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*Research Article*

# Regularity of Functions on the Reduced Quaternion Field in Clifford Analysis

**Ji Eun Kim, Su Jin Lim, and Kwang Ho Shon**

*Department of Mathematics, Pusan National University, Busan 609-735, Republic of Korea*

Correspondence should be addressed to Kwang Ho Shon; khshon@pusan.ac.kr

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We define a new hypercomplex structure of $\mathbb{R}^3$ and a regular function with values in that structure. From the properties of regular functions, we research the exponential function on the reduced quaternion field and represent the corresponding Cauchy-Riemann equations in hypercomplex structures of $\mathbb{R}^3$.

## 1. Introduction

Meglihzon [1], Sudbery [2], and Fueter [3] demonstrated that there are three possible approaches (the Cauchy approach, Weierstrass approach, and Riemann approach) in the theories of functions that would generalize holomorphic functions with respect to several complex variables. Sudbery [2], Soucek [4], and Sommen [5] attempted to research the Cauchy approach using differential forms and differential operators in Clifford analysis. Fueter [3] and Naser [6] studied the properties of quaternionic differential equations as a generalization of the extended Cauchy-Riemann equations in the complex holomorphic function theory. Nôno [7–9] and Sudbery [2] gave a definition and the development of regular functions over the quaternion field. Ryan [10, 11] developed the theories of regular functions in a complex Clifford analysis using a generalization of the Cauchy-Riemann equation. Malonek [12] considered analogously the function theory of hypercomplex variables. He defined the hypercomplex differentiability for the existence of a function over the Clifford algebra and monogenicity based on a generalized Cauchy-Riemann system. Gotô and Nôno [13] and Koriyama et al. [14] dealt with differential operators with the derivative of regular functions in quaternion.

We shall denote by $\mathbb{C}$, $\mathbb{R}$, and $\mathbb{Z}$, respectively, the field of complex numbers, the field of real numbers, and the set of all integers. We [15, 16] showed that any complex-valued harmonic function $f_1$ in a pseudoconvex domain $D$ of $\mathbb{C}^2 \times \mathbb{C}^2$ has a hyperconjugate harmonic function $f_2$ in $D$ such that the quaternion-valued function $f_1 + f_2 j$ is hyperholomorphic in $D$ and gave a regeneration theorem in quaternion analysis in the view of complex and Clifford analysis. Further, we [17, 18] investigated the existence of the hyperconjugate harmonic functions of the octonion number system and some properties of dual quaternion functions.

In this paper, we introduce the Fueter variables on $\mathbb{R}^3$ and investigate a hypercomplex structure of $\mathbb{R}^3$. We define regular functions and obtain the representation of the corresponding Cauchy-Riemann equations for regular functions in the reduced quaternion field.

## 2. Preliminaries

A three-dimensional, noncommutative, and associative real field, called a ternary number system, is constructed by three base elements $e_0, e_1$, and $e_2$ which satisfy

$$\begin{gathered} e_0^2 = 1, \qquad e_1^2 = e_2^2 = -1, \qquad e_1 e_2 = -e_2 e_1, \\ \overline{e_0} = e_0, \qquad \overline{e_r} = -e_r \quad (r = 1, 2). \end{gathered} \tag{1}$$

In addition, let $e_0$ be the identity of a ternary number system and $e_1$ identifies the imaginary unit $\sqrt{-1}$ in the complex field, and

$$\mathbb{C}(\mathbb{T}) := \{z = e_1 z_1 + e_2 z_2 \mid z_1, z_2 \in \mathbb{C}\}, \tag{2}$$

where $z_r = x_r - (1/2)e_r x_0$ $(r = 1, 2)$ and $x_m$ $(m = 0, 1, 2)$ are real variables. They satisfy the equations

$$\overline{z_r w_k} = \overline{w_k z_r} \qquad (r \neq k), \tag{3}$$

where $\overline{z_r} = x_r + (1/2)e_r x_0$ $(r = 1, 2)$, $w_k = y_k - (1/2)e_k y_0$, $\overline{w_k} = y_k + (1/2)e_k y_0$ $(k = 1, 2)$, and $y_m$ $(m = 0, 1, 2)$ are real variables.

For any two elements $z = e_1 z_1 + e_2 z_2$ and $w = e_1 w_1 + e_2 w_2$ of $\mathbb{C}(\mathbb{T})$, their product is given by

$$zw = z \bullet w + z \odot w, \tag{4}$$

where the corresponding commutative inner product $\bullet$ satisfies

$$\begin{aligned} z \bullet w &= \frac{1}{2}(zw + wz) \\ &= -\sum_{r=1}^{2} z_r w_r + \frac{1}{2} e_1 e_2 \left(\overline{z_1} w_2 - \overline{w_2} z_1 + \overline{w_1} z_2 - \overline{z_2} w_1\right) \end{aligned} \tag{5}$$

and the corresponding noncommutative outer product $\odot$ satisfies

$$\begin{aligned} z \odot w &= \frac{1}{2}(zw - wz) \\ &= \frac{1}{2} e_1 e_2 \left(\overline{z_1} w_2 + \overline{w_2} z_1 - \overline{w_1} z_2 - \overline{z_2} w_1\right) \\ &= -w \odot z. \end{aligned} \tag{6}$$

The conjugation $z^*$, the corresponding norm $|z|$, and the inverse $z^{-1}$ of $z$ in $\mathbb{C}(\mathbb{T})$ are given by

$$\begin{aligned} z^* &= \overline{e_1 z_1} + \overline{e_2 z_2}, \\ |z|^2 &= zz^* = z \bullet z^* = \sum_{r=1}^{2} z_r \overline{z_r}, \\ z^{-1} &= \frac{z^*}{|z|^2} \qquad (z \neq 0). \end{aligned} \tag{7}$$

For any element $z$ in $\mathbb{C}(\mathbb{T})$, we have the corresponding exponential function $e^z$ denoted by

$$\exp(z) = \exp(e_1 z_1 + e_2 z_2). \tag{8}$$

**Theorem 1.** *Let $z$ be an arbitrary number in $\mathbb{C}(\mathbb{T})$. Then the corresponding exponential function $\exp(z)$ of $z$ in $\mathbb{C}(\mathbb{T})$ is given as*

$$\exp(z) = \begin{cases} (-1)^k \exp(x_0) \exp(e_2 x_2), & \text{if } x_1 = k\pi, \\ (-1)^t \exp(x_0) \exp(e_1 x_1), & \text{if } x_2 = t\pi, \end{cases} \tag{9}$$

*where $k, t \in \mathbb{Z}$.*

*Furthermore, as hyperbolic functions, one has*

$$\begin{aligned} &\exp(z) \\ &= \begin{cases} (-1)^k \exp(e_2 x_2)(\cosh(x_0) - \sinh(x_0)), & \text{if } x_1 = k\pi, \\ (-1)^t \exp(e_1 x_1)(\cosh(x_0) - \sinh(x_0)), & \text{if } x_2 = t\pi, \end{cases} \end{aligned} \tag{10}$$

*where $k, t \in \mathbb{Z}$.*

*Proof.* For any element $z = e_1 z_1 + e_2 z_2$ of $\mathbb{C}(\mathbb{T})$,

$$\exp(z) = \exp(e_1 z_1 + e_2 z_2) = \exp(e_1 z_1)\exp(e_2 z_2). \tag{11}$$

Since a scalar part of $e_1 z_1$ is $(1/2)x_0$, a vector part of $e_1 z_1$ is $e_1 x_1$, and $|e_1| = 1$, by [19],

$$\begin{aligned} \exp(e_1 z_1) &= \exp\left(\frac{x_0}{2}\right)\left\{\cos(|e_1 x_1|) + \frac{e_1 x_1}{|e_1 x_1|}\sin(|e_1 x_1|)\right\} \\ &= \exp\left(\frac{x_0}{2}\right)\{\cos(x_1) + e_1 \sin(x_1)\} \end{aligned} \tag{12}$$

and, similarly, we have

$$\begin{aligned} \exp(e_2 z_2) &= \exp\left(\frac{x_0}{2}\right)\left\{\cos(|e_2 x_2|) + \frac{e_2 x_2}{|e_2 x_2|}\sin(|e_2 x_2|)\right\} \\ &= \exp\left(\frac{x_0}{2}\right)\{\cos(x_2) + e_2 \sin(x_2)\}. \end{aligned} \tag{13}$$

Then we have

$$\begin{aligned} \exp(z) &= \exp\left(\frac{x_0}{2}\right)\{\cos(x_1) + e_1 \sin(x_1)\} \\ &\quad \times \exp\left(\frac{x_0}{2}\right)\{\cos(x_2) + e_2 \sin(x_2)\} \\ &= \exp(x_0)\{\cos(x_1) + e_1 \sin(x_1)\} \\ &\quad \times \{\cos(x_2) + e_2 \sin(x_2)\} \\ &= \exp(x_0)\{\cos(x_1)\cos(x_2) + e_2 \cos(x_1) \\ &\qquad \times \sin(x_2) + e_1 \sin(x_1)\cos(x_2)\} \\ &\quad + \exp(x_0) e_1 e_2 \sin(x_1)\sin(x_2). \end{aligned} \tag{14}$$

Also, we obtain

$$\begin{aligned} \exp(z) &= \exp(e_2 z_2 + e_1 z_1) \\ &= \exp(e_2 z_2)\exp(e_1 z_1) \\ &= \exp(x_0)\{\cos(x_2) + e_2 \sin(x_2)\} \\ &\quad \times \{\cos(x_1) + e_1 \sin(x_1)\} \\ &= \exp(x_0)\{\cos(x_1)\cos(x_2) + e_2 \cos(x_1) \\ &\qquad \times \sin(x_2) + e_1 \sin(x_1)\cos(x_2)\} \\ &\quad + \exp(x_0) e_2 e_1 \sin(x_1)\sin(x_2). \end{aligned} \tag{15}$$

Since (15) has to be equal to (14), $\sin(x_1)\sin(x_2) = 0$, that is, $\sin(x_1) = 0$ or $\sin(x_2) = 0$. Therefore, $x_1 = k\pi$ or $x_2 = t\pi$, and then $\cos(x_1) = (-1)^k$ or $\cos(x_2) = (-1)^t$, where $k, t \in \mathbb{Z}$. If $x_1 = k\pi$ $(k \in \mathbb{Z})$, then

$$\begin{aligned} \exp(z) &= \exp(x_0)\left\{(-1)^k(\cos(x_2) + e_2 \sin(x_2))\right\} \\ &= (-1)^k \exp(x_0)\exp(e_2 x_2). \end{aligned} \tag{16}$$

Similarly, if $x_2 = t\pi$ $(t \in \mathbb{Z})$, then

$$\begin{aligned}\exp(z) &= \exp(x_0)\left\{(-1)^t\left(\cos(x_1) + e_1\sin(x_1)\right)\right\}\\ &= (-1)^t\exp(x_0)\exp(e_1x_1).\end{aligned} \tag{17}$$

Further, by the Euler formula and the addition rule of trigonometric functions,

$$\begin{aligned}\exp(z) &= \exp(e_1z_1 + e_2z_2) = \exp(e_1z_1)\exp(e_2z_2)\\ &= (\cos(z_1) + e_1\sin(z_1))(\cos(z_2) + e_2\sin(z_2))\\ &= \left\{\cos(x_1)\cos\left(e_1\frac{x_0}{2}\right) + \sin(x_1)\sin\left(e_1\frac{x_0}{2}\right)\right.\\ &\quad + e_1\left(\sin(x_1)\cos\left(e_1\frac{x_0}{2}\right)\right.\\ &\quad\left.\left. - \cos(x_1)\sin\left(e_1\frac{x_0}{2}\right)\right)\right\}\\ &\quad\cdot\left\{\cos(x_2)\cos\left(e_2\frac{x_0}{2}\right) + \sin(x_2)\sin\left(e_2\frac{x_0}{2}\right)\right.\\ &\quad + e_2\left(\sin(x_2)\cos\left(e_2\frac{x_0}{2}\right)\right.\\ &\quad\left.\left. - \cos(x_2)\sin\left(e_2\frac{x_0}{2}\right)\right)\right\}.\end{aligned} \tag{18}$$

Since $\cos(e_r(x_0/2)) = \cosh(x_0/2)$ and $\sin(e_r(x_0/2)) = e_r\sinh(x_0/2)$ $(r = 1, 2)$, we have

$$\begin{aligned}\exp(z) &= \left\{\cos(x_1)\cosh\left(\frac{x_0}{2}\right) + e_1\sin(x_1)\sinh\left(\frac{x_0}{2}\right)\right.\\ &\quad + e_1\left(\sin(x_1)\cosh\left(\frac{x_0}{2}\right)\right.\\ &\quad\left.\left. - e_1\cos(x_1)\sinh\left(\frac{x_0}{2}\right)\right)\right\}\\ &\quad\cdot\left\{\cos(x_2)\cosh\left(\frac{x_0}{2}\right) + e_2\sin(x_2)\sinh\left(\frac{x_0}{2}\right)\right.\\ &\quad + e_2\left(\sin(x_2)\cosh\left(\frac{x_0}{2}\right)\right.\\ &\quad\left.\left. - e_2\cos(x_2)\sinh\left(\frac{x_0}{2}\right)\right)\right\}\\ &= \left\{(\cos(x_1) + e_1\sin(x_1))\right.\\ &\quad\left.\times\left(\cosh\left(\frac{x_0}{2}\right) - \sinh\left(\frac{x_0}{2}\right)\right)\right\}\\ &\quad\times\left\{(\cos(x_2) + e_2\sin(x_2))\right.\\ &\quad\left.\times\left(\cosh\left(\frac{x_0}{2}\right) - \sinh\left(\frac{x_0}{2}\right)\right)\right\}\\ &= (\cos(x_1) + e_1\sin(x_1))(\cos(x_2) + e_2\sin(x_2))\\ &\quad\times\left(\cosh\left(\frac{x_0}{2}\right) - \sinh\left(\frac{x_0}{2}\right)\right)^2.\end{aligned} \tag{19}$$

Since

$$\begin{aligned}&\left(\cosh\left(\frac{x_0}{2}\right) - \sinh\left(\frac{x_0}{2}\right)\right)^2\\ &\quad = \cosh^2\left(\frac{x_0}{2}\right) + \sinh^2\left(\frac{x_0}{2}\right) - 2\cosh^2\left(\frac{x_0}{2}\right)\sinh^2\left(\frac{x_0}{2}\right)\\ &\quad = 1 + 2\frac{\cosh(x_0) - 1}{2} - \sinh(x_0)\\ &\quad = \cosh(x_0) - \sinh(x_0),\end{aligned} \tag{20}$$

we obtain

$$\begin{aligned}\exp(z) &= \exp(e_1z_1)\exp(e_2z_2) = (\cos(x_1) + e_1\sin(x_1))\\ &\quad\times(\cos(x_2) + e_2\sin(x_2))(\cosh(x_0) - \sinh(x_0))\end{aligned} \tag{21}$$

and, similarly,

$$\begin{aligned}\exp(z) &= \exp(e_2z_2)\exp(e_1z_1) = (\cos(x_2) + e_2\sin(x_2))\\ &\quad\times(\cos(x_1) + e_1\sin(x_1))(\cosh(x_0) - \sinh(x_0)).\end{aligned} \tag{22}$$

Since (22) has to be equal to (21), $\sin(x_1)\sin(x_2) = 0$, that is, $\sin(x_1) = 0$ or $\sin(x_2) = 0$. Therefore, $x_1 = k\pi$ or $x_2 = t\pi$, and then $\cos(x_1) = (-1)^k$ or $\cos(x_2) = (-1)^t$, where $k, t \in \mathbb{Z}$. If $x_1 = k\pi$ $(k \in \mathbb{Z})$, then

$$\begin{aligned}\exp(z) &= (\cos(x_2) + e_2\sin(x_2))\\ &\quad\times(-1)^k(\cosh(x_0) - \sinh(x_0))\\ &= (-1)^k\exp(e_2x_2)(\cosh(x_0) - \sinh(x_0)).\end{aligned} \tag{23}$$

Similarly, if $x_2 = t\pi$ $(t \in \mathbb{Z})$, then

$$\begin{aligned}\exp(z) &= (\cos(x_1) + e_1\sin(x_1))\\ &\quad\times(-1)^t(\cosh(x_0) - \sinh(x_0))\\ &= (-1)^t\exp(e_1x_1)(\cosh(x_0) - \sinh(x_0)).\end{aligned} \tag{24}$$

□

*Remark 2.* By Theorem 1 and the properties of the Euler formula, if $x_1 = k\pi$, then we can write

$$\begin{aligned}\exp(z) &= (-1)^k\exp(e_2x_2)(\cosh(x_0) - \sinh(x_0))\\ &= (-1)^k\exp(e_2x_2 - x_0) = (-1)^k\exp\left(e_2\overline{F_2}\right),\end{aligned} \tag{25}$$

also, if $x_2 = t\pi$, then

$$\exp(z) = (-1)^t \exp(e_1 x_1)(\cosh(x_0) - \sinh(x_0)) = (-1)^t \exp(e_1 x_1 - x_0) = (-1)^t \exp(e_1 \overline{F_1}), \tag{26}$$

where $k, t \in \mathbb{Z}$ and $\overline{F_r} = x_r + e_r x_0$ $(r = 1, 2)$ are the conjugate Fueter variables of $F_r = x_r - e_r x_0$ $(r = 1, 2)$ (see [20]).

Let $\Omega$ be an open subset of $\mathbb{R}^3$ and let a function $f(a)$ be defined by the following form on $\Omega$ with values in $\mathbb{C}(\mathbb{T})$:

$$f : \Omega \longrightarrow \mathbb{C}(\mathbb{T}), \tag{27}$$

satisfying

$$a = (x_0, x_1, x_2) \in \Omega \longmapsto f(a) = e_1 f_1(x_0, x_1, x_2) + e_2 f_2(x_0, x_1, x_2) \in \mathbb{C}(\mathbb{T}), \tag{28}$$

where $f_r = u_r - (1/2)e_r u_0$, $\overline{f_r} = u_r + (1/2)e_r u_0$ $(r = 1, 2)$ and $u_m$ $(m = 0, 1, 2)$ are real-valued functions.

From the chain rule, we use the following differential operators:

$$\begin{aligned}\frac{\partial}{\partial A} &:= 2\frac{\partial}{\partial x_0} - \frac{1}{2}e_1\frac{\partial}{\partial x_1} - \frac{1}{2}e_2\frac{\partial}{\partial x_2} = -e_1\frac{\partial}{\partial z_1} - e_2\frac{\partial}{\partial z_2},\\ \frac{\partial}{\partial A^*} &= 2\frac{\partial}{\partial x_0} + \frac{1}{2}e_1\frac{\partial}{\partial x_1} + \frac{1}{2}e_2\frac{\partial}{\partial x_2} = e_1\frac{\partial}{\partial \overline{z_1}} + e_2\frac{\partial}{\partial \overline{z_2}},\end{aligned} \tag{29}$$

where

$$\begin{aligned}\frac{\partial}{\partial z_1} &= \frac{1}{2}\frac{\partial}{\partial x_1} + e_1\frac{\partial}{\partial x_0}, \qquad \frac{\partial}{\partial z_2} = \frac{1}{2}\frac{\partial}{\partial x_2} + e_2\frac{\partial}{\partial x_0},\\ \frac{\partial}{\partial \overline{z_1}} &= \frac{1}{2}\frac{\partial}{\partial x_1} - e_1\frac{\partial}{\partial x_0}, \qquad \frac{\partial}{\partial \overline{z_2}} = \frac{1}{2}\frac{\partial}{\partial x_2} - e_2\frac{\partial}{\partial x_0}\end{aligned} \tag{30}$$

in $\mathbb{C}(\mathbb{T})$. We have the following equations:

$$\begin{aligned}f_r\frac{\partial}{\partial z_r} &= \frac{\partial f_r}{\partial z_r}, \qquad f_r\frac{\partial}{\partial \overline{z_r}} = \frac{\partial f_r}{\partial \overline{z_r}} \quad (r = 1, 2),\\ \overline{f_1}\frac{\partial}{\partial z_2} &= \overline{\left(\frac{\partial f_1}{\partial \overline{z_2}}\right)}, \qquad \overline{f_2}\frac{\partial}{\partial z_1} = \overline{\left(\frac{\partial f_2}{\partial \overline{z_1}}\right)},\\ \overline{f_1}\frac{\partial}{\partial \overline{z_2}} &= \overline{\left(\frac{\partial f_1}{\partial z_2}\right)}, \qquad \overline{f_2}\frac{\partial}{\partial \overline{z_1}} = \overline{\left(\frac{\partial f_2}{\partial z_1}\right)},\end{aligned} \tag{31}$$

and then, the operator $\partial/\partial A$ operates to $f$ as follows:

$$\begin{aligned}\frac{\partial f}{\partial A} &= \left(-e_1\frac{\partial}{\partial z_1} - e_2\frac{\partial}{\partial z_2}\right)(e_1 f_1 + e_2 f_2)\\ &= \frac{\partial f_1}{\partial z_1} + \frac{\partial f_2}{\partial z_2} + e_1 e_2\left(\frac{\partial f_1}{\partial \overline{z_2}} - \frac{\partial f_2}{\partial \overline{z_1}}\right),\\ \frac{\partial f}{\partial A^*} &= \left(e_1\frac{\partial}{\partial \overline{z_1}} + e_2\frac{\partial}{\partial \overline{z_2}}\right)(e_1 f_1 + e_2 f_2)\\ &= -\frac{\partial f_1}{\partial \overline{z_1}} - \frac{\partial f_2}{\partial \overline{z_2}} + e_1 e_2\left(\frac{\partial f_2}{\partial z_1} - \frac{\partial f_1}{\partial z_2}\right),\\ f\frac{\partial}{\partial A} &= (e_1 f_1 + e_2 f_2)\left(-e_1\frac{\partial}{\partial z_1} - e_2\frac{\partial}{\partial z_2}\right)\\ &= f_1\frac{\partial}{\partial z_1} + f_2\frac{\partial}{\partial z_2} + e_1 e_2\left(\overline{f_2}\frac{\partial}{\partial z_1} - \overline{f_1}\frac{\partial}{\partial z_2}\right)\\ &= \frac{\partial f_1}{\partial z_1} + \frac{\partial f_2}{\partial z_2} + e_1 e_2\left\{\overline{\left(\frac{\partial f_2}{\partial \overline{z_1}}\right)} - \overline{\left(\frac{\partial f_1}{\partial \overline{z_2}}\right)}\right\},\\ f\frac{\partial}{\partial A^*} &= (e_1 f_1 + e_2 f_2)\left(e_1\frac{\partial}{\partial \overline{z_1}} + e_2\frac{\partial}{\partial \overline{z_2}}\right)\\ &= -f_1\frac{\partial}{\partial \overline{z_1}} - f_2\frac{\partial}{\partial \overline{z_2}} + e_1 e_2\left(\overline{f_1}\frac{\partial}{\partial \overline{z_2}} - \overline{f_2}\frac{\partial}{\partial \overline{z_1}}\right)\\ &= -\frac{\partial f_1}{\partial \overline{z_1}} - \frac{\partial f_2}{\partial \overline{z_2}} + e_1 e_2\left\{\overline{\left(\frac{\partial f_1}{\partial z_2}\right)} - \overline{\left(\frac{\partial f_2}{\partial z_1}\right)}\right\}.\end{aligned} \tag{32}$$

Thus, we have a corresponding Laplacian in the reduced quaternion $\mathbb{C}(\mathbb{T})$:

$$\Delta_a = \frac{\partial^2}{\partial A \partial A^*} = \frac{\partial^2}{\partial A^* \partial A} = 4\frac{\partial^2}{\partial x_0^2} + \frac{1}{4}\frac{\partial^2}{\partial x_1^2} + \frac{1}{4}\frac{\partial^2}{\partial x_2^2}. \tag{33}$$

*Remark 3.* Let $\Omega$ be an open set of $\mathbb{R}^3$. From the definition of the differential operators in $\mathbb{C}(\mathbb{T})$, we have

$$\begin{aligned}\frac{\partial}{\partial A} \bullet f &= \frac{1}{2}\left(\frac{\partial f_1}{\partial z_1} + \frac{\partial f_2}{\partial z_2} + f_1\frac{\partial}{\partial z_1} + f_2\frac{\partial}{\partial z_2}\right)\\ &\quad + \frac{1}{2}e_1 e_2\left(\overline{f_2}\frac{\partial}{\partial z_1} - \frac{\partial f_2}{\partial \overline{z_1}} + \frac{\partial f_1}{\partial \overline{z_2}} - \overline{f_1}\frac{\partial}{\partial z_2}\right)\\ &= \frac{\partial f_1}{\partial z_1} + \frac{\partial f_2}{\partial z_2} + \frac{1}{2}e_1 e_2\\ &\quad \times\left\{\overline{\left(\frac{\partial f_2}{\partial \overline{z_1}}\right)} - \overline{\left(\frac{\partial f_1}{\partial \overline{z_2}}\right)} - \frac{\partial f_2}{\partial \overline{z_1}} + \frac{\partial f_1}{\partial \overline{z_2}}\right\},\\ \frac{\partial}{\partial A} \odot f &= \frac{1}{2}\left(\frac{\partial f_1}{\partial z_1} + \frac{\partial f_2}{\partial z_2} - f_1\frac{\partial}{\partial z_1} - f_2\frac{\partial}{\partial z_2}\right)\\ &\quad + \frac{1}{2}e_1 e_2\left(\overline{f_1}\frac{\partial}{\partial z_2} + \frac{\partial f_1}{\partial \overline{z_2}} - \frac{\partial f_2}{\partial \overline{z_1}} - \overline{f_2}\frac{\partial}{\partial z_1}\right)\\ &= \frac{1}{2}e_1 e_2\left\{\overline{\left(\frac{\partial f_1}{\partial \overline{z_2}}\right)} + \frac{\partial f_1}{\partial \overline{z_2}} - \frac{\partial f_2}{\partial \overline{z_1}} - \overline{\left(\frac{\partial f_2}{\partial \overline{z_1}}\right)}\right\},\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial A^*}\bullet f &= -\frac{1}{2}\left(\frac{\partial f_1}{\partial \overline{z_1}} + \frac{\partial f_2}{\partial \overline{z_2}} + f_1\frac{\partial}{\partial \overline{z_1}} + f_2\frac{\partial}{\partial \overline{z_2}}\right) \\
&\quad + \frac{1}{2}e_1e_2\left(\frac{\partial f_2}{\partial z_1} - \overline{f_2}\frac{\partial}{\partial \overline{z_1}} + \overline{f_1}\frac{\partial}{\partial \overline{z_2}} - \frac{\partial f_1}{\partial z_2}\right) \\
&= -\frac{\partial f_1}{\partial \overline{z_1}} - \frac{\partial f_2}{\partial \overline{z_2}} + \frac{1}{2}e_1e_2 \\
&\quad \times \left\{\frac{\partial f_2}{\partial z_1} - \overline{\left(\frac{\partial f_2}{\partial z_1}\right)} + \overline{\left(\frac{\partial f_1}{\partial z_2}\right)} - \frac{\partial f_1}{\partial z_2}\right\}, \\
\frac{\partial}{\partial A^*}\odot f &= \frac{1}{2}\left(-\frac{\partial f_1}{\partial \overline{z_1}} - \frac{\partial f_2}{\partial \overline{z_2}} + f_1\frac{\partial}{\partial \overline{z_1}} + f_2\frac{\partial}{\partial \overline{z_2}}\right) \\
&\quad + \frac{1}{2}e_1e_2\left(\frac{\partial f_2}{\partial z_1} + \overline{f_2}\frac{\partial}{\partial \overline{z_1}} - \overline{f_1}\frac{\partial}{\partial \overline{z_2}} - \frac{\partial f_1}{\partial z_2}\right) \\
&= \frac{1}{2}e_1e_2\left\{\frac{\partial f_2}{\partial z_1} + \overline{\left(\frac{\partial f_2}{\partial z_1}\right)} - \overline{\left(\frac{\partial f_1}{\partial z_2}\right)} - \frac{\partial f_1}{\partial z_2}\right\}
\end{aligned} \tag{34}$$

and, therefore,

$$\frac{\partial f}{\partial A} = \frac{\partial}{\partial A}\bullet f + \frac{\partial}{\partial A}\odot f, \qquad \frac{\partial f}{\partial A^*} = \frac{\partial}{\partial A^*}\bullet f + \frac{\partial}{\partial A^*}\odot f. \tag{35}$$

Similarly, we have

$$\begin{aligned}
f\bullet\frac{\partial}{\partial A} &= \frac{1}{2}\left(f_1\frac{\partial}{\partial z_1} + f_2\frac{\partial}{\partial z_2} + \frac{\partial f_1}{\partial z_1} + \frac{\partial f_2}{\partial z_2}\right) \\
&\quad + e_1e_2\left(\overline{f_1}\frac{\partial}{\partial z_2} - \overline{f_2}\frac{\partial}{\partial z_1} + \frac{\partial f_2}{\partial \overline{z_1}} - \frac{\partial f_1}{\partial \overline{z_2}}\right) \\
&= \frac{\partial f_1}{\partial z_1} + \frac{\partial f_2}{\partial z_2} + \frac{1}{2}e_1e_2 \\
&\quad \times \left\{\overline{\left(\frac{\partial f_1}{\partial z_2}\right)} - \overline{\left(\frac{\partial f_2}{\partial z_1}\right)} + \frac{\partial f_2}{\partial \overline{z_1}} - \frac{\partial f_1}{\partial \overline{z_2}}\right\}, \\
f\odot\frac{\partial}{\partial A} &= \frac{1}{2}\left(f_1\frac{\partial}{\partial z_1} + f_2\frac{\partial}{\partial z_2} - \frac{\partial f_1}{\partial z_1} - \frac{\partial f_2}{\partial z_2}\right) \\
&\quad + \frac{1}{2}e_1e_2\left(\overline{f_1}\frac{\partial}{\partial z_2} - \overline{f_2}\frac{\partial}{\partial z_1} - \frac{\partial f_2}{\partial \overline{z_1}} + \frac{\partial f_1}{\partial \overline{z_2}}\right) \\
&= \frac{1}{2}e_1e_2\left\{\overline{\left(\frac{\partial f_1}{\partial z_2}\right)} - \overline{\left(\frac{\partial f_2}{\partial z_1}\right)} - \frac{\partial f_2}{\partial \overline{z_1}} + \frac{\partial f_1}{\partial \overline{z_2}}\right\}, \\
f\bullet\frac{\partial}{\partial A^*} &= \frac{1}{2}\left(-f_1\frac{\partial}{\partial \overline{z_1}} - f_2\frac{\partial}{\partial \overline{z_2}} - \frac{\partial f_1}{\partial \overline{z_1}} - \frac{\partial f_2}{\partial \overline{z_2}}\right) \\
&\quad + \frac{1}{2}e_1e_2\left(\overline{f_1}\frac{\partial}{\partial \overline{z_2}} - \overline{f_2}\frac{\partial}{\partial \overline{z_1}} + \frac{\partial f_2}{\partial z_1} - \frac{\partial f_1}{\partial z_2}\right) \\
&= -\frac{\partial f_1}{\partial \overline{z_1}} - \frac{\partial f_2}{\partial \overline{z_2}} + \frac{1}{2}e_1e_2 \\
&\quad \times \left\{\overline{\left(\frac{\partial f_1}{\partial z_2}\right)} - \overline{\left(\frac{\partial f_2}{\partial z_1}\right)} + \frac{\partial f_2}{\partial z_1} - \frac{\partial f_1}{\partial z_2}\right\}, \\
f\odot\frac{\partial}{\partial A^*} &= \frac{1}{2}\left(-f_1\frac{\partial}{\partial \overline{z_1}} - f_2\frac{\partial}{\partial \overline{z_2}} + \frac{\partial f_1}{\partial \overline{z_1}} + \frac{\partial f_2}{\partial \overline{z_2}}\right) \\
&\quad + \frac{1}{2}e_1e_2\left(\overline{f_1}\frac{\partial}{\partial \overline{z_2}} - \overline{f_2}\frac{\partial}{\partial \overline{z_1}} - \frac{\partial f_2}{\partial z_1} + \frac{\partial f_1}{\partial z_2}\right) \\
&= \frac{1}{2}e_1e_2\left\{\overline{\left(\frac{\partial f_1}{\partial z_2}\right)} - \overline{\left(\frac{\partial f_2}{\partial z_1}\right)} - \frac{\partial f_2}{\partial z_1} + \frac{\partial f_1}{\partial z_2}\right\}
\end{aligned} \tag{36}$$

and, therefore,

$$\begin{aligned}
f\frac{\partial}{\partial A} &= f\bullet\frac{\partial}{\partial A} + f\odot\frac{\partial}{\partial A}, \\
f\frac{\partial}{\partial A^*} &= f\bullet\frac{\partial}{\partial A^*} + f\odot\frac{\partial}{\partial A^*}.
\end{aligned} \tag{37}$$

*Definition 4.* Let $\Omega$ be an open set in $\mathbb{R}^3$ and for any element $a$ in $\Omega$. A function $f(a)$ is said to be *L(R)*-regular on $\Omega$ if the following conditions are satisfied:

(i) $f_r$ $(r = 1, 2)$ are continuously differential functions on $\Omega$, and

(ii) $\partial f(a)/\partial A^* = 0$ $(f(a)(\partial/\partial A^*) = 0)$ on $\Omega$.

In particular, the equation $\partial f/\partial A^* = 0$ of Definition 4 is equivalent to

$$\frac{\partial}{\partial A^*}\bullet f(a) = -\frac{\partial}{\partial A^*}\odot f(a). \tag{38}$$

Moreover, (38) is equivalent to the following system:

$$\begin{aligned}
\frac{\partial f_1}{\partial \overline{z_1}} &= -\frac{\partial f_2}{\partial \overline{z_2}}, \\
\frac{\partial f_1}{\partial z_2} &= \frac{\partial f_2}{\partial z_1}.
\end{aligned} \tag{39}$$

The above system is a corresponding Cauchy-Riemann system in $\mathbb{C}(\mathbb{T})$.

*Remark 5.* From the multiplications of $\mathbb{C}(\mathbb{T})$, the equation $f(\partial/\partial A^*) = 0$ of Definition 4 is equivalent to

$$\frac{\partial}{\partial A^*}\bullet f(a) = \frac{\partial}{\partial A^*}\odot f(a). \tag{40}$$

Also, the above equation (40) is equivalent to the following system:

$$\begin{aligned}
\frac{\partial f_1}{\partial \overline{z_1}} &= -\frac{\partial f_2}{\partial \overline{z_2}}, \\
\overline{f_1}\frac{\partial}{\partial \overline{z_2}} = \overline{f_2}\frac{\partial}{\partial \overline{z_1}} &\Longleftrightarrow \overline{\left(\frac{\partial f_1}{\partial z_2}\right)} = \overline{\left(\frac{\partial f_2}{\partial z_1}\right)}.
\end{aligned} \tag{41}$$

Further, the above system (41) is also a corresponding Cauchy-Riemann system in $\mathbb{C}(\mathbb{T})$. Since the system (39) is equivalent to the system (41), we say that $f(a)$ of Definition 4 is a regular function on $\Omega \subset \mathbb{R}^3$. When the function $f(a)$ is either an $L$-regular function or an $R$-regular function on $\Omega \subset \mathbb{R}^3$, we simply say that $f(a)$ is a regular function on $\Omega \subset \mathbb{R}^3$.

## 3. Properties of Regular Functions with Values in $\mathbb{C}(\mathbb{T})$

We define the derivative $f'(a)$ of $f(a)$ by the following:

$$f'(a) := \frac{\partial f(a)}{\partial A}. \tag{42}$$

**Proposition 6.** *Let $\Omega$ be an open set in $\mathbb{R}^3$ and let a function $f(a)$ be a regular function defined on $\Omega$. Then*

$$\begin{aligned} f'(a) &= -2e_r\left(\frac{\partial f}{\partial z_r} - \frac{\partial f}{\partial \overline{z_r}}\right) = 4\frac{\partial f}{\partial x_0} \\ &= -e_1\frac{\partial f}{\partial x_1} - e_2\frac{\partial f}{\partial x_2} \qquad (r = 1, 2). \end{aligned} \tag{43}$$

*Proof.* From the definition of a regular function (: $(\partial f/\partial A^*) = 0$), we have

$$\frac{\partial f_1}{\partial \overline{z_1}} = -\frac{\partial f_2}{\partial \overline{z_2}}, \qquad \frac{\partial f_1}{\partial z_2} = \frac{\partial f_2}{\partial z_1}. \tag{44}$$

Therefore,

$$\begin{aligned} \frac{\partial}{\partial z} \bullet f &= \frac{\partial f_1}{\partial \overline{z_1}} + 2e_1\frac{\partial u_1}{\partial x_0} + \frac{\partial u_0}{\partial x_0} + \frac{\partial f_2}{\partial \overline{z_2}} + 2e_2\frac{\partial u_2}{\partial x_0} + \frac{\partial u_0}{\partial x_0} \\ &\quad + \frac{1}{2}e_1e_2\left(\overline{f_2}\frac{\partial}{\partial \overline{z_1}} + 2e_1\frac{\partial u_2}{\partial x_0} + e_2e_1\frac{\partial u_0}{\partial x_0} - \frac{\partial f_2}{\partial z_1}\right) \\ &\quad + \frac{1}{2}e_1e_2\left(2e_1\frac{\partial u_2}{\partial x_0} - e_1e_2\frac{\partial u_0}{\partial x_0} + \frac{\partial f_1}{\partial z_2} - 2e_2\frac{\partial u_1}{\partial x_0}\right. \\ &\qquad + e_2e_1\frac{\partial u_0}{\partial x_0} - \overline{f_1}\frac{\partial}{\partial \overline{z_2}} \\ &\qquad \left. -2e_2\frac{\partial u_1}{\partial x_0} - e_1e_2\frac{\partial u_0}{\partial x_0}\right) \\ &= 4\frac{\partial f}{\partial x_0} + \frac{1}{2}e_1e_2\overline{\left(\frac{\partial f_2}{\partial z_1}\right)} - \frac{1}{2}e_1e_2\overline{\left(\frac{\partial f_1}{\partial z_2}\right)}, \\ \frac{\partial}{\partial z} \odot f &= \frac{1}{2}e_1e_2 \\ &\quad \times \left(\overline{f_1}\frac{\partial}{\partial \overline{z_2}} + 2e_2\frac{\partial f_1}{\partial x_0} + \frac{\partial f_1}{\partial z_2} - 2e_2\frac{\partial f_1}{\partial x_0} - \frac{\partial f_2}{\partial z_1}\right. \\ &\qquad \left. +2e_1\frac{\partial f_2}{\partial x_0} - \overline{f_2}\frac{\partial}{\partial \overline{z_1}} - 2e_1\frac{\partial f_2}{\partial x_0}\right) \\ &= 2e_1e_2\left(e_1\frac{\partial f_1}{\partial x_0} - e_1\frac{\partial f_1}{\partial x_0} - e_2\frac{\partial f_2}{\partial x_0} + e_2\frac{\partial f_2}{\partial x_0}\right) \\ &\quad - \frac{1}{2}e_1e_2\overline{\left(\frac{\partial f_2}{\partial z_1}\right)} + \frac{1}{2}e_1e_2\overline{\left(\frac{\partial f_1}{\partial z_2}\right)} = 0. \end{aligned} \tag{45}$$

Hence, we obtain the equation

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z} \bullet f + \frac{\partial}{\partial z} \odot f = 4\frac{\partial f}{\partial x_0}. \tag{46}$$

Similarly, by calculating the derivative $f'(z)$ of $f(z)$,

$$\begin{aligned} \frac{\partial}{\partial z} \bullet f &= \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \\ &\quad + \frac{1}{2}e_1e_2\left\{\overline{\left(\frac{\partial f_2}{\partial z_1}\right)} - \overline{\left(\frac{\partial f_1}{\partial z_2}\right)} + \frac{\partial f_1}{\partial x_2} - \frac{\partial f_2}{\partial x_1}\right\}, \\ \frac{\partial}{\partial z} \odot f &= \left(e_1\frac{\partial f_2}{\partial x_1} - e_1\frac{\partial f_2}{\partial x_1} - e_2\frac{\partial f_1}{\partial x_2} + e_2\frac{\partial f_1}{\partial x_2}\right) \\ &\quad - \frac{1}{2}e_1e_2\overline{\left(\frac{\partial f_2}{\partial z_1}\right)} + \frac{1}{2}e_1e_2\overline{\left(\frac{\partial f_1}{\partial z_2}\right)}. \end{aligned} \tag{47}$$

Therefore, we have the equation

$$\begin{aligned} \frac{\partial f}{\partial z} &= \frac{\partial}{\partial z} \bullet f + \frac{\partial}{\partial z} \odot f = -e_1\left(e_1\frac{\partial f_1}{\partial x_1} + e_2\frac{\partial f_2}{\partial x_1}\right) \\ &\quad - e_2\left(e_2\frac{\partial f_2}{\partial x_2} + e_1\frac{\partial f_1}{\partial x_2}\right) = -e_1\frac{\partial f}{\partial x_1} - e_2\frac{\partial f}{\partial x_2}. \end{aligned} \tag{48}$$

Further, using the same procedure, we obtain the equations

$$\frac{\partial f}{\partial z} = -2e_r\left(\frac{\partial f}{\partial z_r} - \frac{\partial f}{\partial \overline{z_r}}\right) \quad (r = 1, 2). \tag{49}$$

□

**Proposition 7.** *Let $\Omega$ be an open set in $\mathbb{R}^3$. If $f(a)$ is a regular function on $\Omega$, then we have*

$$\frac{\partial^n f}{\partial A^n} = 4^n\frac{\partial^n f}{\partial x_0^n}, \tag{50}$$

*where $n$ is a positive integer.*

*Proof.* Since $f$ is a regular function on $\Omega$ with values in $\mathbb{C}(\mathbb{T})$, by Definition 4,

$$\frac{\partial}{\partial A^*}\left(4\frac{\partial f}{\partial x_0}\right) = 4\frac{\partial}{\partial x_0}\left(\frac{\partial f}{\partial A^*}\right) = 0. \tag{51}$$

Hence, $\partial f/\partial x_0$ is a regular function with values in $\mathbb{C}(\mathbb{T})$. From Proposition 6, we have

$$\frac{\partial^2 f}{\partial A^2} = \frac{\partial}{\partial A}\left(\frac{\partial f}{\partial \mathrm{A}}\right) = 4^2\frac{\partial^2 f}{\partial x_0^2}. \tag{52}$$

By repeating the above process, we can obtain the equation

$$\frac{\partial^n f}{\partial A^n} = 4^n\frac{\partial^n f}{\partial x_0^n}. \tag{53}$$

□

We let

$$\square_a = \sum_{r=1}^{2}\frac{\partial^2}{\partial z_r\partial \overline{z_r}} = 2\frac{\partial^2}{\partial x_0^2} + \frac{1}{4}e_1\frac{\partial^2}{\partial x_1^2} + \frac{1}{4}e_2\frac{\partial^2}{\partial x_2^2} \tag{54}$$

on an open set $\Omega$ in $\mathbb{R}^3$.

**Theorem 8.** *Let $\Omega$ be an open set in $\mathbb{R}^3$. If $f$ is a regular function on $\Omega$, then the following equation holds true:*

$$\square_a f(a) = -\frac{1}{8}\frac{\partial^2 f(a)}{\partial A^2}. \quad (55)$$

*Proof.* Since $f$ is a regular function on $\Omega$, we have the following system:

$$4\frac{\partial u_0}{\partial x_0} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}, \frac{\partial u_2}{\partial x_1} = \frac{\partial u_1}{\partial x_2},$$
$$4\frac{\partial u_r}{\partial x_0} = -\frac{\partial u_0}{\partial x_r} \quad (r = 1, 2). \quad (56)$$

By the definition of $\square_a$, we have

$$\begin{aligned}\square_a f &= \left(2\frac{\partial^2}{\partial x_0^2} + \frac{1}{4}e_1\frac{\partial^2}{\partial x_1^2} + \frac{1}{4}e_2\frac{\partial^2}{\partial x_2^2}\right)(u_0 + e_1u_1 + e_2u_2)\\ &= 2\frac{\partial^2 u_0}{\partial x_0^2} + 2e_1\frac{\partial^2 u_1}{\partial x_0^2} + 2e_2\frac{\partial^2 u_2}{\partial x_0^2} - \frac{\partial^2 u_1}{\partial x_0\partial x_1}\\ &\quad + e_1\frac{\partial^2 u_0}{\partial x_0\partial x_1} - \frac{\partial^2 u_2}{\partial x_0\partial x_2} + e_2\frac{\partial^2 u_0}{\partial x_0\partial x_2}\\ &= -2\frac{\partial^2 u_0}{\partial x_0^2} - 2e_1\frac{\partial^2 u_1}{\partial x_0^2} - 2e_2\frac{\partial^2 u_2}{\partial x_0^2} = -2\frac{\partial^2 f}{\partial x_0^2}.\end{aligned} \quad (57)$$

From Proposition 7, we have $\partial^2 f/\partial A^2 = 4^2(\partial^2 f/\partial x_0^2)$. Hence, by calculating and comparing the above polynomials, we obtain that $\square_a f$ is equal to $-(1/8)(\partial^2/\partial A^2)f$. □

Next, we consider a differential form

$$\omega = 4dx_1 \wedge dx_2 - e_1dx_0 \wedge dx_2 + e_2dx_0 \wedge dx_1. \quad (58)$$

**Theorem 9.** *Let $\Omega$ be an open set in $\mathbb{R}^3$ and let $U$ be any domain on $\Omega$ with a smooth distinguished boundary $bU$ such that $U \subset \Omega$. If $f$ is a regular function on $\Omega$, then one has*

$$\int_{bU} \omega f = 0, \quad (59)$$

*where $\omega f$ is the reduced quaternionic product of the form $\omega$ on the function $f(a)$.*

*Proof.* Since $\omega f = 4fdx_1\wedge dx_2 - e_1fdx_0\wedge dx_2 + e_2fdx_0\wedge dx_1$, we have

$$\begin{aligned}d(\omega f) &= 4\frac{\partial f}{\partial x_0}dx_0 \wedge dx_1 \wedge dx_2 + e_1\frac{\partial f}{\partial x_1}dx_0 \wedge dx_1 \wedge dx_2\\ &\quad + e_2\frac{\partial f}{\partial x_2}dx_0 \wedge dx_1 \wedge dx_2\\ &= 4\frac{\partial(e_1f_1 + e_2f_2)}{\partial x_0}dI + e_1\frac{\partial(e_1f_1 + e_2f_2)}{\partial x_1}dI\\ &\quad + e_2\frac{\partial(e_1f_1 + e_2f_2)}{\partial x_2}dI\\ &= \left\{\left(4\frac{\partial u_0}{\partial x_0} - \frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2}\right) + e_1\left(4\frac{\partial u_1}{\partial x_0} + \frac{\partial u_0}{\partial x_1}\right)\right.\\ &\quad \left. + e_2\left(4\frac{\partial u_2}{\partial x_0} + \frac{\partial u_0}{\partial x_2}\right) + e_1e_2\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)\right\} dI,\end{aligned} \quad (60)$$

where $dI = dx_0 \wedge dx_1 \wedge dx_2$ in $U$. From the corresponding Cauchy-Riemann system (39) for $f(a)$ in $\mathbb{C}(\mathbb{T})$, we have the system (56). Hence, $d(\omega f) = 0$ and, therefore, by Stokes theorem, we obtain the following result:

$$\int_{bU} \omega f = \int_U d(\omega f) = 0. \quad (61)$$

□

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.


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