline{u})_n\right)_i\right]\right| \\
&\quad \le \left|u^{ij}u_{tij} + \frac{n+2-\beta}{1-|Du|^2}u_i u_{ti}\right| \\
&\qquad + \left|\frac{x_t}{R}\left(u^{ij}u_{nij} + \frac{n+2-\beta}{1-|Du|^2}u_i u_{ni}\right)\right| \\
&\qquad + \left|u^{ij}\left(\underline{u}_{tij} - \frac{x_t}{R}\underline{u}_{nij}\right)\right| + \frac{n+2-\beta}{1-|Du|^2}\left|u_i\left(\underline{u}_{ti} + \frac{x^t}{R}\underline{u}_{ni}\right)\right| \\
&\qquad + \left|u^{ij}\left(\frac{\delta_{it}}{R}u_{nj} + \frac{\delta_{jt}}{R}u_{ni}\right)\right| + \left|u^{ij}\left(\frac{\delta_{it}}{R}\underline{u}_{nj} + \frac{\delta_{jt}}{R}\underline{u}_{ni}\right)\right| \\
&\qquad + \frac{n+2-\beta}{1-|Du|^2}\left|u_i \cdot \frac{\delta_{it}}{R}(u-\underline{u})_n\right| \\
&\quad \le c\cdot\operatorname{tr}\left(u^{ij}\right)\cdot\left(\left|D^3\underline{u}\right| + \frac{1}{R}\left|D^2\underline{u}\right|\right) \\
&\qquad + c\cdot\frac{n+2-\beta}{1-|D^2\underline{u}|^2}\left[\left|D^2\underline{u}\right| + \frac{1}{R^2}\right] + c \le \frac{c}{R^2}\operatorname{tr}\left(u^{ij}\right) + c.
\end{aligned} \tag{38}$$

□

In the next lemma, we introduce a function $\mathscr{V}$, which will be the main part of a barrier function to prove Theorem 7.

Lemma 9. *There exists a positive constant ε independent of R such that*

$$\mathscr{V} := (u-\underline{u}) + \frac{1}{\sqrt{R}}d - \frac{1}{2R^{5/4}}d^2 \tag{39}$$

fulfills the estimates

$$\begin{aligned}
L\mathscr{V} &\le -\varepsilon R^{-(1/n)(\beta-(7/4))} - \varepsilon R^{-(5/4)}\operatorname{tr}\left(u^{ij}\right), \quad \text{in } \Omega_\delta, \\
\mathscr{V} &\ge 0, \quad \text{on } \partial\Omega_\delta
\end{aligned} \tag{40}$$

provided that $\delta = R^{3/4}$ and R is sufficiently large. Here $d = R - |x|$ is the distance from ∂B_R.

Proof. In view of $\delta = R^{3/4}$, for $x \in \partial\Omega_\delta$, $d = R - |x| \le \delta = R^{3/4}$, and $u \ge \underline{u}$, we have

$$\mathscr{V} = (u-\underline{u}) + \frac{1}{\sqrt{R}}d - \frac{1}{2R^{5/4}}d^2 \ge \frac{1}{\sqrt{R}}d - \frac{1}{2R^{5/4}}d^2 \ge 0 \quad \text{on } \partial\Omega_\delta. \tag{41}$$

We fix $x \in \Omega_\delta$ and set $\nu = x/|x|$. Let τ, τ' belong to an orthonormal basis for the orthogonal complement of ν which we choose such that the submatrix $(u^{\tau\tau'})$ is diagonal. Assume that ν and τ, τ' correspond to the indices n and $1, \dots, n-1$, respectively. We use the Einstein summation convention for τ, τ'. The matrix u^{ij} is positive, and thus testing with the vectors $\nu \pm \tau$ gives

$$\left|u^{\nu\tau}\right| \le \frac{1}{2}\left(u^{\nu\nu} + u^{\tau\tau}\right). \tag{42}$$

In view of

$$\begin{aligned} u^{\nu\nu} &= u^{ij}\frac{x_i}{|x|}\frac{x_j}{|x|}, \\ \operatorname{tr}\left(u^{\tau\tau'}\right) &= \operatorname{tr}\left(u^{ij}\right) - u^{\nu\nu} = u^{ij}\left(\delta_{ij} - \frac{x_i}{|x|}\frac{x_j}{|x|}\right). \end{aligned} \tag{43}$$

Direct computations using (17) give

$$\begin{aligned} Lu &= u^{ij}u_{ij} + (n+2-\beta)\frac{|Du|^2}{1-|Du|^2} \\ &\le (n+2-\beta)\frac{|Du|^2}{1-|Du|^2} + c. \end{aligned} \tag{44}$$

By (8), (16), and (42) we have

$$\begin{aligned} L\underline{u} &= u^{ij}\underline{u}_{ij} + \frac{n+2-\beta}{1-|Du|^2}u_i\underline{u}_i \\ &= \frac{n+2-\beta}{1-|Du|^2}u_i\underline{u}_i + u^{ij}\left[|x|_{ij} + (\underline{u}-|x|)_{ij}\right] \\ &= \frac{n+2-\beta}{1-|Du|^2}u_i\underline{u}_i + \frac{1}{|x|}u^{ij}\left(\delta_{ij} - \frac{x_i}{|x|}\frac{x_j}{|x|}\right) + u^{ij}(\underline{u}-|x|)_{ij} \\ &= \frac{n+2-\beta}{1-|Du|^2}u_i\underline{u}_i + \frac{1}{|x|}\operatorname{tr}\left(u^{\tau\tau'}\right) + u^{\tau\tau'}(\underline{u}-|x|)_{\tau\tau'} \\ &\quad + 2u^{\tau\nu}(\underline{u}-|x|)_{\tau\nu} + u^{\nu\nu}(\underline{u}-|x|)_{\nu\nu} \\ &\ge \frac{n+2-\beta}{1-|Du|^2}u_i\underline{u}_i + \left(\frac{1}{|x|} - \frac{c}{|x|^2}\right)\operatorname{tr}\left(u^{\tau\tau'}\right) - \frac{c}{|x|^2}u^{\nu\nu}, \\ Ld &= u^{ij}(R-|x|)_{ij} + \frac{n+2-\beta}{1-|Du|^2}u_i(R-|x|)_i \\ &= -\frac{1}{|x|}\operatorname{tr}\left(u^{\tau\tau'}\right) - \frac{n+2-\beta}{1-|Du|^2}\frac{u_ix_i}{|x|}, \\ Ld^2 &= u^{ij}\left((R-|x|)^2\right)_{ij} + \frac{n+2-\beta}{1-|Du|^2}u_i\left((R-|x|)^2\right)_i \\ &\ge -\frac{2R}{|x|}\operatorname{tr}\left(u^{\tau\tau'}\right) + 2\operatorname{tr}\left(u^{ij}\right) \\ &\quad - \frac{n+2-\beta}{1-|Du|^2}u_i\left(\frac{2Rx_i}{|x|} - 2x_i\right) \\ &= -2\frac{d}{|x|}\operatorname{tr}\left(u^{\tau\tau'}\right) + 2u^{\nu\nu} - \frac{n+2-\beta}{1-|Du|^2}u_i\left(\frac{2Rx_i}{|x|} - 2x_i\right). \end{aligned} \tag{45}$$

Then,

$$\begin{aligned} L\mathscr{V} &= L(u-\underline{u}) + \frac{1}{\sqrt{R}}Ld - \frac{1}{2R^{5/4}}Ld^2 \\ &\le c + \frac{n+2-\beta}{1-|Du|^2}u_i\left(u_i - \underline{u}_i\right) \\ &\quad + \frac{n+2-\beta}{1-|Du|^2}\frac{u_ix_i}{R^{5/4}|x|}\left(d - R^{3/4}\right) \\ &\quad - \left[\frac{1}{|x|} - \frac{c}{|x|^2} + \frac{1}{|x|}\left(\frac{1}{\sqrt{R}} - \frac{d}{R^{5/4}}\right)\right]\operatorname{tr}\left(u^{\tau\tau'}\right) \\ &\quad - \left(\frac{1}{R^{5/4}} - \frac{c}{|x|^2}\right)u^{\nu\nu}. \end{aligned} \tag{46}$$

Thus, for R large enough, we have

$$L\mathscr{V} \le cR^{1/2} - \frac{1}{2R}\operatorname{tr}\left(u^{\tau\tau'}\right) - \frac{1}{2R^{5/4}}u^{\nu\nu}. \tag{47}$$

Expanding the determinant and using that $(u^{\tau\tau'})$ is a diagonal matrix give

$$\begin{aligned} \det\left(u^{ij}\right) &= \det\begin{pmatrix} u^{11} & 0 & \cdots & 0 & u^{1n} \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & u^{n-1\,n-1} & u^{n-1\,n} \\ u^{1n} & \cdots & \cdots & u^{n-1\,n} & u^{nn} \end{pmatrix} \\ &= \prod_i u^{ii} - \sum_\tau \left|u^{n\tau}\right|^2 \prod_{\tau'\ne\tau} u^{\tau'\tau'} \le \prod_i u^{ii}. \end{aligned} \tag{48}$$

By the inequality for arithmetic and geometric means,

$$\begin{aligned} &\frac{1}{R}\operatorname{tr}\left(u^{\tau\tau'}\right) + \frac{1}{R^{5/4}}u^{\nu\nu} \\ &\quad \ge n\left[\left(\frac{1}{R}\right)^{n-1}\cdot\frac{1}{R^{5/4}}\prod_i u^{ii}\right]^{1/n} \\ &\quad \ge n\left[\left(\det\left(u_{ij}\right)\right)\right]^{-(1/n)}R^{-(1/n)(n+(1/4))}. \end{aligned} \tag{49}$$

Hence for large R,

$$\begin{aligned} L\mathscr{V} &\le cR^{1/2} - \frac{1}{4R}\operatorname{tr}\left(u^{\tau\tau'}\right) - \frac{1}{4R^{5/4}}u^{\nu\nu} - \frac{1}{4R}\operatorname{tr}\left(u^{\tau\tau'}\right) \\ &\quad - \frac{1}{4R^{5/4}}u^{\nu\nu} \\ &\le cR^{1/2} - c\left[\det(u_{ij})\right]^{-(1/n)}R^{-(1/n)(n+(1/4))} - \frac{1}{4R^{5/4}}\operatorname{tr}\left(u^{ij}\right) \\ &= cR^{1/2} - c\left[\left(1-|Du|^2\right)^{(n+2-\beta)/2}\right]^{-(1/n)}R^{-(1/n)(n+(1/4))} \\ &\quad - \frac{1}{4R^{5/4}}\operatorname{tr}\left(u^{ij}\right) \end{aligned}$$

$$\leq cR^{1/2} - cR^{(3-2\beta-2n)/4n} - \frac{1}{4R^{5/4}}\operatorname{tr}\left(u^{ij}\right)$$
$$\leq -cR^{(3-2\beta-2n)/4n} - \frac{1}{4R^{5/4}}\operatorname{tr}\left(u^{ij}\right). \tag{50}$$

Note that we have used the fact $(3-2\beta-2n)/4n > 1/2$ in the last inequality, which is from the assumption $\beta < (3/2) - 2n$. □

Lemma 10. *There exists a positive constant A independent of R such that*

$$\Theta := \mathcal{V} + A\cdot\frac{1}{R^2}\cdot\left|x-x_0\right|^2 \pm T\left(u-\underline{u}\right) \tag{51}$$

satisfies

$$L\Theta \leq 0, \quad \text{in } \Omega_\delta,$$
$$\Theta \geq 0, \quad \text{on } \partial\Omega_\delta, \tag{52}$$

where $\delta = R^{3/4}$ and $\mathcal{V}$ is as in Lemma 9.

Proof. According to Lemma 9, the fact $\Theta \geq 0$ on $\partial\Omega_\delta$ follows from

$$A\cdot\frac{1}{R^2}\cdot\left|x-x_0\right|^2 \pm T\left(u-\underline{u}\right) \geq 0 \quad \text{on } \partial\Omega_\delta, \tag{53}$$

which can be attained by choosing A sufficiently large. The property $L\Theta \leq 0$ now follows from the inequality

$$-\varepsilon R^{(3-2\beta-2n)/4n} - \varepsilon R^{-5/4}\operatorname{tr}\left(u^{ij}\right) + cAR^{-(1/4)} + c$$
$$+ c\cdot\frac{1+A}{R^2}\operatorname{tr}\left(u^{ij}\right) \leq 0, \tag{54}$$

which holds for R large enough. □

Proof of Theorem 7. The maximum principle applied to (52) yields that $\Theta \geq 0$ in Ω_δ. Since $\Theta(x_0) = 0$, it follows that

$$\Theta_\nu\left(x_0\right) \geq 0 \tag{55}$$

with $\nu = -x_0/|x_0|$. Thus we get

$$\mathcal{V}_\nu\left(x_0\right) \geq \left|\left(T\left(u-\underline{u}\right)\right)_\nu\right|\left(x_0\right). \tag{56}$$

That is,

$$\left[-(u-\underline{u})_n - \frac{1}{\sqrt{R}}(R-|x|)_n + \frac{1}{R^{5/4}}(R-|x|)_n^2\right]\left(x_0\right)$$
$$\geq \left|(u-\underline{u})_{tn} + \frac{x_t}{R}(u-\underline{u})_{nn}\right|\left(x_0\right) = \left|(u-\underline{u})_{tn}\right|\left(x_0\right), \tag{57}$$

which, together with (8) and (14), implies

$$\left|u_{tn}\left(x_0\right)\right| \leq \left|\underline{u}_{tn}\left(x_0\right)\right| + \left|(u-\underline{u})_n\right| + \frac{1}{\sqrt{R}} \leq \frac{c}{\sqrt{R}}. \tag{58}$$

That is, (28) holds. □

Theorem 11 (double normal C^2-estimates at the outer boundary). *Under the assumption of Theorem 2 and the notation of Theorem 7, we have*

$$\left|u_{\nu\nu}\left(x_0\right)\right| \leq c. \tag{59}$$

Proof. As the proof of Lemma 9, we fix $x_0 \in \partial B_R$ and set $\nu = x_0/|x_0|$. Let τ, τ' belong to an orthonormal basis for the orthogonal complement of ν which we choose such that the submatrix $(u^{\tau\tau'})$ is diagonal. Assume that ν and τ, τ' correspond to the indices n and $1,\ldots,n-1$, respectively. We expand the determinant,

$$\left(1-|Du|^2\right)^{(n+2-\beta)/2}$$
$$= \det\left(u_{ij}\right) = u_{nn}\cdot\prod_{i<n}u_{ii} - \sum_{k<n}u_{kn}^2\cdot\prod_{k\neq i<n}u_{ii}$$
$$= u_{nn}\cdot\prod_{i<n}u_{ii} - \prod_{i<n}u_{ii}\sum_{k<n}u_{kn}^2\frac{1}{u_{kk}}. \tag{60}$$

Then, for $\beta < (3/2) - 2n$, we have

$$u_{nn} = \frac{\left(1-|Du|^2\right)^{(n+2-\beta)/2}}{\prod_{i<n}u_{ii}} + \sum_{k<n}\frac{u_{kn}^2}{u_{kk}}$$
$$\leq \frac{cR^{-((n+2-\beta)/2)}}{(c/R)^{n-1}} + \sum_{k<n}\frac{\left(c/\sqrt{R}\right)^2}{c/R} \leq c. \tag{61}$$

□

Proof of Theorem 2. It follows from Theorems 6, 7, and 11 that $\|u^R\|_{C^2}$ are uniformly bounded in R. By the standard Krylov/Shafanov theory, Schauder regularity theory, and a diagonal sequence argument, we obtain a smooth locally strictly convex solution u to (3)–(5) on exterior domain $\mathbb{R}^n \setminus D$. □

4. Proof of Theorem 3

In this section, we prove Theorem 3, which gives a simple example of a barrier construction.

Proof of Theorem 3. We introduce functions

$$\varphi(\tau) = a\rho_0^2\tau^{-3},$$
$$\eta(r) = -\int_{\rho_0}^{r}\left(\int_{\rho}^{\infty}\varphi(\tau)\,d\tau\right)d\rho, \tag{62}$$

where $0 < a < 1$ will be determined. We define $\underline{u}$ by

$$\underline{u}: \mathbb{R}^n \setminus B_{\rho_0} \longrightarrow \mathbb{R}$$
$$x \longmapsto |x| - \rho_0 + \eta(|x|). \tag{63}$$

Then, for $r \geq \rho_0$,

$$0 < -\eta'(r) = \int_r^\infty \varphi(\tau)\,d\tau = \frac{a}{2}\rho_0^2 r^{-2} \leq \frac{a}{2} < 1. \tag{64}$$

Obviously, $\underline{u} = 0$ on ∂B_{ρ_0}. Moreover,

$$\sup |\underline{u} - r| \leq \rho_0 + \int_{\rho_0}^{\infty} \left(\int_{\rho}^{\infty} \varphi(\tau)\, d\tau \right) d\rho,$$
$$|D(\underline{u} - r)| = |\eta'(r)| = O\left(\frac{1}{r}\right), \tag{65}$$
$$|D^2(\underline{u} - r)| + |D^3\underline{u}| = O\left(\frac{1}{r^2}\right),$$

where $r = |x|$. Therefore, $\underline{u}$ is close to a cone in the sense of (7) and satisfies the regularity conditions (8) and (17).

We compute the Gauss curvature of graph $\underline{u}$ as follows:

$$\underline{u}_i = (1 + \eta')\frac{x_i}{r}, \qquad |D\underline{u}| = 1 + \eta',$$
$$\underline{u}_{ij} = (1 + \eta')\frac{1}{r}\left(\delta_{ij} - \frac{x_i x_j}{r^2}\right) + \eta''(r)\frac{x_i x_j}{r^2} \tag{66}$$
$$\det D^2\underline{u} = \frac{\eta''(r)}{r^{n-1}}(1 + \eta')^{n-1} = \frac{\varphi}{r^{n-1}}(1 + \eta')^{n-1}.$$

Take $0 < a < \min\{1, 1/(2\rho_0)^2\}$. Using the assumption $\beta < 0$ and the fact that

$$|D\underline{u}|^2 = (1 + \eta')^2 = 1 - a\rho_0^2 r^{-2} + \frac{a^2}{4}\rho_0^4 r^{-4} \geq 1 - a\rho_0^2 r^{-2}, \tag{67}$$

we conclude that

$$\frac{\det D^2\underline{u}}{(1 - |D\underline{u}|^2)^{(n+2-\beta)/2}}$$
$$= \varphi(r)(1 + \eta')^{n-1} \cdot \frac{1}{(1 - (1 + \eta')^2)^{(n+2-\beta)/2}} \cdot \frac{1}{r^{n-1}} \tag{68}$$
$$\geq a\rho_0^2 r^{-3} 2^{1-n} (a\rho_0^2 r^{-2})^{-(n+2-\beta)/2} \cdot r^{1-n}$$
$$= 2^{1-n} a^{-(n+\beta)/2} \rho_0^{\beta-n} r^{-\beta} \geq (2\sqrt{a}\rho_0)^{-n} \geq 1.$$

Thus,

$$\det D^2\underline{u} \geq (1 - |D\underline{u}|^2)^{(n+2-\beta)/2}, \quad |x| > \rho_0,$$
$$\underline{u} = 0, \quad |x| = \rho_0. \tag{69}$$

The theorem is completed. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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