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Research Article

A New Type of Coincidence and Common Fixed Point Theorem with Applications

Shyam Lal Singh,[1] Raj Kamal,[2] Manuel De la Sen,[3] and Renu Chugh[2]

[1] *Gurukula K. Vishwavidyalaya, Haridwar 249404, India*
[2] *Department of Mathematics, Maharshi Dayanand University, Rohtak 124001, India*
[3] *Institute of Research and Development Processes, University of Basque Country, Campus of Leioa (Bizkaia), Apartado 644, 48080 Bilbao, Spain*

Correspondence should be addressed to Shyam Lal Singh; vedicmri@gmail.com

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Coincidence and common fixed point theorems for a class of Ćirić-Suzuki hybrid contractions involving a multivalued and two single-valued maps in a metric space are obtained. Some applications including the existence of a common solution for certain class of functional equations arising in a dynamic programming are also discussed.

1. Introduction

Consistent with [1] (see also [2, 3]), Y denotes an arbitrary nonempty set, (X, d) a metric space, and $CL(X)$ (resp., $CB(X)$), the collection of all nonempty closed (resp., closed bounded) subsets of X. The hyperspace $(CL(X), H)$ (resp., $(CB(X), H)$) is called the generalized Hausdorff (resp., the Hausdorff) metric space induced by the metric d on X.

For nonempty subsets A, B of X, $d(A, B)$ denotes the gap between the subsets A and B, while

$$\rho(A,B) = \sup\{d(a,b) : a \in A, b \in B\},$$
$$BN(X) = \{A : \emptyset \neq A \subseteq X \text{ and the diameter of } A \text{ is finite}\}. \tag{1}$$

As usual, we write $d(x, B)$ (resp., $\rho(x, B)$) for $d(A, B)$ (resp., $\rho(A, B)$) when $A = \{x\}$.

For the sake of brevity, we follow the following notations, wherein S, f, and g are maps to be defined specifically in a particular context, while x and y are elements of some specific domain:

$$M(S; fx, gy) = \max\left\{d(fx, gy), d(fx, Sx), d(gy, Sy), \frac{d(Sx, gy) + d(Sy, fx)}{2}\right\};$$

$$M(S; fx, fy) = \max\left\{d(fx, fy), d(fx, Sx), d(fy, Sy), \frac{d(Sx, fy) + d(Sy, fx)}{2}\right\};$$

$$M(Sx, Sy) = \max\left\{d(x, y), d(x, Sx), d(y, Sy), \frac{d(y, Sx) + d(x, Sy)}{2}\right\};$$

$$M_1(fx, gy) = \max\left\{d(x,y), d(x,fx), d(y,gy), \frac{d(y,fx)+d(x,gy)}{2}\right\}. \tag{2}$$

The Banach contraction principle (Bcp) plays an important role in nonlinear analysis and has numerous generalizations and several applications (see, e.g., [1–21] and others). Nadler Jr. [1] (see also [22]) initiated the study of multivalued Banach contractions in metric spaces. In view of its numerous applications, the Nadler multivalued contraction theorem received enormous attention (see, e.g., [2, 3, 7, 8, 11–15, 17–21, 23–36] and references thereof).

The following result [13, p. 250] extends and generalizes many results due to Fisher [37], Goebel [38], Kubiak [29], and others.

Theorem 1. *Let $S : Y \to CL(X)$ and $f, g : Y \to X$ be such that $S(Y) \subseteq f(Y) \cap g(Y)$, and one of $S(Y)$, $f(Y)$ or $g(Y)$ is a complete subspace of X. Assume there exists $r \in [0, 1)$ such that for every $x, y \in Y$,*

$$H(Sx, Sy) \le rM(S; fx, gy). \tag{3}$$

Then

(i) *S and f have a coincidence point v in Y,*

(ii) *S and g have a coincidence point w in Y.*

Further, if $Y = X$, then

(iii) *S and f have a common fixed point v provided that fv is a fixed point of f, and f and S commute at v;*

(iv) *S and g have a common fixed point w provided that gw is a fixed point of g, and g and S commute at w;*

(v) *S, f, and g have a common fixed point provided that (iii) and (iv) both are true.*

We remark that certain contractive conditions studied for $S : Y \to CL(X)$ and $f, g : Y \to X$ by Ćirić [5], Covitz and Nadler Jr. [16], Czerwik [6], Fisher [37], Goebel [38], Jungck [17], Kubiak [29], Naimpally et al. [8], Pathak [15], Pathak et al. [9], Petrusel and Rus [10], Reich [11], and Rus [3] are included in the following condition:

$$H(Sx, Sy) \le rM(S; fx, gy), \tag{4}$$

for every $x, y \in Y$, where $0 \le r < 1$.

In particular, (4) with $Y = X$ and $f = g =$ the identity map on X was studied by Ćirić [5].

Recently, Suzuki [39, Th. 2] obtained a remarkable generalization of the Bcp. The same has been extended to multivalued maps by Kikkawa and Suzuki [30] in the following manner.

Theorem 2. *Define a strictly decreasing function $\eta : [0, 1) \to ((1/2), 1]$ by*

$$\eta(r) = \frac{1}{1+r}. \tag{5}$$

Let (X, d) be a complete metric space and $S : Y \to CB(X)$. Assume there exists $r \in [0, 1)$ such that for every $x, y \in Y$,

$$\eta(r)\, d(x, Sx) \le d(x, y) \quad \text{implies } H(Sx, Sy) \le rd(x, y). \tag{6}$$

Then there exists $z \in X$ such that $z \in Sz$.

Subsequently, some interesting extensions and generalizations of Theorem 2 were obtained among others by Abbas et al. [23], Dhompongsa and Yingtaweesittikul [24], Dorić and Lazović [25], Kamal et al. [18], Moţ and Petruşel [26], Singh and Mishra [27, 31, 36], and Singh et al. [28, 32, 33].

The importance of Suzuki contraction theorem [39, Th. 2] and subsequently obtained coincidence and fixed point theorems (cf. [23–28, 30–33, 36] and others) for maps in metric spaces satisfying Suzuki-type contractive conditions is that the contractive conditions are required to be satisfied not for all points of the domain.

In all that follows we take a nonincreasing function φ from $[0, 1)$ onto $(0, 1]$ defined by

$$\varphi(r) = \begin{cases} 1 & \text{if } 0 \le r < \dfrac{1}{2} \\ 1 - r & \text{if } \dfrac{1}{2} \le r < 1. \end{cases} \tag{7}$$

Recently, Singh et al. [33] obtained the following coincidence and common fixed point theorem which is a generalization of a result of Dorić and Lazović [25].

Theorem 3. *Let $S : Y \to CL(X)$ and $f : Y \to X$ be such that $S(Y) \subseteq f(Y)$. Assume there exists $r \in [0, 1)$ such that for every $x, y \in Y$,*

$$\begin{aligned} &\varphi(r)\, d(fx, Sx) \le d(fx, fy) \\ &\text{implies } H(Sx, Sy) \le rM(S; fx, fy). \end{aligned} \tag{8}$$

If one of $S(Y)$ or $f(Y)$ is a complete subspace of X, then there exists a point $z \in Y$ such that $fz \in Sz$.

Further, if $Y = X$ and fz is a fixed point of f, then fz is a fixed point of S provided that f is IT-commuting with S at z.

In this paper, we obtain a coincidence and common fixed point theorem (cf. Theorem 6) extending and generalizing Theorems 1, 2, 3, and several others. We also deduce the existence of common solution for a certain class of functional equations arising in dynamic programming. Examples are given to justify theorems and applications.

2. Main Results

The following definition is due to Itoh and Takahashi [19] (see also [27]).

Definition 4. Let $S : X \to CL(X)$ and $f : X \to X$. Then the hybrid pair (S, f) is IT-commuting at $z \in X$ if $fSz \subseteq Sfz$.

We remark that IT-commuting maps are more general than commuting maps [34, p. 2]. However, a pair of maps $f, g : X \to X$ are IT-commuting (also called weakly compatible by Jungck and Rhoades [20]) at $x \in X$ if $fgx = gfx$ when $fx = gx$.

We will need the following lemma essentially due to Nadler Jr. [1] (see also [5], [2, p. 61], [35, p. 4], [3, p. 76]).

Lemma 5. *If $A, B \in CL(X)$ and $a \in A$, then for each $\varepsilon > 0$, there exists $b \in B$ such that $d(a, b) \le H(A, B) + \varepsilon$.*

Let $C(S, f)$ denote the collection of all coincidence points of S and f; that is, $C(S, f) = \{z \in Y : fz \in Sz\}$ when $S : Y \to CL(X)$ and $f : Y \to X$; and $C(S, f) = \{z \in Y : fz = Sz\}$ when $S, f : Y \to X$.

The following is the main result of this section.

Theorem 6. *Let $S : Y \to CL(X)$ and $f, g : Y \to X$ be such that $S(Y) \subseteq f(Y) \cap g(Y)$. Assume there exists $r \in [0, 1)$ such that for every $x, y \in Y$,*

$$\varphi(r) \min \{d(fx, Sx), d(gy, Sy)\} \le d(fx, gy) \tag{9a}$$

implies

$$H(Sx, Sy) \le rM(S; fx, gy). \tag{9b}$$

If one of $S(Y)$, $f(Y)$, or $g(Y)$ is a complete subspace of X, then

(I) *$C(S, f)$ is nonempty; that is, there exists a point $z \in Y$ such that $fz \in Sz$.*

(II) *$C(S, g)$ is nonempty; that is, there exists a point $z_1 \in Y$ such that $gz_1 \in Sz_1$.*

Further if, $Y = X$, then

(III) *S and f have a common fixed point provided that the maps S and f are IT-commuting just at coincidence point z and fz is fixed point of f;*

(IV) *S and g have a common fixed point provided that the maps S and g are IT-commuting just at coincidence point z_1 and gz_1 is fixed point of g;*

(V) *S, f, and g have a common fixed point provided that both (III) and (IV) are true.*

Proof. Without loss of generality, we may take $r > 0$ and f, g nonconstant maps.

Let $\varepsilon > 0$ be such that $\beta = r + \varepsilon < 1$. We construct two sequences $\{x_n\}$ in Y and $\{y_n\}$ in X as follows.

Let $x_0 \in Y$ and $y_0 = gx_1 \in Sx_0$. By Lemma 5, there exists $y_1 = fx_2 \in Sx_1$ such that

$$d(fx_2, gx_1) \le H(Sx_0, Sx_1) + \varepsilon M(S; fx_0, gx_1). \tag{10}$$

Similarly, there exists $y_2 = gx_3 \in Sx_2$ such that

$$d(fx_2, gx_3) \le H(Sx_2, Sx_1) + \varepsilon M(S; fx_2, gx_1). \tag{11}$$

Continuing in this manner, we find a sequence $\{y_n\}$ in X such that

$$\begin{gathered} y_{2n} = gx_{2n+1} \in Sx_{2n}, \qquad y_{2n+1} = fx_{2n+2} \in Sx_{2n+1}, \\ d(fx_{2n}, gx_{2n+1}) \le H(Sx_{2n}, Sx_{2n-1}) + \varepsilon M(S; fx_{2n}, gx_{2n-1}), \\ d(fx_{2n+2}, gx_{2n+1}) \le H(Sx_{2n}, Sx_{2n+1}) + \varepsilon M(S; fx_{2n}, gx_{2n+1}). \end{gathered} \tag{12}$$

Now, we show that for any $n \in N$,

$$d(y_{2n}, y_{2n-1}) \le \beta d(y_{2n-1}, y_{2n-2}). \tag{13}$$

Suppose if $d(gx_{2n-1}, Sx_{2n-1}) \ge d(fx_{2n}, Sx_{2n})$, then

$$\varphi(r) \min \{d(fx_{2n}, Sx_{2n}), d(gx_{2n-1}, Sx_{2n-1})\} \le d(fx_{2n}, gx_{2n-1}). \tag{14}$$

Therefore, by the assumption,

$$\begin{aligned} d(fx_{2n}, gx_{2n+1}) &\le H(Sx_{2n}, Sx_{2n-1}) + \varepsilon M(S; fx_{2n}, gx_{2n-1}) \\ &\le rM(S; fx_{2n}, gx_{2n-1}) + \varepsilon M(S; fx_{2n}, gx_{2n-1}) \\ &= \beta M(S; fx_{2n}, gx_{2n-1}) \\ &= \beta \max \Big\{ d(fx_{2n}, gx_{2n-1}), d(fx_{2n}, Sx_{2n}), \\ &\qquad d(gx_{2n-1}, Sx_{2n-1}), \\ &\qquad \frac{d(gx_{2n-1}, Sx_{2n}) + d(fx_{2n}, Sx_{2n-1})}{2} \Big\}. \end{aligned} \tag{15}$$

This yields (13).

Suppose if $d(fx_{2n}, Sx_{2n}) \ge d(gx_{2n-1}, Sx_{2n-1})$, then

$$\varphi(r) \min \{d(fx_{2n}, Sx_{2n}), d(gx_{2n-1}, Sx_{2n-1})\} \le d(fx_{2n}, gx_{2n-1}). \tag{16}$$

Therefore, by the assumption,

$$\begin{aligned}
&d\left(fx_{2n}, gx_{2n+1}\right)\\
&\quad\le H\left(Sx_{2n}, Sx_{2n-1}\right) + \varepsilon M\left(S; fx_{2n}, gx_{2n-1}\right)\\
&\quad\le rM\left(S; fx_{2n}, gx_{2n-1}\right) + \varepsilon M\left(S; fx_{2n}, gx_{2n-1}\right)\\
&\quad= \beta M\left(S; fx_{2n}, gx_{2n-1}\right)\\
&\quad= \beta \max\left\{d\left(fx_{2n}, gx_{2n-1}\right), d\left(fx_{2n}, Sx_{2n}\right),\right.\\
&\qquad\qquad d\left(gx_{2n-1}, Sx_{2n-1}\right),\\
&\qquad\qquad \left.\frac{d\left(gx_{2n-1}, Sx_{2n}\right) + d\left(fx_{2n}, Sx_{2n-1}\right)}{2}\right\}\\
&\quad\le \beta \max\left\{d\left(fx_{2n}, gx_{2n-1}\right), d\left(fx_{2n}, gx_{2n+1}\right)\right\},
\end{aligned} \tag{17}$$

yielding (13). So, in both cases, we obtain (13). In an analogous manner, we show that

$$d\left(y_{2n+1}, y_{2n}\right) \le \beta d\left(y_{2n}, y_{2n-1}\right). \tag{18}$$

We conclude from (13) and (18) that for any $n \in N$,

$$d\left(y_{n+1}, y_n\right) \le \beta d\left(y_n, y_{n-1}\right). \tag{19}$$

Therefore the sequence $\{y_n\}$ is Cauchy. Assume that the space $g(Y)$ is complete. Notice that the sequence $\{y_{2n}\}$ is contained in $g(Y)$ and has a limit in $g(Y)$. Call it u. Let $z \in f^{-1}u$. Then $z \in Y$ and $fz = u$. The subsequence $\{y_{2n+1}\}$ also converges to u. Let $z_1 \in g^{-1}u$. Then

$$gz_1 = u. \tag{20}$$

Now we show that for any $gy \in X - \{fz\}$,

$$d\left(u, Sy\right) \le r \max\left\{d\left(u, gy\right), d\left(gy, Sy\right)\right\}, \tag{21}$$

and for any $fy \in X - \{gz\}$,

$$d\left(u, Sy\right) \le r \max\left\{d\left(u, fy\right), d\left(fy, Sy\right)\right\}. \tag{22}$$

Since $fx_{2n} \to fz$, there exists $n_0 \in N$ (naturals) such that

$$d\left(fx_{2n}, fz\right) \le \frac{1}{3} d\left(fz, gy\right) \quad \text{for } gy \ne fz \text{ and all } n \ge n_0. \tag{23}$$

Also, since $gx_{2n+1} \to fz$, there exists $n_1 \in N$ such that

$$\begin{gathered}
d\left(gx_{2n+1}, fz\right) \le \frac{1}{3} d\left(fz, gy\right)\\
\text{for } gy \ne fz \text{ and all } n \ge n_1.
\end{gathered} \tag{24}$$

Then, as in [39, p. 1862] (see also [25]),

$$\begin{aligned}
&\varphi(r)\, d\left(fx_{2n}, Sx_{2n}\right)\\
&\quad\le d\left(fx_{2n}, Sx_{2n}\right) \le d\left(fx_{2n}, gx_{2n+1}\right)\\
&\quad\le \frac{2}{3} d\left(fz, gy\right) = d\left(fz, gy\right) - \frac{1}{3} d\left(fz, gy\right)\\
&\quad\le d\left(fz, gy\right) - d\left(fx_{2n}, fz\right) \le d\left(fx_{2n}, gy\right).
\end{aligned} \tag{25}$$

Therefore,

$$\varphi(r)\, d\left(fx_{2n}, Sx_{2n}\right) \le d\left(fx_{2n}, gy\right). \tag{26}$$

Now, either $d(fx_{2n}, Sx_{2n}) \le d(gy, Sy)$ or $d(gy, Sy) \le d(fx_{2n}, Sx_{2n})$.

In each case, by (26) and the assumption,

$$\begin{aligned}
&d\left(fx_{2n+1}, Sy\right)\\
&\quad\le H\left(Sx_{2n}, Sy\right) \le rM\left(S; fx_{2n}, gy\right).\\
&\quad\le r \max\left\{d\left(fx_{2n}, gy\right), d\left(fx_{2n}, Sx_{2n}\right), d\left(gy, Sy\right),\right.\\
&\qquad\qquad \left.\frac{d\left(fx_{2n}, Sy\right) + d\left(gy, Sx_{2n}\right)}{2}\right\}.
\end{aligned} \tag{27}$$

Making $n \to \infty$,

$$\begin{aligned}
&d\left(u, Sy\right)\\
&\quad\le r \max\left\{d\left(u, gy\right), d\left(u, u\right), d\left(gy, Sy\right),\right.\\
&\qquad\qquad \left.\frac{d\left(u, Sy\right) + d\left(u, gy\right)}{2}\right\}\\
&\quad\le r \max\left\{d\left(u, gy\right), d\left(gy, Sy\right), \frac{d\left(u, Sy\right) + d\left(u, gy\right)}{2}\right\}\\
&\quad= r \max\left\{d\left(u, gy\right), d\left(gy, Sy\right)\right\}.
\end{aligned} \tag{28}$$

This yields (21); that is,

$$d\left(fz, Sy\right) \le r \max\left\{d\left(fz, gy\right), d\left(gy, Sy\right)\right\}. \tag{29}$$

Analogously, we can prove (22); that is,

$$d\left(gz_1, Sy\right) \le r \max\left\{d\left(gz_1, fy\right), d\left(fy, Sy\right)\right\}. \tag{30}$$

Now, we show that $C(S, f)$ is nonempty.

We first consider the case $0 \le r < 1/2$.

Suppose $fz \notin Sz$. Then as in [24, p. 6], let $ga \in Sz$ be such that $2rd(ga, fz) < d(Sz, fz)$.

Since $ga \in Sz$ implies $ga \ne fz$, we have from (21) and (22),

$$d\left(fz, Sa\right) \le r \max\left\{d\left(fz, ga\right), d\left(ga, Sa\right)\right\}. \tag{31}$$

On the other hand, since $\varphi(r)d(fz, Sz) \le d(fz, Sz) \le d(fz, ga)$,

$$\varphi(r) \min\left\{d\left(fz, Sz\right), d\left(ga, Sa\right)\right\} \le d\left(fz, ga\right). \tag{32}$$

Therefore, by the assumption (13),

$$\begin{aligned} d(ga, Sa) &\le H(Sz, Sa) \\ &\le r\max\left\{d(fz, ga), d(fz, Sz), d(ga, Sa), \frac{d(fz, Sa) + d(ga, Sz)}{2}\right\} \\ &= r\max\{d(fz, ga), d(ga, Sa)\}. \end{aligned} \tag{33}$$

This gives $d(ga, Sa) \le H(Sz, Sa) \le rd(fz, ga) < d(fz, ga)$.

So by (31), $d(fz, Sa) \le rd(fz, ga)$. Thus, by the assumption,

$$\begin{aligned} d(fz, Sz) &\le d(fz, Sa) + H(Sz, Sa) \\ &\le rd(fz, ga) + rd(fz, ga) \\ &= 2rd(fz, ga) < d(fz, Sz). \end{aligned} \tag{34}$$

This contradicts $fz \notin Sz$. Consequently, $fz \in Sz$, and $C(S, f)$ is nonempty.

In an analogous manner, we can prove in the case $0 \le r < 1/2$ that $C(S, g)$ is nonempty.

We now consider the case $1/2 \le r < 1$. We first show that

$$\begin{aligned} &H(Sz, Sy) \\ &\le r\max\left\{d(fz, gy), d(fz, Sz), d(gy, Sy), \frac{d(gy, Sz) + d(fz, Sy)}{2}\right\}. \end{aligned} \tag{35}$$

Assume that $fz \ne gy$. Then for every $n \in N$, there exists $z_n \in Sy$ such that

$$d(fz, z_n) \le d(fz, Sy) + \frac{1}{n}d(fz, gy). \tag{36}$$

Therefore,

$$\begin{aligned} d(gy, Sy) &\le d(gy, z_n) \\ &\le d(gy, fz) + d(fz, z_n) \\ &\le d(gy, fz) + d(fz, Sy) + \frac{1}{n}d(fz, gy). \end{aligned} \tag{37}$$

So using (31), the inequality (37) implies

$$\begin{aligned} d(gy, Sy) \le{}& d(fz, gy) + r\max\{d(fz, gy), d(gy, Sy)\} \\ &+ \frac{1}{n}d(fz, gy). \end{aligned} \tag{38}$$

If $d(fz, gy) \ge d(gy, Sy)$, then (38) gives

$$\begin{aligned} d(gy, Sy) &\le d(fz, gy) + rd(fz, gy) + \frac{1}{n}d(fz, gy) \\ &= \left(1 + r + \frac{1}{n}\right)d(fz, gy). \end{aligned} \tag{39}$$

Making $n \to \infty$,

$$d(gy, Sy) \le (1 + r)\, d(fz, gy). \tag{40}$$

Thus,

$$\begin{aligned} \varphi(r)\, d(gy, Sy) &= (1 - r)\, d(gy, Sy) \\ &\le \left(\frac{1}{1 + r}\right) d(gy, Sy) \le d(fz, gy). \end{aligned} \tag{41}$$

Then

$$\varphi(r)\min\{d(fz, Sz), d(gy, Sy)\} \le d(fz, gy), \tag{42}$$

and by the assumption,

$$\begin{aligned} &H(Sz, Sy) \\ &\le r\max\left\{d(fz, gy), d(fz, Sz), d(gy, Sy), \frac{d(gy, Sz) + d(fz, Sy)}{2}\right\}. \end{aligned} \tag{43}$$

If $d(fz, gy) < d(gy, Sy)$, then (38) gives

$$d(gy, Sy) \le d(fz, gy) + rd(gy, Sy) + \frac{1}{n}d(fz, gy); \tag{44}$$

that is, $(1 - r)d(gy, Sy) \le (1 + 1/n)d(fz, gy)$.

Making $n \to \infty$, $\varphi(r)d(gy, Sy) \le d(fz, gy)$.

Then $\varphi(r)\min\{d(fz, Sz), d(gy, Sy)\} \le d(fz, gy)$, and by the assumption, we get (43).

Since $d(Sz, fx_{2n+2}) \le H(Sz, Sx_{2n+1})$, taking $y = x_{2n+1}$ in (43) and passing to the limit, we obtain

$$d(Sz, fz) \le rd(fz, Sz). \tag{45}$$

This gives $fz \in Sz$; that is, z is a coincidence point of f and S. Analogously, $gz \in Sz$. Thus, (I) and (II) are completely proved.

Further, if $Y = X$, fz is a fixed point of f, and S and f are IT-commuting at z, then $fSz \subseteq Sfz$. Therefore, $fz \in Sz$ implies $ffz \in fSz \subseteq Sfz$, so $fz \in Sfz$. This proves that $u = fz$ is a common fixed point of f and S. This proves (III). Analogously, S and g have a common fixed point gz_1. Therefore (20) implies that u is a common fixed point of S and g. This proves (IV). Now (V) is immediate. □

Remark 7. In Theorem 6, the hypothesis "fz is a fixed point of f" is essential for the existence of a common fixed point of S and f (see also [8]). Similarly, the hypothesis "gz_1 is a fixed point of g" is essential for the existence of a common fixed point of S and g. Further, the contractive condition for three maps $S : Y \to CL(X)$ and $f, g : Y \to X$ studied by Abbas et al. [23] are included in the assumptions of Theorem 6.

Corollary 8. *Theorem 2.*

Proof. It comes from Theorem 6 when $g = f$. □

The following result due to Dorić and Lazović [25] generalizing many fixed point theorems is obtained as a special case from Theorem 6 when $Y = X$ and f and g are the identity map on X.

Corollary 9. *Let (X, d) be a complete metric space and $S : X \to CL(X)$. Assume there exists $r \in [0,1)$ such that for every $x, y \in X$,*

$$\varphi(r)\, d(x, Sx) \le d(x, y) \quad \text{implies } H(Sx, Sy) \le rM(Sx, Sy). \tag{46}$$

Then there exists an element $z \in X$ such that $z \in Sz$.

The following result extends and generalizes coincidence and fixed point theorems of Fisher [37], Goebel [38], Jungck [17], and others.

Corollary 10. *Let $f, g, P : Y \to X$ be such that $P(Y) \subseteq f(Y) \cap g(Y)$. Let $P(Y)$ or $f(Y)$ or $g(Y)$ be a complete subspace of X. Assume there exists $r \in [0,1)$ such that for every $x, y \in Y$,*

$$\varphi(r) \min\{d(fx, Px), d(gy, Py)\} \le d(fx, gy), \tag{47}$$

implies

$$d(Px, Py) \le rM(P; fx, gy). \tag{48}$$

Then $C(P, f)$ and $C(P, g)$ are nonempty. Further, if $Y = X$ and if P commutes with f and g at a common coincidence point, then f, g, and P have a unique common fixed point; that is, there exists a unique point $z \in X$ such that $fz = gz = Pz = z$.

Proof. Set $Sx = \{Px\}$ for every $x \in Y$. Then it easily comes from Theorem 6 that $C(P, f)$ and $C(P, g)$ are nonempty. Further, if $Y = X$ and P commutes with f and g at z, then $ffz = fPz = Pfz$ and $ggz = gPz = Pgz$.

Also $\varphi(r)\min\{d(fz, Pz), d(ffz, Pfz)\} = 0 \le d(fz, ffz)$, and this implies

$$\begin{aligned} d(Pz, Pfz) &\le r \max\Big\{d(fz, ffz), d(fz, Pz), d(ffz, Pfz), \\ &\qquad \frac{d(fz, Pfz) + d(ffz, Pz)}{2}\Big\} \\ &= rd(Pz, Pfz). \end{aligned} \tag{49}$$

This says that fz is fixed point of f and P. Analogously gz is fixed point of g and P. The uniqueness of the common fixed point follows easily. □

Corollary 11. *Let (X, d) be a complete metric space and let $f, g : X \to X$ be an onto maps. Assume there exists $r \in [0,1)$ such that for every $x, y \in X$,*

$$\varphi(r)\min\{d(x, fx), d(y, gy)\} \le d(fx, gy) \quad \text{implies } d(x, y) \le rM_1(fx, gy). \tag{50}$$

Then f and g have a unique common fixed point.

Proof. It comes from Corollary 10 when $Y = X$ and P is the identity map on X. □

Corollary 12. *Let (X, d) be a complete metric space and let $f : X \to X$ be onto maps. Assume there exists $r \in [0,1)$ such that for every $x, y \in X$,*

$$\varphi(r)\, d(x, fx) \le d(fx, fy) \quad \text{implies } d(x, y) \le rM(fx, fy). \tag{51}$$

Then f has a unique fixed point.

Proof. It comes from Corollary 11 when $f = g$. □

The following example shows that Theorem 6 is indeed more general than Theorem 1.

Example 13. Consider a metric space $X = \{(0,0), (0,1), (1,0), (1,2), (2,1)\}$, where d is defined by

$$d[(x_1, x_2), (y_1, y_2)] = |x_1 - y_1| + |x_2 - y_2|. \tag{52}$$

Let S, f and $g : X \to X$ be such that

$$\begin{aligned} S(x_1, x_2) &= \begin{cases} (0,0) & \text{if } (x_1, x_2) \ne (1,2), (2,1) \\ (0,1) & \text{if } (x_1, x_2) = (1,2) \\ (1,0) & \text{if } (x_1, x_2) = (2,1), \end{cases} \\ f(x_1, x_2) &= (x_2, x_1) \quad \forall (x_1, x_2) \in X, \\ g(x_1, x_2) &= \begin{cases} (x_1, x_2) & \text{if } (x_1, x_2) \ne (1,0) \\ (0,1) & \text{if } (x_1, x_2) = (1,0). \end{cases} \end{aligned} \tag{53}$$

It is readily verified that

$$\begin{aligned} &d(Sx, Sy) \\ &\le \frac{1}{2} \max\Big\{d(fx, gy), d(fx, Sx), d(gy, Sy), \\ &\qquad \frac{d(Sx, gy) + d(Sy, fx)}{2}\Big\}, \end{aligned} \tag{54}$$

for all $(x, y) \in X$ except for $x, y \in \{(1,2), (2,1)\}$ with $r = 1/2$.

For $x, y \in \{(1,2), (2,1)\}$, condition (3) yields $2 \le 2r$, which contradicts $0 \le r < 1$. Therefore, the condition (3) of Theorem 1 is not satisfied. So, in order to see that the maps S, f, and g satisfy the assumption of Theorem 6, we notice that the condition (9a) of Theorem 6 does not hold for $x, y \in \{(1,2), (2,1)\}$. Indeed, for $(x, y) = ((1,2), (2,1))$,

$$\begin{aligned} &\varphi(r)\min\{d(fx, Sx), d(gy, Ty)\} \\ &\quad = \varphi(r)\min\{d(f(1,2), S(1,2)), d(g(2,1), T(2,1))\} \\ &\quad = \varphi(r)\min\{2, 2\} = 2\varphi(r). \end{aligned} \tag{55}$$

That is, $\varphi(r)\min\{d(fx, Sx), d(gy, Ty)\} = 1 > 0 = d(fx, gy)$.

This violates (9a) when $\varphi(r) = 1/2$ (as $r = 1/2$). Similarly (9a) is also not true for $(x, y) = ((2,1), (1,2))$. It is easily seen that all other hypotheses of Theorem 6 are also true.

Now we give an application of Corollary 10.

Theorem 14. *Let $S : Y \to BN(X)$ and $f, g : Y \to X$ be such that $S(Y) \subseteq f(Y) \cap g(Y)$, and let one of $S(Y)$, $f(Y)$, or $g(Y)$ be a complete subspace of X. Assume there exists $r \in [0,1)$ such that for every $x, y \in Y$,*

$$\varphi(r)\min\{\rho(fx,Sx),\rho(gy,Sy)\} \le d(fx,gy), \tag{56}$$

implies

$$\begin{aligned}\rho(Sx,Sy) &\le r\max\Big\{d(fx,gy),\rho(fx,Sx),\rho(gy,Sy),\\ &\qquad \frac{d(fx,Sy)+d(gy,Sx)}{2}\Big\}.\end{aligned} \tag{57}$$

Then $C(S,f)$ and $C(S,g)$ are nonempty.

Proof. Choose $\lambda \in (0,1)$. Define single-valued maps $h_1, h_2 : X \to X$ as follows. For each $x \in X$, let h_1x be a point of Sx which satisfies

$$d(fx,h_1x) \ge r^{\lambda}\rho(fx,Sx). \tag{58}$$

Similarly, for each $y \in X$, let h_2y be a point of Sy such that

$$d(gy,h_2y) \ge r^{\lambda}\rho(gy,Sy). \tag{59}$$

Since $h_1x \in Sx$ and $h_2y \in Sy$,

$$d(fx,h_1x) \le \rho(fx,Sx), \qquad d(gy,h_2y) \le \rho(gy,Sy). \tag{60}$$

So (56) gives

$$\begin{aligned}&\varphi(r)\min\{d(fx,h_1x),d(gy,h_2y)\}\\ &\quad\le \varphi(r)\min\{\rho(fx,Sx),\rho(gy,Sy)\} \le d(fx,gy),\end{aligned} \tag{61}$$

and this implies (57). Therefore,

$$\begin{aligned}&d(h_1x,h_2y)\\ &\le \rho(Sx,Sy)\\ &\le r\cdot r^{-\lambda}\max\Big\{r^{\lambda}d(fx,gy),r^{\lambda}\rho(fx,Sx),r^{\lambda}\rho(gy,Sy),\\ &\qquad\frac{r^{\lambda}d(fx,Sy)+r^{\lambda}d(gy,Sx)}{2}\Big\}\\ &\le r^{1-\lambda}\max\Big\{d(fx,gy),d(fx,h_1x),d(gy,h_2y),\\ &\qquad\frac{d(fx,h_2y)+d(gy,h_1x)}{2}\Big\}.\end{aligned} \tag{62}$$

So (61), namely, $\varphi(r')\min\{d(fx,h_1x),d(gy,h_2y)\} \le d(fx,gy)$, implies

$$\begin{aligned}d(h_1x,h_2y) &\le r'\max\Big\{d(fx,gy),d(fx,h_1x),d(gy,h_2y),\\ &\qquad\frac{d(fx,h_2y)+d(gy,h_1x)}{2}\Big\},\end{aligned} \tag{63}$$

where $r' = r^{1-\lambda} < 1$.

Hence, by Corollary 10, there exist $z_1, z_2 \in Y$ such that $h_1z_1 = fz_1$ and $h_2z_2 = gz_2$. This implies that z_1 is a coincidence point of f and S, and z_2 is a coincidence point of g and S. □

Corollary 15. *Let $S : Y \to BN(X)$ and $f : Y \to X$ be such that $S(Y) \subseteq f(Y)$, and let $S(Y)$ or $f(Y)$ be a complete subspace of X. Assume there exists $r \in [0,1)$ such that for every $x, y \in Y$,*

$$\varphi(r)\rho(fx,Sx) \le d(fx,fy), \tag{64}$$

implies

$$\begin{aligned}\rho(Sx,Sy) &\le r\max\Big\{d(fx,fy),\rho(fx,Sx),\rho(fy,Sy),\\ &\qquad\frac{d(fx,Sy)+d(fy,Sx)}{2}\Big\}.\end{aligned} \tag{65}$$

Then there exists $z \in Y$ such that $fz \in Sz$.

Proof. It comes from Theorem 14 when $g = f$. □

Corollary 16. *Let X be a complete metric space and let $S : X \to BN(X)$. Assume there exists $r \in [0,1)$ such that for every $x, y \in X$,*

$$\varphi(r)\rho(x,Sx) \le d(x,y), \tag{66}$$

implies

$$\begin{aligned}\rho(Sx,Sy) &\le r\max\Big\{d(x,y),\rho(x,Sx),\rho(y,Sy),\\ &\qquad\frac{d(x,Sy)+d(y,Sx)}{2}\Big\}.\end{aligned} \tag{67}$$

Then there exists a unique point $z \in X$ such that $z \in Sz$.

Proof. It comes from Theorem 14 that S has a fixed point when $f = g$ is the identity map on X. The uniqueness of the fixed point follows easily. □

3. Applications

Throughout this section, we assume that U and V are Banach spaces, $W \subseteq U$, and $D \subseteq V$. Let R denote the field of reals,

$\tau : W \times D \to W$, $g, g' : W \times D \to R$, and $G, F_1, F_2 : W \times D \times R \to R$. Considering W and D as the state and decision spaces, respectively, the problem of dynamic programming reduces to the problem of solving the functional equations:

$$p = \sup_{y \in D} \{g(x, y) + G(x, y, p(\tau(x, y)))\}, \quad x \in W, \tag{68a}$$

$$q_i = \sup_{y \in D} \{g'(x, y) + F_i(x, y, q(\tau(x, y)))\}, \quad x \in W, \quad i = 1, 2. \tag{68b}$$

Indeed, in the multistage process, some functional equations arise in a natural way (cf. Bellman [40] and Bellman and Lee [41]; see also [6, 9, 15, 28, 33, 42–45]). In this section, we study the existence of a common solution of the functional equations (68a) and (68b) arising in the dynamic programming.

Let $B(W)$ denote the set of all bounded real-valued functions on W. For an arbitrary $h \in B(W)$, define $\|h\| = \sup_{x \in W} |h(x)|$. Then $(B(W), \|\cdot\|)$ is a Banach space. Suppose that the following conditions hold:

(DP-1) G, F_1, F_2, g, and g' are bounded.

(DP-2) Let $\varphi(r)$ be considered as in the previous sections. Assume that there exists $r \in [0, 1)$ such that for every $(x, y) \in W \times D$, $h, k \in B(W)$ and $t \in W$,

$$\varphi(r) \min\{|J_1 h(t) - Ah(t)|, |J_2 k(t) - Ak(t)|\} \leq |J_1 h(t) - J_2 k(t)|, \tag{69}$$

implies

$$|G(x, y, h(t)) - G(x, y, k(t))| \leq rM(A; J_1 h, J_2 k), \tag{70}$$

where

$$M(A; J_1 h, J_2 k) = \max\left\{ |J_1 h(t) - J_2 k(t)|, |J_1 h(t) - Ah(t)|, |J_2 k(t) - Ak(t)|, \frac{|J_1 h(t) - Ak(t)| + |J_2 k(t) - Ah(t)|}{2} \right\}, \tag{71}$$

and A, J_1, and J_2 are defined as follows:

$$\begin{aligned} Ah(x) &= \sup_{y \in D} \{g(x, y) + G(x, y, h(\tau(x, y)))\}, \quad x \in W, \quad h \in B(W), \\ J_i h(x) = q_i &= \sup_{y \in D} \{g'(x, y) + F_i(x, y, h(\tau(x, y)))\}, \quad x \in W, \quad h \in B(W), \quad i = 1, 2. \end{aligned} \tag{72}$$

(DP-3) For any $h, k \in B(W)$, there exists $u, v \in B(W)$ such that

$$Ah(x) = J_1 u(x), \quad Ak(x) = J_2 v(x), \quad x \in W. \tag{73}$$

(DP-4) There exists $h, k \in B(W)$ such that

$$\begin{aligned} J_1 h(x) = Ah(x) \quad &\text{implies } J_1 Ah(x) = AJ_1 h(x), \\ J_2 k(x) = Ak(x) \quad &\text{implies } J_2 Ak(x) = AJ_2 k(x). \end{aligned} \tag{74}$$

Theorem 17. *Assume the conditions (DP-1)–(DP-4). Let $J(B(W))$ be a closed convex subspace of $B(W)$. Then the functional equations* (68a) *and* (68b), *$i = 1, 2$, have a unique bounded common solution in $B(W)$.*

Proof. For any $h, k \in B(W)$, let $d(h, k) = \sup\{|h(x) - k(x)| : x \in W\}$. Then $(B(W), d)$ is a complete metric space. By virtue of (DP-3) and (DP-4), $A(B(W)) \subseteq J_1(B(W)) \cap J_2(B(W))$ and the map A is IT-commuting with J_1 and J_2 at coincidence points.

Let λ be an arbitrary positive number and $h_1, h_2 \in B(W)$. Pick $x \in W$, and choose $y_1, y_2 \in D$ such that

$$Ah_j < g(x, y_j) + G(x, y_j, h_j(x_j)) + \lambda, \quad j = 1, 2, \tag{75}$$

where $x_j = \tau(x, y_j)$. Further,

$$Ah_1 \geq g(x, y_2) + G(x, y_2, h_1(x_2)), \tag{76}$$

$$Ah_2 \geq g(x, y_1) + G(x, y_1, h_2(x_1)). \tag{77}$$

Therefore, the first inequality in (DP-2) becomes

$$\varphi(r) \min\{|J_1 h_1(x) - Ah_1(x)|, |J_2 h_2(x) - Ah_2(x)|\} \leq |J_1 h_1(x) - J_2 h_2(x)|, \tag{78}$$

and this together with (75), (77), and (78) implies

$$\begin{aligned} Ah_1 - Ah_2 &< G(x, y_1, h_1(x_1)) - G(x, y_1, h_2(x_1)) + \lambda \\ &\leq |G(x, y_1, h_1(x_1)) - G(x, y_1, h_2(x_1))| + \lambda \\ &\leq rM(A; J_1 h_1, J_2 h_2) + \lambda. \end{aligned} \tag{79}$$

Similarly, (75), (76), and (78) imply

$$Ah_2(x) - Ah_1(x) \leq rM(A; J_1 h_1, J_2 h_2) + \lambda. \tag{80}$$

So, from (79) and (80), we obtain

$$|Ah_1(x) - Ah_2(x)| \leq rM(A; J_1 h_1, J_2 h_2) + \lambda. \tag{81}$$

As $\lambda > 0$ is arbitrary and (81) is true for any $x \in W$, taking supremum, we find from (78) and (81) that

$$\varphi(r) \min\{d(J_1 h_1, Ah_1), d(J_2 h_2, Ah_2)\} \leq d(J_1 h_1, J_2 h_2), \tag{82}$$

implies

$$d(Ah_1, Ah_2) \leq rM(A; J_1 h_1, J_2 h_2). \tag{83}$$

Therefore, Corollary 10 applies, wherein A, J_1 and J_2 correspond, respectively, to the maps P, f, and g. So (A, J_1) and (A, J_2) have a unique common fixed point h^*; that is, $h^*(x)$ is the unique bounded common solution of the functional equations (68a) and (68b), $i = 1, 2$. □

Now we furnish an example in support of Theorem 17.

Example 18. Let $X = Y = R$ be a Banach space endowed with the standard norm $\|\cdot\|$ defined by $\|x\| = |x|$, for all $x \in X$.

Suppose $W = [0,1] \subseteq X$ be the state space and $D = [0,\infty) \subseteq Y$ the decision space. Define $\tau : W \times D \to W$ by

$$\tau(x,y) = \frac{x}{y^2+1}, \quad x \in W,\ y \in D. \tag{84}$$

For any $h, k \in B(W)$ and $i = 1, 2$, define $p, q_i : W \to R$ by

$$p(x) = q_i(x) = x^2 + \frac{1}{2}. \tag{85}$$

Define $G, F_1, F_2, g, g' : W \times D \times R \to R$ by

$$\begin{aligned} G(x,y,t) &= \frac{1}{4}\left\{\frac{x}{(x+1)(y+1)}\sin\frac{y}{y+1} + 2\right\}, \\ F_1(x,y,t) &= \frac{1}{2x+y+1} + \frac{1}{2}\sin t, \\ F_2(x,y,t) &= \frac{1}{2x+3y+1} + \frac{1}{2}\sin t, \\ g(x,y) &= \frac{x^2y^2}{x+y^2}, \qquad g'(x,y) = \frac{x^2y^5}{x+y^5}. \end{aligned} \tag{86}$$

Notice that G, F_1, F_2, g, and g' are bounded. Also

$$\begin{aligned} J_1h(x) &= \sup_{y\in D}\left\{g'(x,y) + F_1(x,y,h(\tau(x,y)))\right\}, \\ &= x^2 + \frac{1}{2} = q_1(x); \\ &\qquad x \in W, \quad h \in B(W), \\ J_2k(x) &= \sup_{y\in D}\left\{g'(x,y) + F_2(x,y,k(\tau(x,y)))\right\}, \\ &= x^2 + \frac{1}{2} = q_2(x); \\ &\qquad x \in W, \quad h \in B(W), \\ Ah(x) &= \sup_{y\in D}\left\{g(x,y) + G(x,y,h(\tau(x,y)))\right\}, \\ &= x^2 + \frac{1}{2} = p(x); \\ &\qquad x \in W, \quad h \in B(W), \\ Ak(x) &= \sup_{y\in D}\left\{g(x,y) + G(x,y,k(\tau(x,y)))\right\}, \\ &= x^2 + \frac{1}{2} = p(x); \\ &\qquad x \in W, \quad h \in B(W). \end{aligned} \tag{87}$$

Now

$$\begin{aligned} &\varphi(r)\min\left\{\left|J_1h(t) - Ah(t)\right|, \left|J_2k(t) - Ak(t)\right|\right\} \\ &\quad = \varphi(r)\min\left\{\left|q_1(x) - p(x)\right|, \left|q_2(x) - p(x)\right|\right\} \\ &\quad = 0 = \left|J_1h(t) - J_2k(t)\right|. \end{aligned} \tag{88}$$

Thus,

$$\begin{aligned} &\varphi(r)\min\left\{|Jh(t) - Ah(t)|, |Jk(t) - Ak(t)|\right\} \\ &\quad = |Jh(t) - Jk(t)|, \end{aligned} \tag{89}$$

and this implies

$$\left|G(x,y,h(t)) - G(x,y,k(t))\right| = 0 \le rM(A; Jh(t), Jk(t)). \tag{90}$$

Finally, for any $h, k \in B(W)$ with $Ah = Jh$, we have

$$AJh = p(x) = q(x) = JJh = JAh; \tag{91}$$

that is, $JAh = AJh$, and with $Ak = Jk$, we have $AJk = p(x) = q(x) = JJk = JAk$; that is, $JAk = AJk$.

Thus, all the hypotheses of Theorem 17 are satisfied. So the system of (68a) and (68b) has a unique solution in $B(W)$.

Corollary 19. *Suppose that the following conditions hold.*

(i) G, F, g, and g' are bounded.

(ii) Assume there exists $r \in [0,1)$ such that for every $(x,y) \in W \times D$, $h, k \in B(W)$ and $t \in W$,

$$\varphi(r)|Jh(t) - Ah(t)| \le |Jh(t) - Jk(t)|, \tag{92}$$

implies

$$\begin{aligned} &\left|G(x,y,h(t)) - G(x,y,k(t))\right| \\ &\quad \le r\max\left\{|Jh(t) - Jk(t)|, |Jh(t) - Ah(t)|,\right. \\ &\qquad\qquad |Jk(t) - Ak(t)|, \\ &\qquad\qquad \left.\frac{|Jh(t) - Ak(t)| + |Jk(t) - Ah(t)|}{2}\right\}, \end{aligned} \tag{93}$$

where A and J are defined as follows:

$$\begin{aligned} Ah(x) &= \sup_{y\in D}\left\{g(x,y) + G(x,y,h(\tau(x,y)))\right\}, \\ &\qquad x \in W, \quad h \in B(W), \\ Jh(x) &= q = \sup_{y\in D}\left\{g'(x,y) + F(x,y,h(\tau(x,y)))\right\}, \\ &\qquad x \in W, \quad h \in B(W). \end{aligned} \tag{94}$$

(iii) For any $h, k \in B(W)$, there exists $u, v \in B(W)$ such that

$$Ah(x) = Ju(x), \quad Ak(x) = Jv(x), \quad x \in W. \tag{95}$$

(iv) There exists $h, k \in B(W)$ such that

$$\begin{aligned} Jh(x) = Ah(x) \quad &\text{implies } JAh(x) = AJh(x), \\ Jk(x) = Ak(x) \quad &\text{implies } JAk(x) = AJk(x). \end{aligned} \tag{96}$$

Then the functional equations (68a) *and* (68b) *with $F_1 = F_2 = F$ possess a unique bounded common solution in W.*

Proof. It comes from Theorem 17 when $F_1 = F_2 = F$. □

Now we derive the the following result due to Dorić and Lazović [25], which in turn extends certain results from [41, 42].

Corollary 20. *Suppose that the following conditions hold.*

(i) G and g are bounded.

(ii) There exists $r \in [0, 1)$ such that for every $(x, y) \in W \times D$, $h, k \in B(W)$ and $t \in W$,

$$\varphi(r)|h(t) - Kh(t)| \le |h(t) - k(t)| \tag{97}$$

implies

$$\begin{aligned} &\left|G(x, y, h(t)) - G(x, y, k(t))\right| \\ &\quad \le r \max M(K, K; h(t), k(t)), \end{aligned} \tag{98}$$

where K is defined as

$$\begin{aligned} Ah(x) = \sup_{y\in D} \{g(x, y) + G(x, y, h(\tau(x, y)))\}, \\ x \in W, \quad h \in B(W). \end{aligned} \tag{99}$$

Then the functional equation (68a) *with $G_1 = G_2 = G$ possesses a unique bounded solution in W.*

Proof. It comes from Corollary 19 when $g' = 0$, $\tau(x, y) = x$ and $F(x; y; t) = t$ as the assumption (DP-3) becomes redundant in this context. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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